Functional Analysis
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Lecture No. 60
Exercises - Part 1

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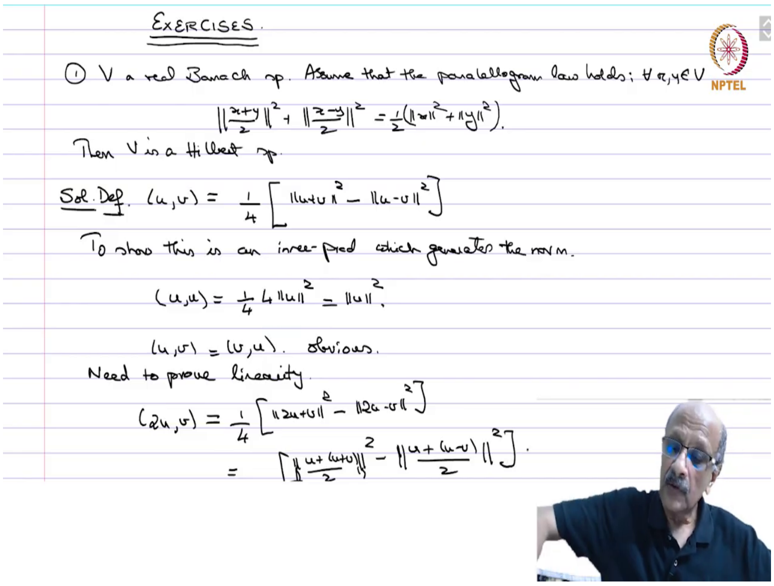

Exercises.

Problem 1. V a real Banach space and assume that the parallelogram law holds, that means $\forall x,\ y\ \in V$, we have $\left|\left|\frac{x+y}{2}\right|\right|^2 + \left|\left|\frac{x-y}{2}\right|\right|^2 = \frac{1}{2}(||x||^2 + ||y||^2)$. Then V is a Hilbert space. So, this is the real version of Fréchet–von Neumann–Jordan theorem, which says that if you have the parallelogram law, then the norm must come from the inner product.

Solution. So, we have to define the inner product. So, we worked backwards and tried to get a formula for the inner product in terms of the norm. Define $(u,\ v) = \frac{1}{4}(||u + v||^2 - ||u - v||^2)$.

We have to check this is an inner product which generates the norm. This is very easy because if you take $(u,\ u) = \frac{1}{4}(4||u||^2 - 0) = ||u||^2$, so it certainly generates the norm, there is no problem with that. And also, it is easy to see that $(u,\ v) = (v,\ u)$, that is obvious from the definition. So, we need to prove linearity. We do that in two stages, all several stages. Let us take first $(2u,\ v)\ = \frac{1}{4}(||2\,u\ +\ v||^2\ -\ ||2\,u\ -\ v||^2)$.

$$(2u,\ v)\ = \frac{1}{4}(||2\,u\ +\ v||^2\ -\ ||2\,u\ -\ v||^2)\ =\ \left(\left\|\frac{u+(u+v)}{2}\right\|^2 - \left\|\frac{u+(u-v)}{2}\right\|^2\right)$$

Now, we are in a position to use the parallelogram law.

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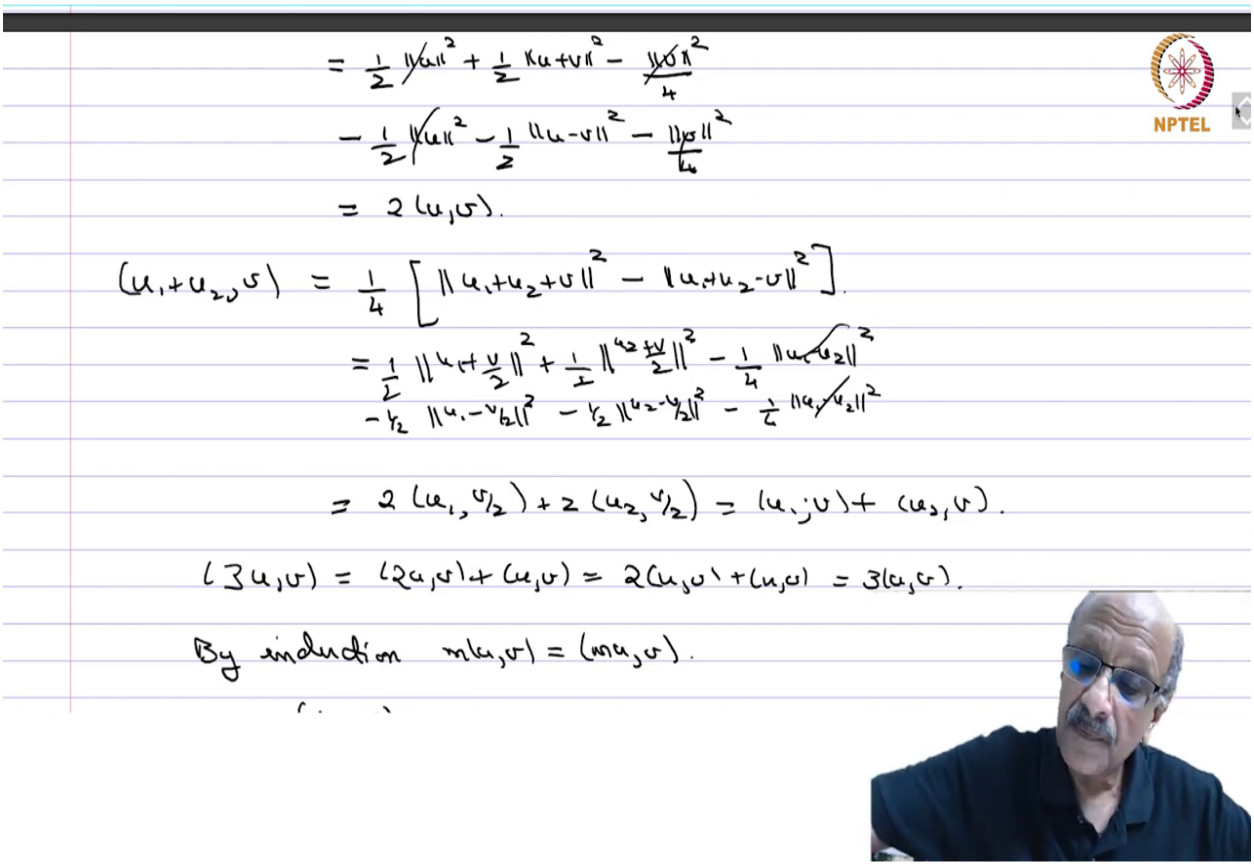

