Functional Analysis
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The Institute of Mathematical Sciences
Lecture No. 61

Exercises – Part 2

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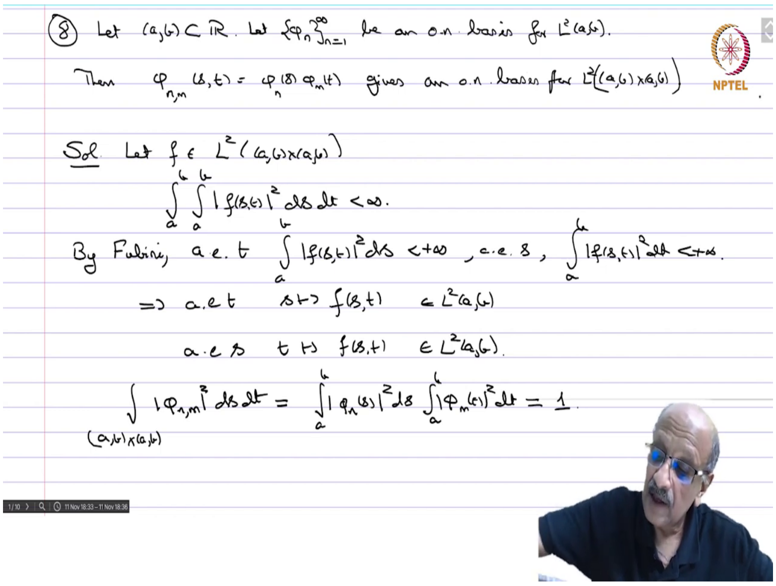
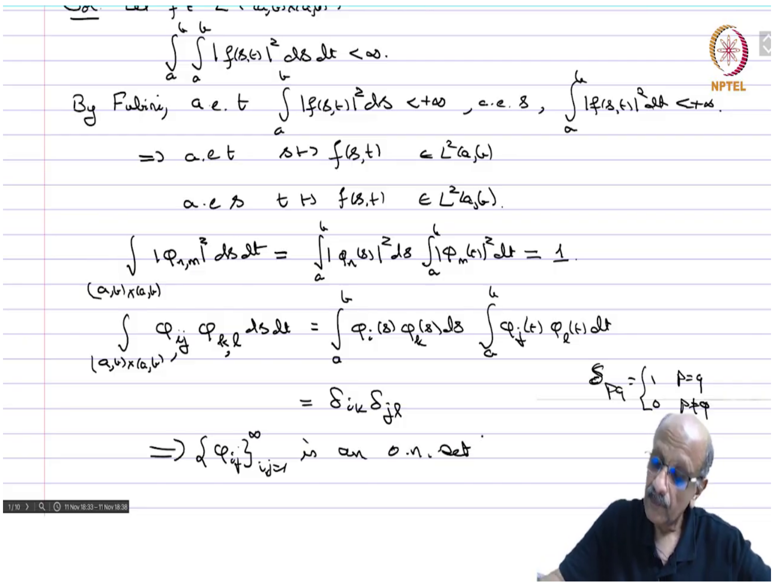




We continue with the exercises. We are at number 8.

8. Let $(a, b) \subset \mathbb{R}$, let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis for $L^2(a, b)$. Recall that L^2 is a separable space therefore, it has a countable orthonormal basis. Then $\phi_{n,m}(s, t) = \phi_n(s)\, \phi_m(t)$gives an orthonormal basis for $L^2((a, b) \times (a, b))$.

Solution: Let $f \in L^2((a, b) \times (a, b))$ then we have, $\int_a^b \int_a^b |f(s, t)|^2 ds\, dt < \infty$

By Fubini's theorem, for a. e. t, $\int_a^b |f(s, t)|^2 ds <+ \infty$, a. e. s, $\int_a^b |f(s, t)|^2 dt <+ \infty$.

$\Rightarrow$a. e. t, $s \mapsto f(s, t) \in L^2(a, b)$ -------------------(*)

and similarly a. e. s, $t \mapsto f(s, t) \in L^2(a, b)$ ------------------------(**)

This is just a direct application of Fubini's theorem.