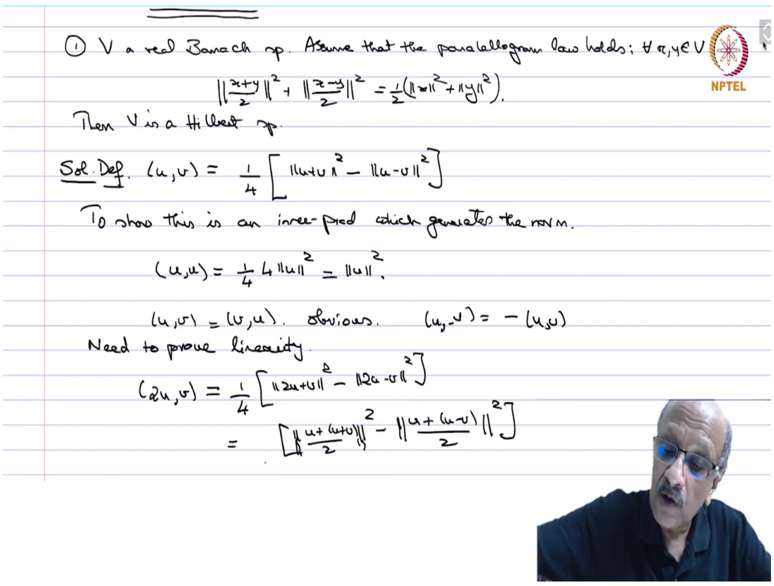

Therefore, $(2u,\ v)\ =\ \frac{1}{2}||u||^2 + \frac{1}{2}||u\ +\ v||^2 - \frac{||v||^2}{4} - \frac{1}{2}||u||^2 - \frac{1}{2}||u\ -\ v||^2 + \frac{||v||^2}{4}$

$$= \frac{1}{2}||u\ +\ v||^2 - \frac{1}{2}||u\ -\ v||^2 = 2(u,\ v)$$

So, $(2u,\ v)\ =\ 2(u,\ v)$. That is one relationship.

$$(u_1 + u_2, v) = \frac{1}{4}(||u_1 + u_2 + v||^2 - ||u_1 + u_2 - v||^2)$$

$$= \left(||(u_1 + \frac{v}{2}) + (u_2 + \frac{v}{2})||^2 - ||(u_1 - \frac{v}{2}) + (u_2 - \frac{v}{2})||^2\right)$$

$$= \frac{1}{2}||u_1 + \frac{v}{2}||^2 + \frac{1}{2}||u_2 + \frac{v}{2}||^2 - \frac{1}{4}||u_1 - u_2||^2$$

$$- \frac{1}{2}||u_1 - \frac{v}{2}||^2 + \frac{1}{2}||u_2 - \frac{v}{2}||^2 + \frac{1}{4}||u_1 - u_2||^2$$

$$= 2\left(u_1, \frac{v}{2}\right) + 2\left(u_2, \frac{v}{2}\right) = (u_1, v) + (u_2, v)$$

Now, we have to show that $(\alpha u, v) = \alpha(u, v)$.

We have shown it for $\alpha = 2$. Now, if you have $(3u, v) = (2u, v) + (u, v) = 3(u, v)$. So, now, if you take any integer the m, so by induction $(mu, v) = m(u, v)$. And $(u, -v) =- (u, v)$, from the definition. These two implies $(mu, v) = m(u, v)\ \forall m \in \mathbb{Z}$.

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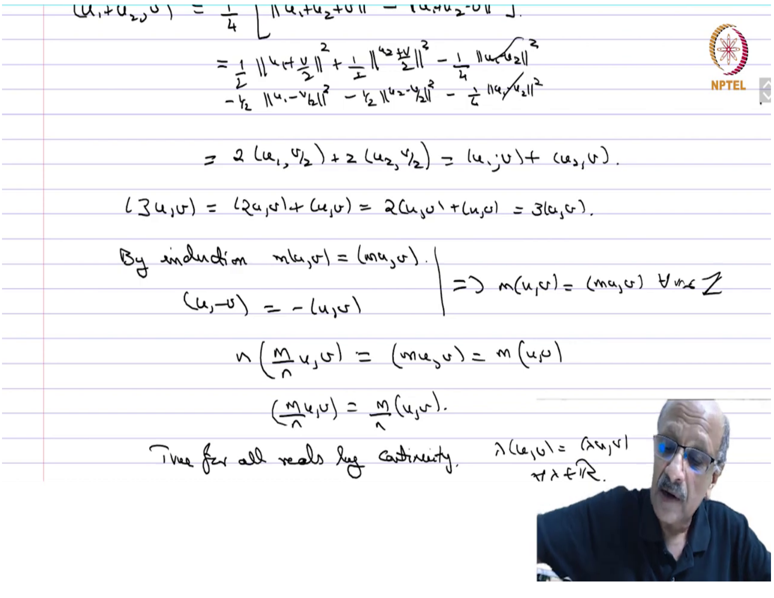

Then $n(\frac{m}{n}u, v) = (mu, v) = m(u, v)$, so $(\frac{m}{n}u, v) = \frac{m}{n}(u, v)$. So, it is true for all rationales. Therefore, true for all reals by continuity. The inner product which we defined is also continuous and therefore, if you take any real number approximated by rational numbers and therefore, you have $(\alpha u, v) = \alpha(u, v)\ \forall\alpha \in \mathbb{R}$. So, we have that

$(u_1 + u_2, v) = (u_1, v) + (u_2, v)$, (u, v) is symmetric and $\forall\lambda \in \mathbb{R}\quad (\lambda u, v) = \lambda(u, v)$. Therefore, the norm is generated by an inner product and therefore, we have that it becomes a Hilbert space.