Let us consider $\int_{(a,b)\times(a,b)} |\phi_{n,m}|^2 ds\, dt$

Since the functions are variables separable, this just breaks up into two integrals as follows

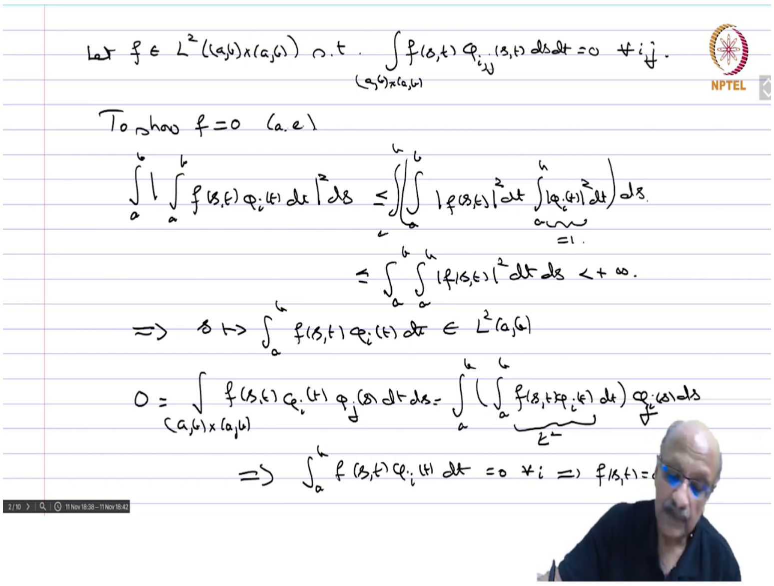
$$\int_{(a,b)\times(a,b)} |\phi_{n,m}|^2 ds\, dt = \int_a^b |\phi_n(s)|^2 ds \int_a^b |\phi_n(t)|^2 dt = 1$$

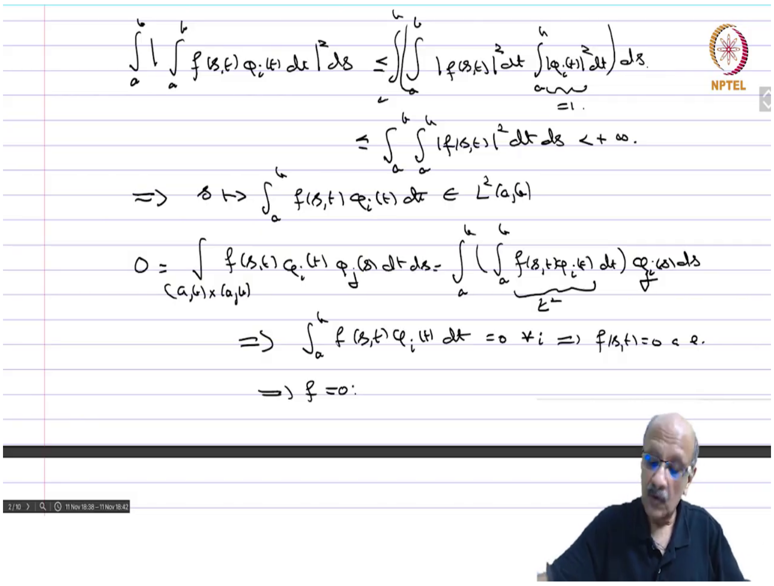
Now, let us consider the orthogonality,

$$\int_{(a,b)\times(a,b)} \phi_{i,j}\, \phi_{k,l}\, ds\, dt = \int_a^b \phi_i(s)\, \phi_k(s) ds \int_a^b \phi_j(t)\, \phi_l(t) dt = \delta_{ik}\delta_{jl},$$

where $\delta_{pq} = 1,\ if\ p = q$ and $\delta_{pq} = 0,\ if\ p \neq q$. δ_{pq} is kronecker symbol.

Therefore, unless $i = j$, $k = l$ this will always be 0. Even if one of the two indices is different this will be equal to 0 and therefore, this shows that $\{\phi_{i,j}\}_{i,j=1}^{\infty}$ is an orthonormal set.





So, now, we have to show that it is complete. Let $f \in L^2((a, b) \times (a, b))$ such that

$$\int_{(a,b)\times(a,b)} f(s,t)\,\phi_{i,j}(s,\,t)\,ds\,dt = 0 \quad \forall i,\,j.$$

Then we have to show $f = 0,\ a.\ e.$ Of course, that means it is 0 in the two sets. Let us take

$$\int_a^b |\int_a^b f(s,t)\,\phi_i(t)\,dt|^2\,ds \ \le \int_a^b(\int_a^b |f(s,t)|^2 dt \int_a^b |\phi_i(t)|^2 dt)ds$$ [by the Cauchy Schwarz inequality]

$$\leq \int_a^b \int_a^b |f(s,t)|^2 dt\, ds < +\infty. \text{ [} \int_a^b |\phi_i(t)|^2 dt = 1\text{, because it is an}$$

orthonormal sequence basis]

$$\Rightarrow s \mapsto \int_a^b f(s,t)\, \phi_i(t)\, dt \in L^2(a,b)$$

So, we have shown that the integral is an $L^2(a,b)$. So, now we have

$$0 = \int_{(a,b)\times(a,b)} f(s,t)\, \phi_i(t)\, \phi_j(s)\, dt\, ds = \int_a^b \left(\int_a^b f(s,t)\, \phi_i(t)\, dt\right) \phi_j(s)\ \ ds\ \forall i,\ j.$$

Now we can apply Fubini's because modulus is only integrable and therefore, there is no problem at all. Now, $\int_a^b f(s,t)\, \phi_i(t)\, dt$ is given to be an L^2 function and its inner product with every ϕ_j is 0. So, this implies that $\int_a^b f(s,t)\, \phi_i(t)\, dt = 0,\ \forall\, i$ and this implies that $f(s,t) = 0$ $a.\ e.$once again because $s \mapsto (s,t)\, f(s,t) \in L^2$. So, from statements (*) and (**) you get that that $f(s,t)$ has to be 0 almost everywhere. So, this implies that $f = 0$ and therefore, we have a complete orthonormal basis.

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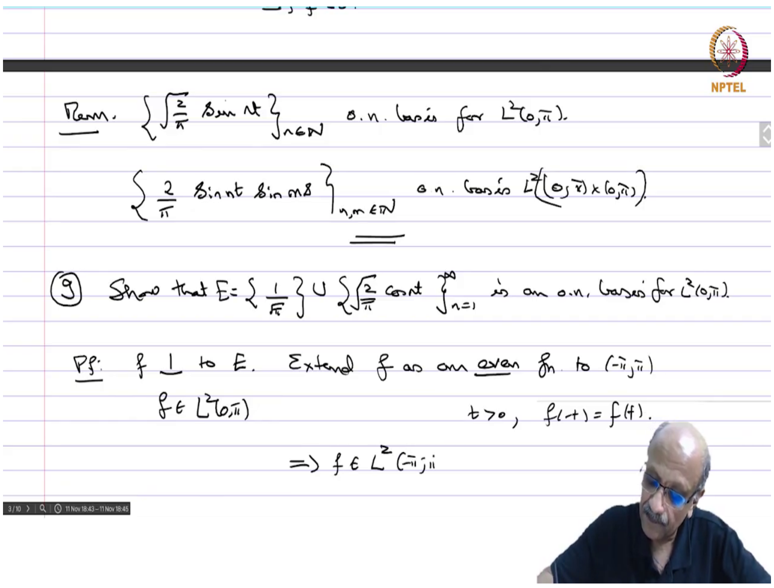

Remark we saw that $\{\sqrt{\frac{2}{\pi}}\ \sin\ nt\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(0,\ \pi)$. So, we have

$\{\frac{2}{\pi}\ \sin\ nt\ \sin\ ms\}_{n,m\in\mathbb{N}}$ is an orthonormal basis for $L^2((0,\ \pi)\ \times\ (0,\ \pi))$.

9. Show that $E\ = \left\{\frac{1}{\sqrt{\pi}}\right\} \cup \left\{\sqrt{\frac{2}{\pi}}\ \cos nt\right\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(0,\ \pi)$.

Proof. Take $f\ \perp\ E$ that means, f is orthogonal every member of the set E. Now, extend f as an even functions to $(-\ \pi,\ \pi)$. So, for $t\ >\ 0$, you have $f(-\ t)\ =\ f(t)$. So, that is what you define, then you have this extension $f\ \in\ L^2(-\ \pi,\ \pi)$.