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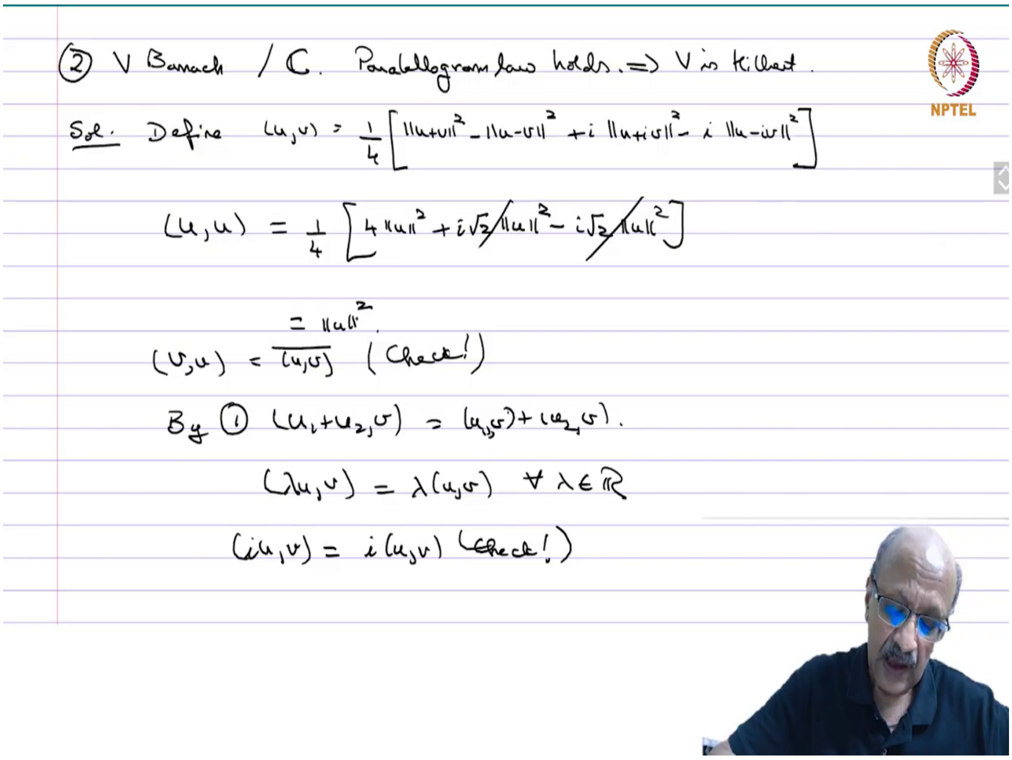

Problem 2. Now, in the complex case, V Banach over $\mathbb{C}$. Parallelogram law holds $\Rightarrow V$ is Hilbert. That means the norm comes from an inner product.

Solution. Now, you define the inner product in the following way.

$$(u, v) = \frac{1}{4}\left(||u + v||^2 - ||u - v||^2 + i\,||u + i\,v||^2 - i\,||u - i\,v||^2\right).$$

It is very similar to the real one. So, if you think $||u + v||^2 - ||u - v||^2$ is the real part, the imaginary part also looks very much like the real part. If you now take

$(u, u) = \frac{1}{4}\left(4||u||^2 - 0 + i\sqrt{2}\,||u||^2 - i\sqrt{2}||u||^2\right) = ||u||^2$. Now, check $(v, u) = \overline{(u, v)}$, it is just a straightforward computation, you just have to pull out i and so on and there is no such thing. Now by 1, you have $(u_1 + u_2, v) = (u_1, v) + (u_2, v)$, i.e. linearity. So, now, also you have that $\forall\lambda \in \mathbb{R} \quad (\lambda\, u,\ v) = \lambda(u,\ v)$, because again this inner product looks exactly like the previous case and therefore, we can repeat all those arguments and then you see.

And now, you just take $(i\, u,\ v) = i(u,\ v)$ Check.

These are just straightforward algebraic manipulations and therefore, you have $\forall\lambda \in \mathbb{C} \quad (\lambda\, u,\ v) = \lambda(u,\ v)$, and therefore, you have linearity in the first variables and you have conjugacy and you have that the norm is produced and therefore, this becomes a Hilbert space.