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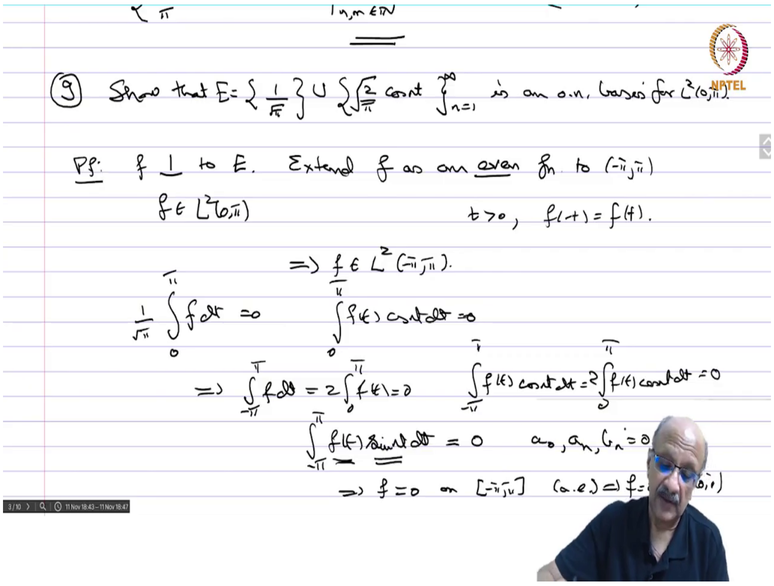

So, if you now take. So, you are given that $\frac{1}{\sqrt{\pi}}\int_{0}^{\pi} f\, dt = 0$ and $\int_{0}^{\pi} f(t) \cos nt\, dt = 0$, and this implies that $\int_{-\pi}^{\pi} f\, dt = 2\int_{0}^{\pi} f\, dt = 0$ and similarly,

$\int_{-\pi}^{\pi} f(t) \cos nt\, dt = 2\int_{0}^{\pi} f(t) \cos nt\, dt = 0$ because both are even functions.

$\int_{-\pi}^{\pi} f(t) \sin nt\, dt = 0$ as $f(t)$ is even and $sin\, nt$ is odd. So, the product is odd and therefore, the integral is 0. So, a_0, a_n, b_n are all 0 $\forall\, n$ and therefore, you have $f = 0$ on $[-\pi, \pi]$ almost everywhere, of course, this implies $f = 0$ on $[0, \pi]$ almost everywhere and therefore, f is 0 element of L^2, consequently, $E = \left\{\frac{1}{\sqrt{\pi}}\right\} \cup \left\{\sqrt{\frac{2}{\pi}} \cos nt\right\}_{n=1}^{\infty}$ is a complete orthonormal basis. So, you can write a Fourier series corresponding to this you can expand any L^2 function in $[0, \pi]$ in terms of the constant and the cosines and that is called the Fourier Cosine series. This test we had the Fourier sine series.

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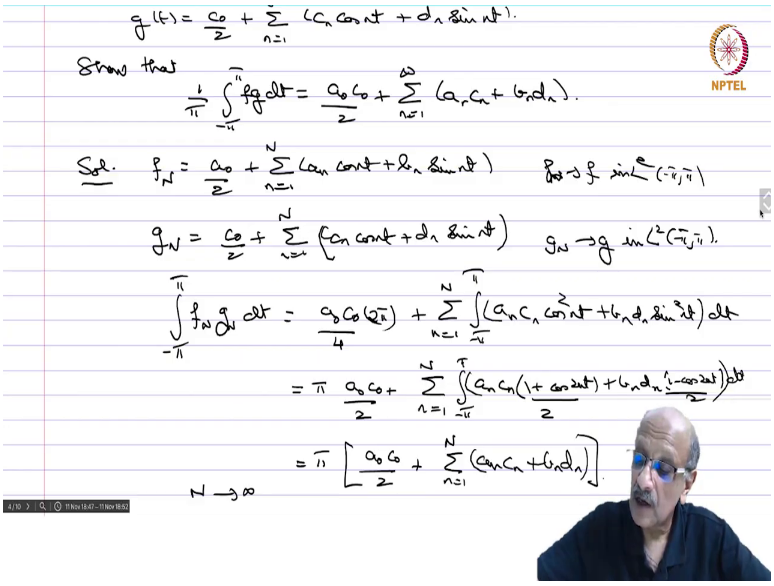

10. Let $f,\ g\ \in L^2(-\ \pi,\ \pi)$, and $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nt\ +\ b_n \sin nt\right)$ and