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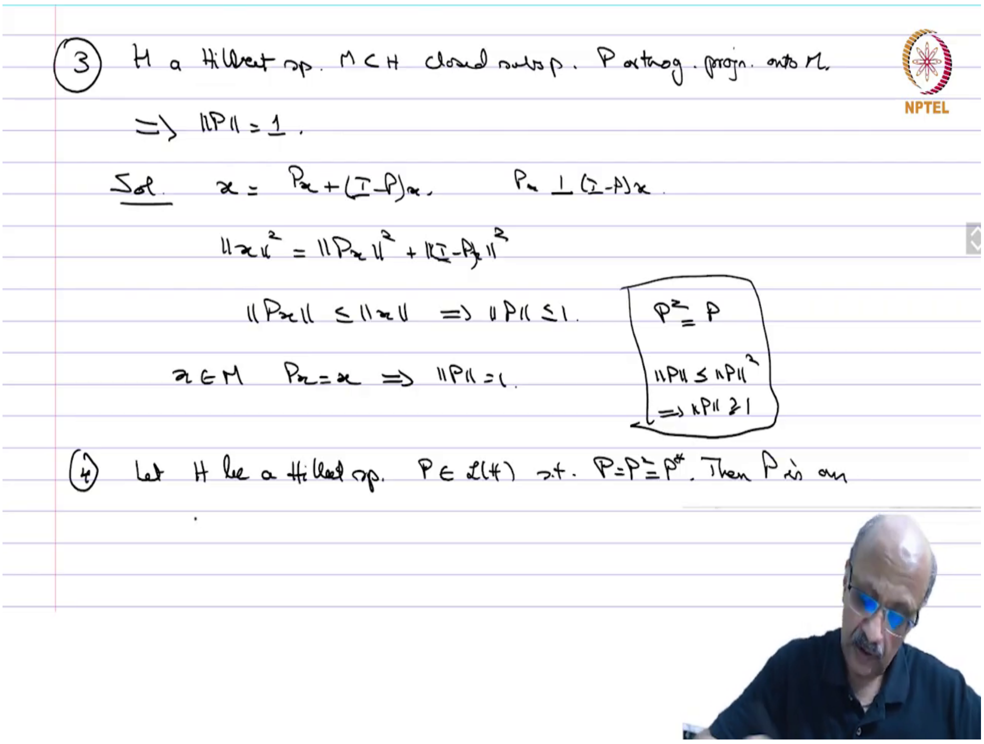

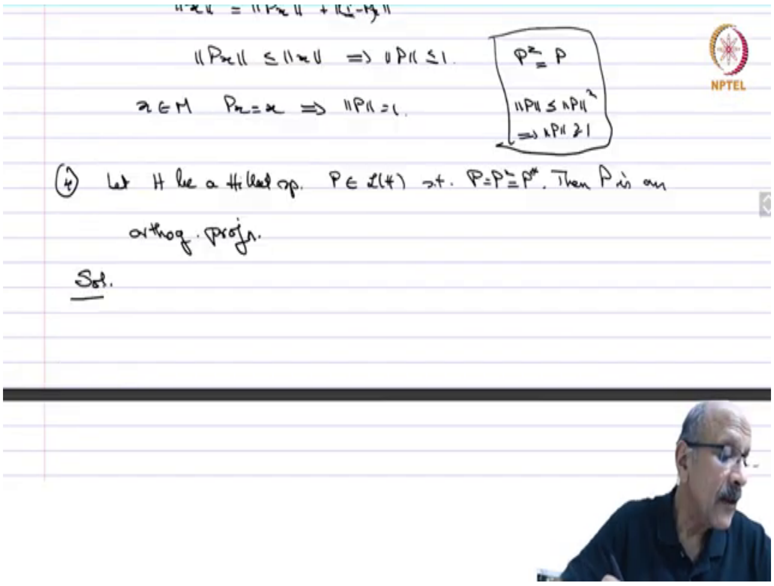

Problem 3. H a Hilbert space, $M \subset H$ is closed subspace, and P orthogonal projection onto M, then this implies that $||P|| = 1$.

Solution. P is orthogonal projection. So, $x = P\,x + (I - P)\,x$. And you have the $P\,x \perp (I - P)\,x$ because it is an orthogonal projection. And therefore, $||x||^2 = ||P\,x||^2 + ||(I - P)\,x||^2$, the cross term will not exist.

Just in other words, a version of the Pythagoras theorem. You have $||P\,x|| \le ||x||$ and therefore $||P|| \le 1$. Now you can do the other thing, if $x \in M$, then you have $P\,x = x$. And therefore, this implies $||P\,|| = 1$. Another way is to show that since it is a projection, you have $P^2 = P$ for $||P\,|| \le ||P\,||^2 \Rightarrow ||P\,|| \ge 1$, therefore you have $||P\,|| = 1$. So, you can do it in either of these two ways.

Problem 4. This I already mentioned in the class, but did not do it fully, I asked you to verify it. Let H be a Hilbert space and $P \in L(H)$ such that $P = P^2 = P^*$, then P is an orthogonal projection.

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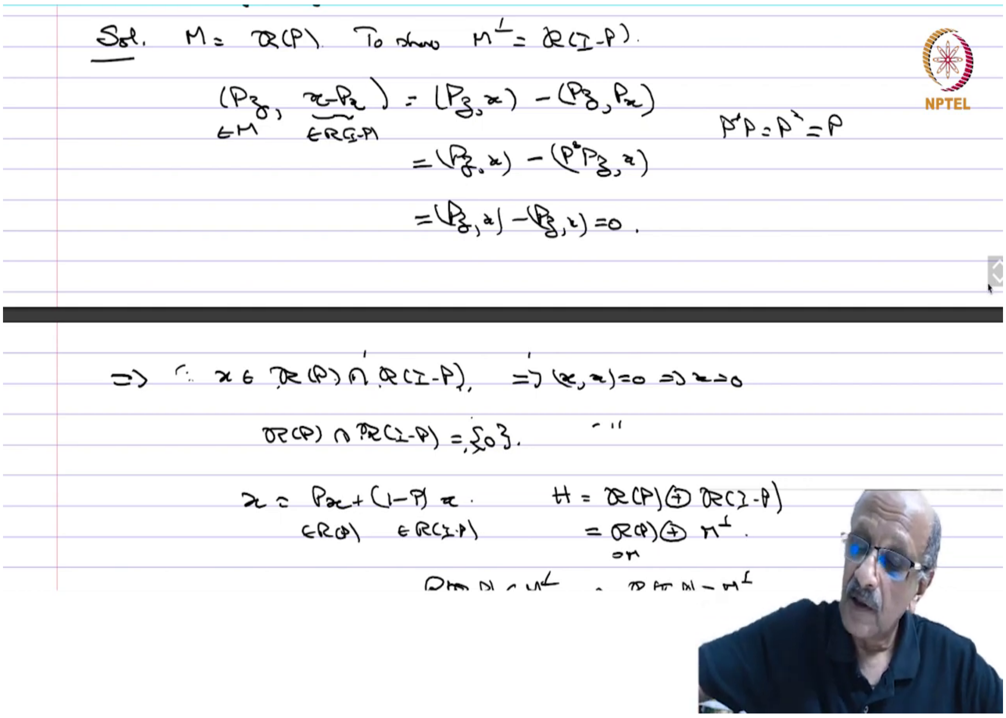

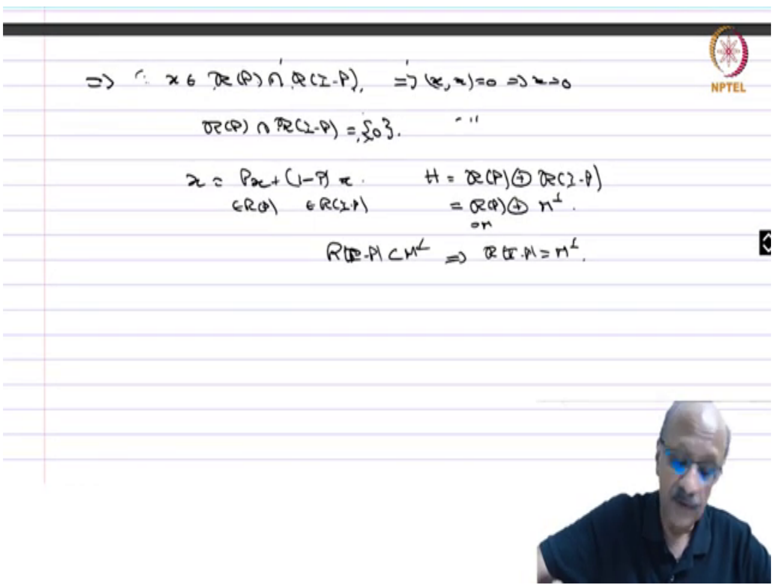

Solution. $M = R(P)$ and to show $M^{\perp} = R(I - P)$. So, that will show that it is an orthogonal projection. Let us take $P z \in M$ & $x - P x \in R(I - P)$, and see what the inner product is.

$$(P z,\ x - P x) = (P z,\ x) - (P z,\ P x) = (P z,\ x) - (P^{*} P z,\ x)$$

$$= (P z,\ x) - (P z,\ x) = 0$$

$\Rightarrow R(I - P) \subseteq M^{\perp}$. Now, you can complete the proof in many ways. Let me do one of them. We have to show that $M^{\perp} = R(I - P)$, so let me take the Hahn–Banach method. Let

$y \in M^{\perp}$, such that $(y, z - Pz) = 0, \ \forall z \in H$. That means it vanishes on $R(I - P)$, then I want to show that it vanishes everywhere and therefore, I think we will not do this because that only shows it is dense, but it will not show that it is in fact equal. Let us do another way.

So, we have that if $x \in R(P) \cap R(I - P)$. Now, $R(I - P)$ is orthogonal to $R(P)$ and therefore $(x, x) = 0 \Rightarrow x = 0$. So, $R(P) \cap R(I - P) = \{0\}$. And then any x can be written as $x = P x + (I - P)x$, where $Px \in R(P)$ and $(I - P)x \in R(I - P)$.

And therefore, you have $H = R(P) \oplus R(I - P) = R(P) \oplus M^{\perp}$, $i.e.$ $R(I - P) \subset M^{\perp}$.

Therefore this implies $R(I - P) = M^{\perp}$. So, that completes the proof.

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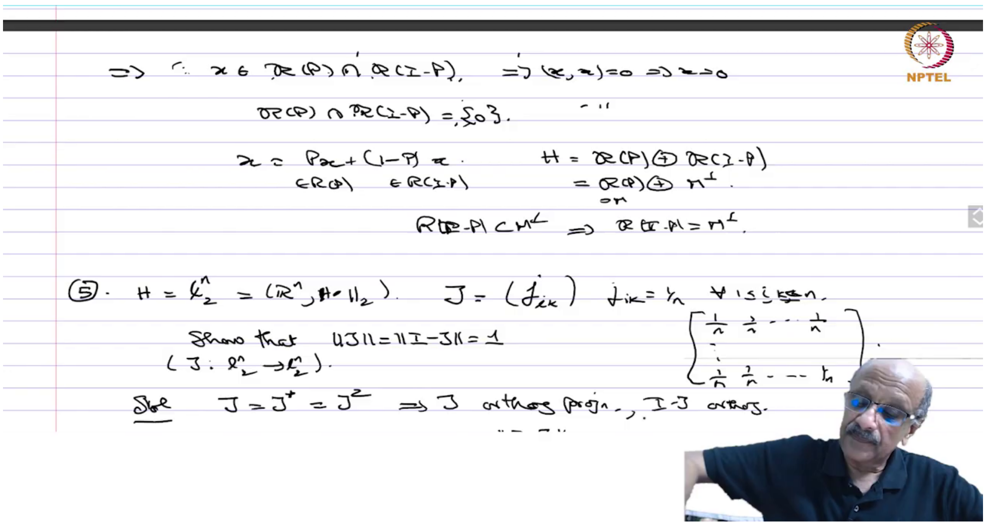

Problem 5. Let us take $H = l_2^n = (\mathbb{R}^n, ||.||_2)$. And then you look at the matrix $J = (j_{ik}),\ j_{ik} = \frac{1}{n}, \forall 1 \le i \le k \le n$. So, there is a matrix with all the same entries everywhere. Show that $||J|| = ||I - J|| = 1$. ($J : l_2^n \to l_2^n$ is a linear mapping and therefore, it has a norm.)