$g(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty}\left(c_n \cos nt\ +\ d_n \sin nt\right)$. Then show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f\, g\, dt \ = \ \frac{a_0 c_0}{2} + \sum_{n=1}^{N}(a_n c_n + b_n d_n).$$

Solution Take $f_N = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n \cos nt\ +\ b_n \sin nt\right),\ \ f_N \to f\ \ in\ \ L^2(-\ \pi,\ \pi).$

$$g_N = \frac{c_0}{2} + \sum_{n=1}^{N}\left(c_n \cos nt\ +\ d_n \sin nt\right),\ \ g_N \to g\ \ in\ \ L^2(-\ \pi,\ \pi).$$

$$\int_{-\pi}^{\pi} f_n g_n\, dt \ = \ \frac{a_0 c_0}{4}\, 2\pi \ + \ \sum_{n=1}^{N}\int_{-\pi}^{\pi} (a_n c_n \ \cos^2 nt \ + b_n d_n \sin^2 nt\,)\, dt$$

$$\left[\ \int_{-\pi}^{\pi} \cos nt\ \sin nt \ = 0\right]$$

$$= \ \frac{a_0 c_0}{2}\,\pi \ + \ \sum_{n=1}^{N}\int_{-\pi}^{\pi} \left(a_n c_n \left(\frac{1+\cos 2\,nt}{2}\right) + b_n d_n \left(\frac{1-\cos 2\,nt}{2}\right)\right) dt$$

$$= \pi \left[\frac{a_0 c_0}{2} + \sum_{n=1}^{N} (a_n c_n + b_n d_n) \right]$$

And now, you let $N \to \infty$. So, on both sides here you will get

$\int_{-\pi}^{\pi} f\, g\, dt = \pi \left[\frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} (a_n c_n + b_n d_n) \right]$. So, that will prove the theorem that will give you the solution.

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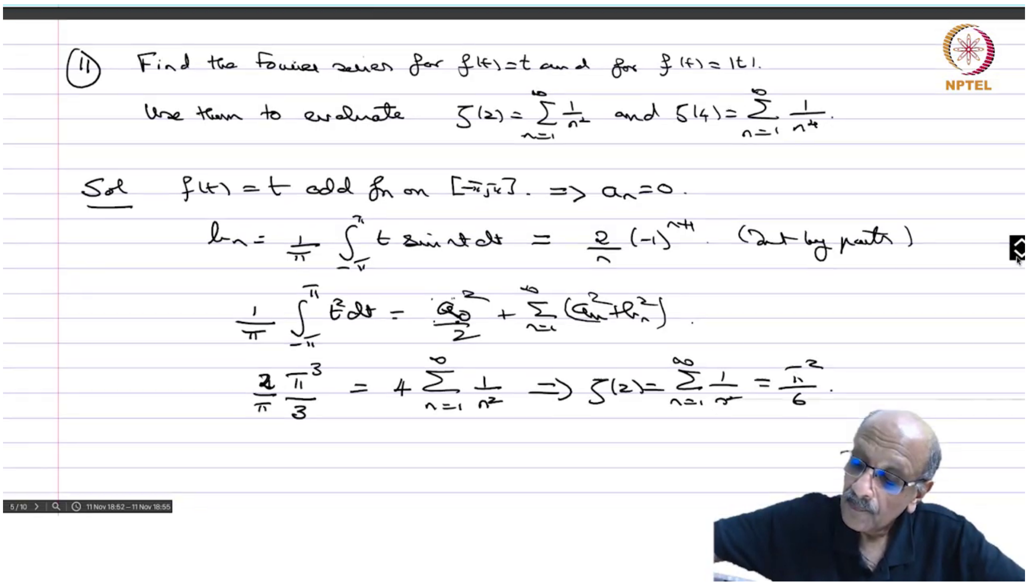