Solution. By inspection, we have $J = J^* = J^2$. So, if you multiply J by J, once again you will get J. So, this implies J is an orthogonal projection. Therefore $||J|| = 1$, and similarly $I - J$

is also an orthogonal projection. Because if P is an orthogonal projection, $I - P$ is an orthogonal projection and therefore $||\ J\ || = 1 = ||I - J||$, so that is simply done.

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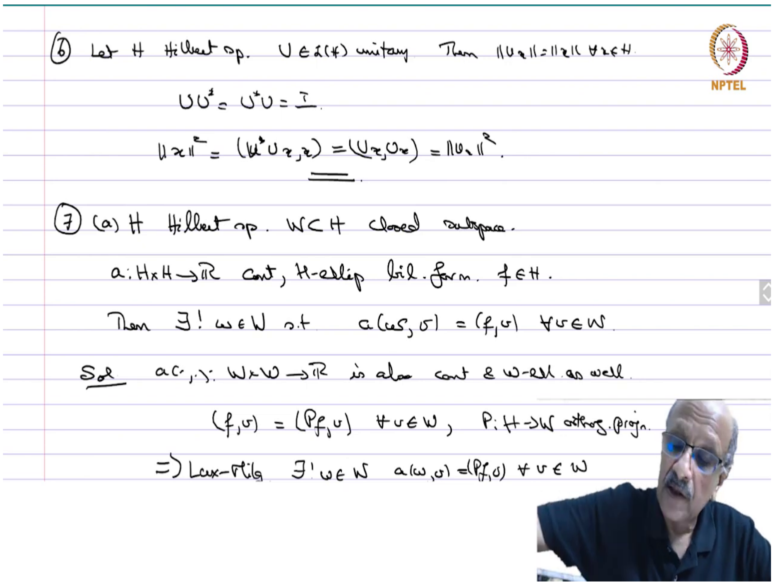


Problem 6. Let H be a Hilbert space and $U \in L(H)$ unitary. Then $||U\ x|| = ||x||, \forall\ x \in H$. That is $U$is an isometry we have already seen. You have $U\ U^{*} = U^{*}U\ = I$. And therefore, we have

$||x||^{2} = (U^{*}U\ x,\ x) = (U\ x,\ U\ x) = ||U\ x||^{2}$ and therefore, you get this. So, this we have already seen in the previous lecture, where we computed the spectrum of a unitary operator.

Problem 7. (a) H Hilbert space $W \subset H$ closed subspace, a: $H \times H \rightarrow \mathbb{R}$ continuous, H-elliptic bilinear form, and $f \in H$. Then there exists a unique $w \in W$ such that $a(w,\ v) = (f,\ v)\ \ \forall\ v \in W$. So, we prove this, I mentioned this as a remark, we prove the Lax Milgram lemma for the whole of H. And now, we are saying that is true even in a, so solution.

Solution. a: $W \times W \rightarrow \mathbb{R}$ is also continuous because it is continuous on H and w-elliptic as well, [with H-elliptic it is also w-elliptic because W is only a subspace]. Now, if you take

$(f,\ v) = (P\ f,\ v)\ \forall v \in W$, where $P: H \rightarrow W$ is the orthogonal projection. So, by Lax-Milgram because W is a Hilbert space on its own, there exists a unique $w \in W$ such that $a(w,\ v) = (P\ f,\ v)\ \ \forall\ v \in W$ and $(P\ f,\ v) = (f,\ v)$. And therefore, we have shown that there exists a unique solution.(Refer Slide Time: 24:32)

$\Rightarrow$ Lax-Milg. $\exists!\, w \in W \quad a(w,v) = (f,v) \ \forall v \in W$

$= (f,v)$

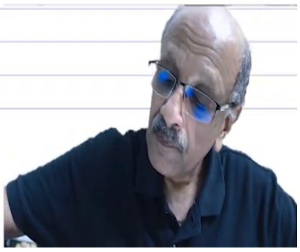
(b) $\|w\| \leq \frac{\|f\|}{\alpha}$.

$$\alpha \|w\|^2 \leq a(w,w) = (f,w) \leq \|f\|\,\|w\| \Rightarrow \|w\| \leq \frac{\|f\|}{\alpha}.$$

(c) Let $u \in H$ be the unique soln. of

$$a(u,v) = (f,v) \quad \forall v \in H.$$

Then $\|u-w\| \leq \frac{M}{\alpha} \inf_{v \in W} \|u-v\|$.



$$\alpha \|w\|^2 \leq a(w,w) = (f,w) \leq \|f\|\,\|w\| \Rightarrow \|w\| \leq \frac{\|f\|}{\alpha}.$$

(c) Let $u \in H$ be the unique soln. of

$$a(u,v) = (f,v) \quad \forall v \in H.$$

Then $\|u-w\| \leq \frac{M}{\alpha} \inf_{v \in W} \|u-v\|$.

$a(u, v-w) = (f, v-w)$

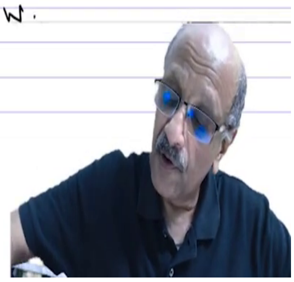
$a(w, v-w) = (f, v-w)$

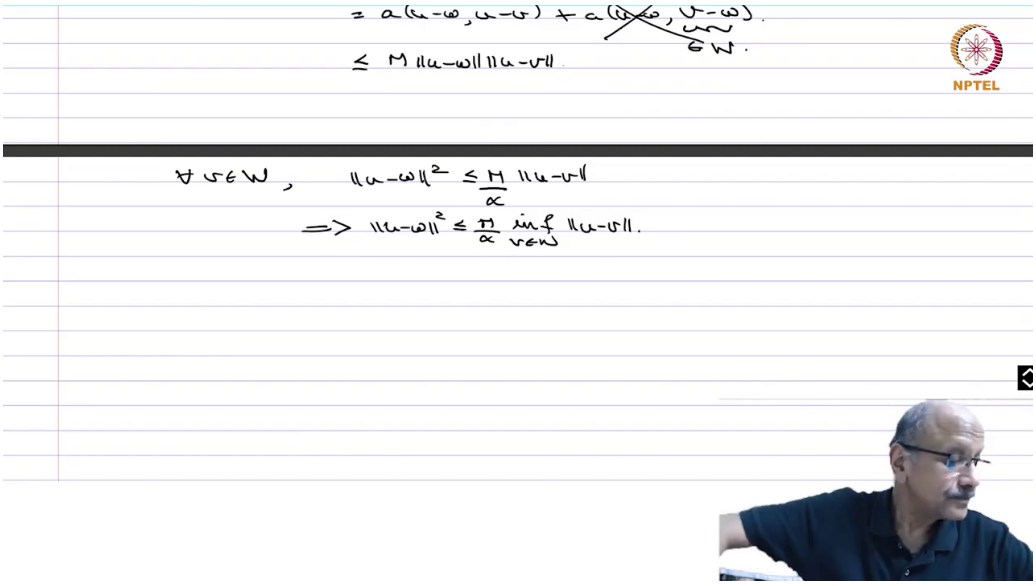
Sol $\alpha \|u-w\|^2 \leq a(u-w, u-w)$

$$= a(u-w, u-v+v-w) \quad \forall v \in W$$

$$= a(u-w, u-v) + \cancel{a(u-w, \underbrace{v-w}_{\in W})}.$$

$$\leq M \|u-w\|\,\|u-v\|$$





(b) $||w|| \leq \frac{||f||}{\alpha}$. $\alpha\, ||w||^2 \leq\ a(w,\ w)$ that is the ellipticity condition and $a(w,\ w)\ =\ (f,\ w)$ because it solves the equation. By Cauchy Schwarz, $(f,\ w)\ \leq\ ||f||\ ||w||$. Therefore, this shows that $||w||\ \leq\ \frac{||f||}{\alpha}$. See that this estimate is independent of the subspace. So, whatever may be the subspace whether you are solving this problem, the solution has the same estimate for the norm.

(c) Let $u\ \in\ H$ be the unique solution of $a(u,\ v)\ =\ (f,\ v)\quad \forall\, v\ \in\ H$. So, you take the full problem of the entire space, then $||u\ -\ w||^2 \leq \frac{M}{\alpha}\ \inf_{v\,\in W}||u\ -\ v||$.

Solution. $\alpha||u\ -\ w||^2 \leq a(u\ -\ w,\ u\ -\ w)\ =\ a(u\ -\ w,\ u\ -\ v\ +\ v\ -\ w)\ \ \forall v\ \in\ W$.

$$=\ a\,(u\ -\ w,\ u\ -\ v)\ +\ a(u\ -\ w,\ v\ -\ w)$$

Now, $a(u,\ v\ -\ w)\ =\ (f,\ v\ -\ w)$ because u is the solution of the whole space and $a(w,\ v\ -\ w)\ =\ (f,\ v\ -\ w)$ because it is a solution in the subspace. So, $a(u\ -\ w,\ v\ -\ w)\ =\ 0$ and $a\,(u\ -\ w,\ u\ -\ v)\ \leq\ M\,||u\ -\ w||\ ||u\ -\ v||$. Therefore, for every $v\ \in\ W$, you have $||u\ -\ w||^2 \leq \frac{M}{\alpha}||u\ -\ v||$. And therefore, I can take, this is true for all $v\ \in\ W$. So, this implies $||u\ -\ w||^2 \leq \frac{M}{\alpha}\ \inf_{v\,\in W}||u\ -\ v||$.

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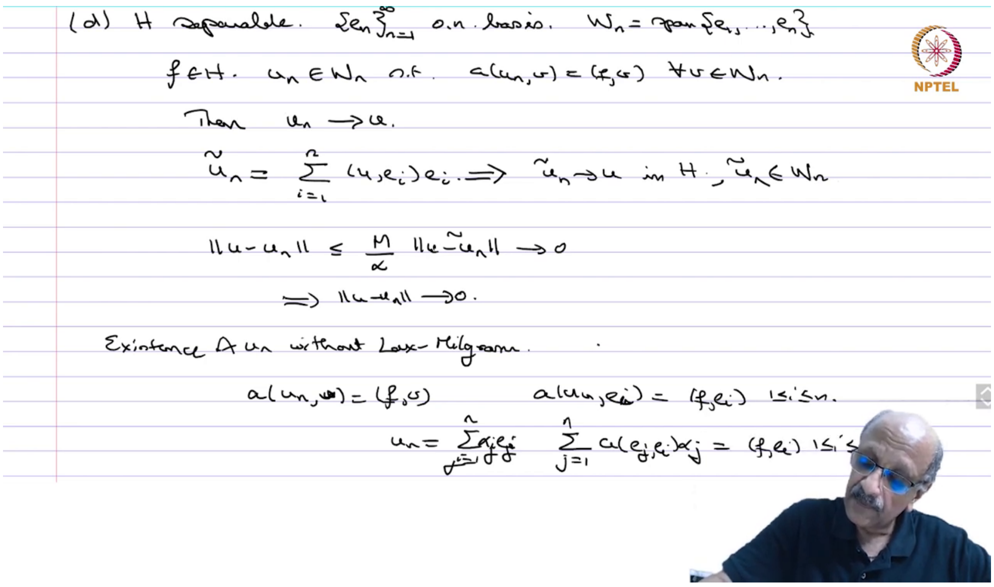

(d) H separable and $\{e_n\}_{n=1}^{\infty}$ orthonormal basis. $W_n = span\{e_1, \ldots, e_n\}$ is a finite dimension of closed subspace. $f \in H$. $u_n \in W_n$ such that $a(u_n, v) = (f, v)$, $\forall v \in W_n$.