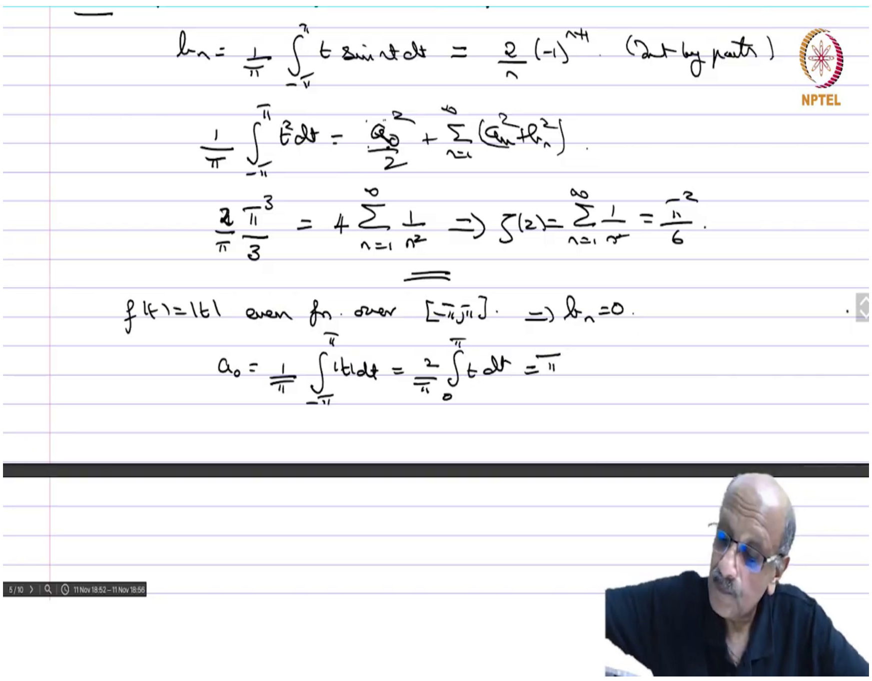

11. so find the Fourier series for $f(t) = t$ and for $f(t) = |t|$. Use them to evaluate

$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution. So, $f(t) = t$ is an odd function on $[-\pi, \pi]$.

So, this means that all the $a_n = 0$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin nt\, dt$

and you do an integration by parts, which will give you $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin nt\, dt = \frac{2}{n}(-1)^{n+1}$.

Now, $\frac{1}{\pi}\int_{-\pi}^{\pi} t^2\, dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \Rightarrow \frac{2}{\pi}\frac{\pi^3}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \zeta(2) = \frac{\pi^2}{6}$.

Now, $f(t) = |t|$ is an even function on $[-\pi, \pi]$. and therefore $b_n = 0$. Therefore,

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\, dt = \frac{2}{\pi}\int_{0}^{\pi} t\, dt = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$.

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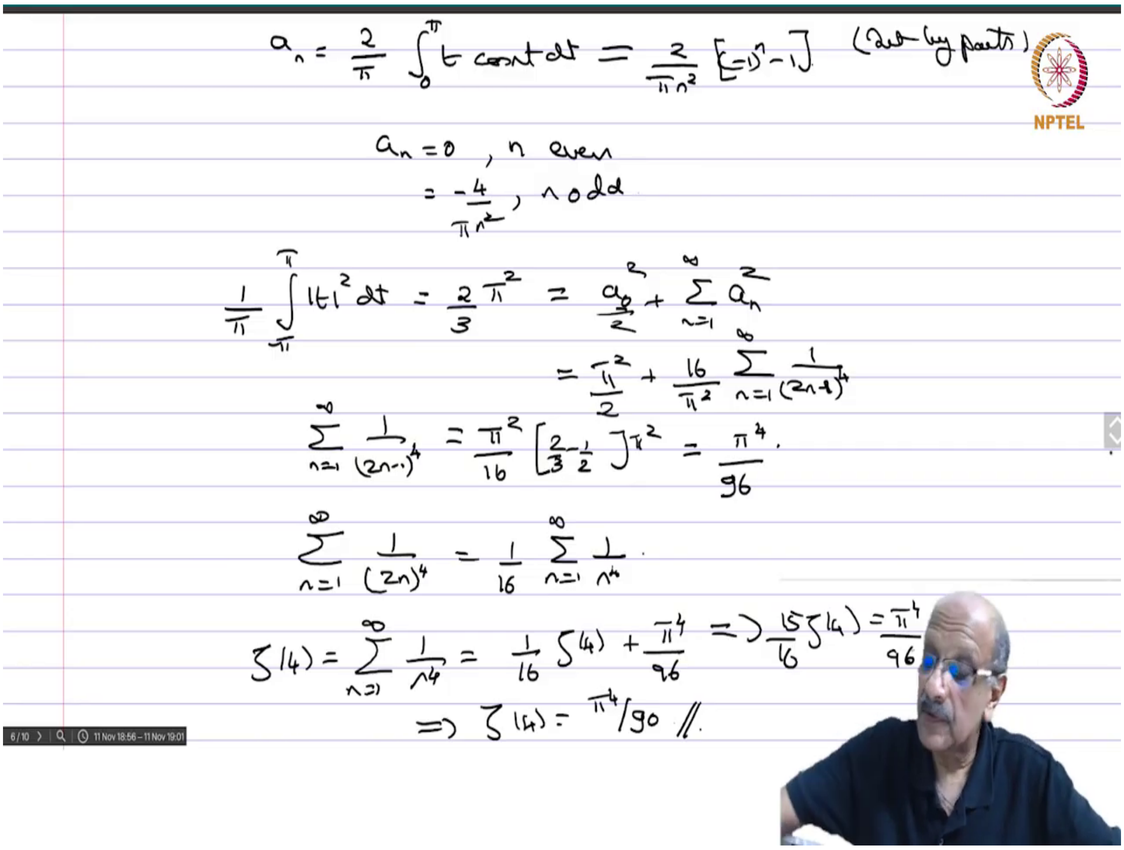