Then $u_n \to u$, what is u? u is a solution in the whole space, $a(u, v) = (f, v)\ \forall v \in H$. So, this tells you that these spare solutions are approximations of the original solutions.

$\tilde{u}_n = \sum_{i=1}^{n} (u, e_i)\, e_i \Rightarrow \tilde{u}_n \to u$ in H, $\tilde{u}_n \in W_n$ this is what we know because that is u is nothing but the infinite series associated with this. Therefore,

$||u - u_n|| \le \frac{M}{\alpha} ||u - \tilde{u}_n|| \to 0 \Rightarrow ||u - u_n|| \to 0$. (by the previous problem $||u - u_n|| \le \frac{M}{\alpha} \inf ||u - v||$ for any v, I have taken $v = \tilde{v}_n$)

Remark. We can even throw the Lax-Milgram Lemma directly from this fact. So, we can prove the existence of u_n directly and therefore, you can prove Lax-Milgram Lemma also. How? So, if I want to find the existence of u_n without Lax-Milgram, i.e., we want to find u_n such that, $a\,(u_n, v) = (f, v)$. So, by linearity it is enough to check for the basis element.

So, you take $a\,(u_n, e_i) = (f, e_i),\ 1 \le i \le n$. Now, $u_n = \sum_{j=1}^{n} \alpha_j e_j$, substituting this we have $\sum_{j=1}^{n} a\,(e_j, e_i)\,\alpha_j = (f, e_i),\ 1 \le i \le n.$

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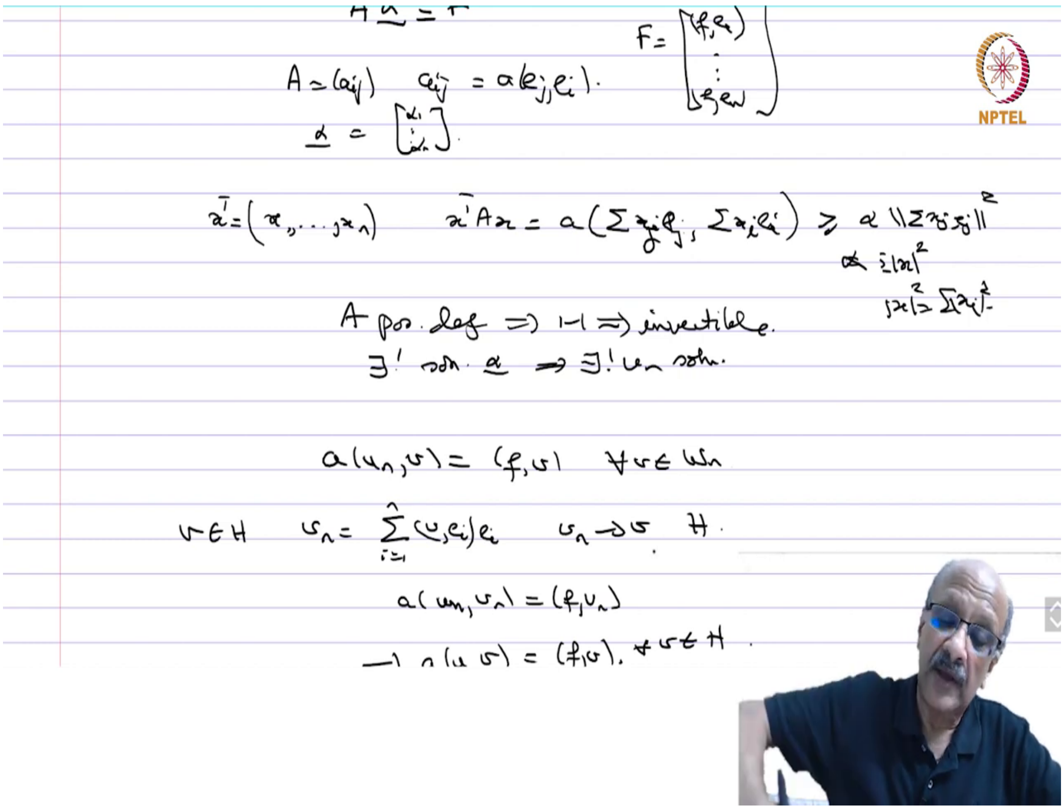

And therefore, you get a linear system, $A\,\alpha = F$, where $A = (a_{ij}) = a\,(e_i, e_j)$. α is unknown vector $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ and $F = ((f, e_1), \ldots, (f, e_n))^T$. So, if you solve this linear system, you will get α, so you will get u_n and that will prove. So, all you need to show is that matrix A is invertible. Take, $x = (x_1, \ldots, x_n)^T$. Then $x^T A\, x = a\left(\sum x_j e_j, \sum x_i e_i\right) \ge \alpha \left\|\sum x_j e_j\right\|^2 = \alpha\,|x|^2, \quad |x|^2 = \sum |x_i|^2$, since the e_j's are orthonormal. Therefore, A is positive definite and therefore implies 1-1 $\Rightarrow$ invertible and therefore, there exists a unique solution $\alpha \Rightarrow$ implies there exists a unique solution u_n. And now, because $u_n \to u$, you can also now prove the Lax-Milgram lemma, you have $a(u_n, v) = (f, v), \ \forall v \in W_n$. So, given any $v \in H$, you have $v_n = \sum_{i=1}^{n} (v, e_i)e_i$.

And therefore, $v_n \to v$ in H and therefore, you have $(a\, u_n, v_n) = (f, v_n)$. And now, pass to the limit because a is continuous being linear form, this $u_n \to u$ and $v_n \to v$ and therefore, this gives you f $[a(u, v) = (f, v)]$ and therefore, we have produced a solution for the original for every $v \in H$. So therefore, we have proved Lax-Milgram Lemma by this approximation process, at least for the separable Hilbert space. So, we will continue the exercises.