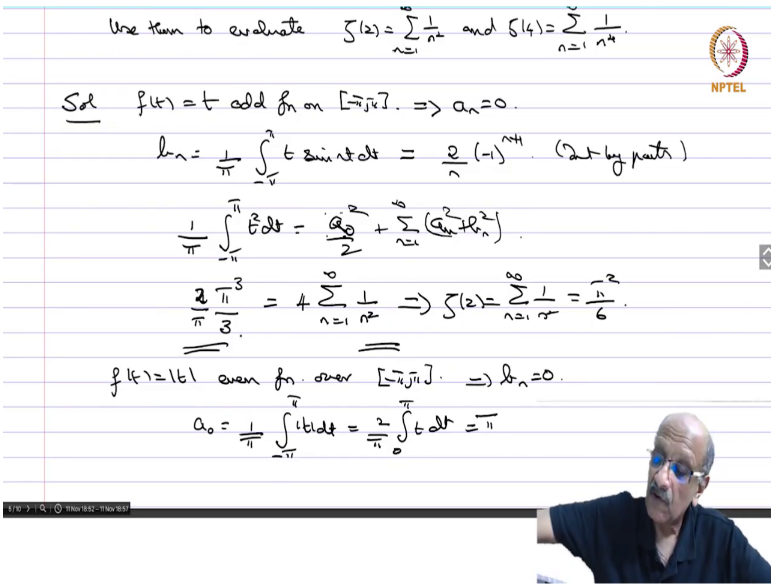

$a_n = \int_0^{\pi} t\cos nt\,dt = \frac{2}{\pi n^2}[(-1)^n - 1]$ (integration by parts). So, you can check this calculation. This is a routine calculation. So, you have $a_n = 0$, if n is even

$a_n = -\frac{4}{n\pi^2}$, if n is odd.

So, only the odd a_n's are surviving all the even a_n's are 0.

So, $\frac{1}{\pi}\int_{-\pi}^{\pi} |t|^2\, dt = \frac{2}{3}\pi^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$

So, $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^2}{16}\left[\frac{2}{3} - \frac{1}{2}\right]\pi^2 = \frac{\pi^4}{96}$.

Now, $\sum_{n=1}^{\infty} \frac{1}{(2n)^4} = \frac{1}{16}\sum_{n=1}^{\infty} \frac{1}{n^4}$.

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{16}\zeta(4) + \frac{\pi^4}{96} \Rightarrow \frac{15}{16}\zeta(4) = \frac{\pi^4}{96} \Rightarrow \zeta(4) = \frac{\pi^4}{90} .$$

So, that is how you compute. In fact, all the even powers can be computed. All the even zeta values can be computed and you know the Riemann zeta function is very important.

And it is not known for the odd powers we do not know very much. In fact, the big result is, was proved sometime in the 1990s that $\zeta(3)$ is a rational number. So, this is was proved, but we do not know anything about $\zeta(n)$ for n odd, the even once can be all computed in using Fourier series in terms of what are called the Bernoulli numbers. So, we will continue the exercises again.