Functional Analysis
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Lecture 6
Isomorphism

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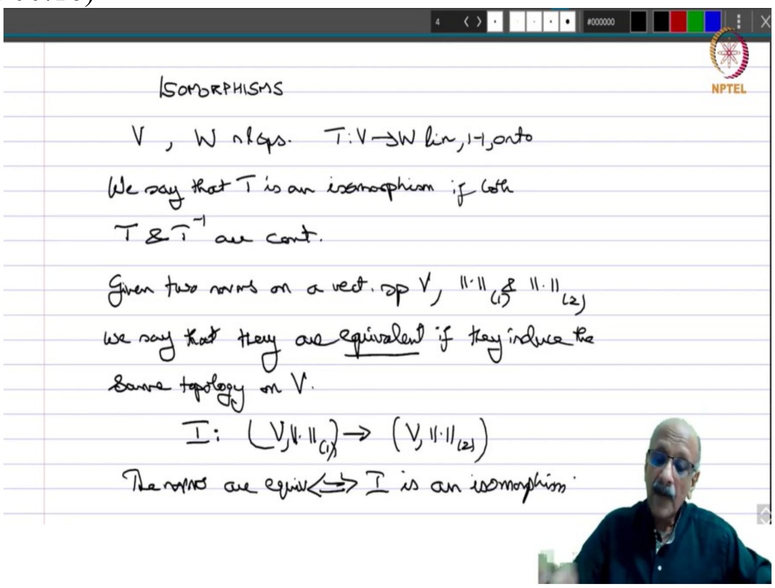

Isomorphism. We will now talk about isomorphisms. Let V and W be norm linear spaces and $T: V \mapsto W$ linear 1 - 1 and onto. In linear algebra, this would mean that it is an isomorphism between the two vector spaces. But, now we are in the question of norm linear spaces, and therefore we say that, T is a isomorphism if both T and T^{-1} inverse are continuous.

So, whenever we talk of isomorphism in the context of norm linear spaces, we mean a 1 -1, onto, linear map which is continuous and the inverse is also continuous. So given two norms $\|.\|_{(1)}, \|.\|_{(2)}$ on a vector space V (I am putting a bracket now, so that we do not confuse with the $\|.\|_1, \|.\|_2$ which we defined earlier). We say that $\|.\|_{(1)} \wedge \|.\|_{(2)}$ are equivalent if they induce the same topology on V. This means that, if I have V with $\|.\|_{(1)} \wedge \|.\|_{(2)}$, then I have two norm linear spaces. I look at the identity map between these two spaces i.e., $I:$¿. Notice that, the norms are equivalent if and only if I is an isomorphism.

Now, if I is an isomorphism, then we have that $\|.\|_{(2)} \leq k_1 \|.\|_{(1)}$ and also $\|.\|_{(1)} \leq k_2 \|.\|_{(2)}$ for all $x \in V$. Therefore, we can say for all $x \in V, C_1 \|.\|_{(1)} \leq \|.\|_{(2)} \leq C_2 \|.\|_{(1)}$ for some $C_1, C_2 > 0$.

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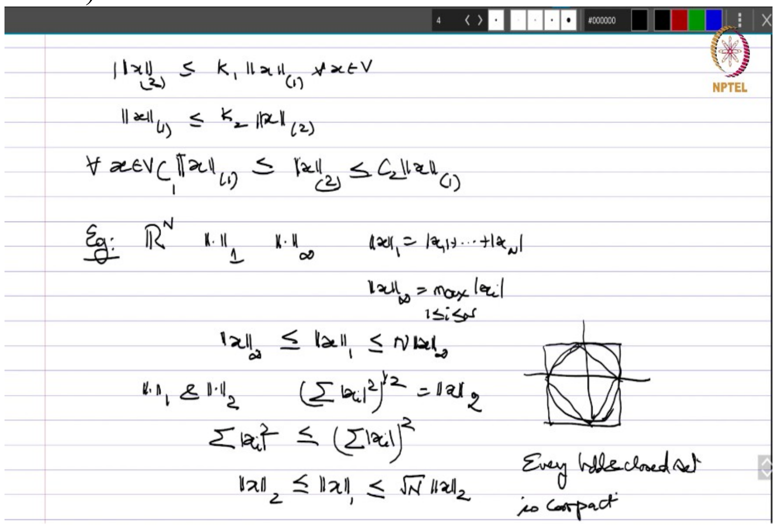

So, this is a necessary and sufficient condition for two norms to be equivalent which means that the topologies are the same, or in other words, the identity map is an isomorphism.

Let us look at some examples.

Example 1 Let us take R^N with the $\|.\|_1$ and $\|.\|_\infty$. What is $\|.\|_1$? $\|x\|_1 = |x_1| + \ldots + |x_N|$ and $\|x\|_\infty = \max_{i=1,\ldots,N} |x_i|$. Then, we have that $\|x\|_\infty \leq \|x\|_1$ and $\|x\|_1 \leq N\|x\|_\infty$ (as each $|x_i|$ is less than the maximum $\|x\|_\infty$). And therefore, these two norms are equivalent.

Now, let us look at $\|.\|_1$ and $\|.\|_2$. So, what is $\|.\|_2$? $\|x\|_2 = \left[\sum_{i=1,\ldots N} |x_i|^2\right]^{\frac{1}{2}}$. So, we have that $\sum_{i=1,\ldots N} |x_i|^2 \leq \left[\sum_{i=1,\ldots N} |x_i|\right]^2$ (because the cross terms are missing, only the squares are there). Therefore, from this it immediately follows that $\|x\|_2 \leq \|x\|_1$. If you apply the Cauchy–Schwarz inequality, then $\|x\|_1 \leq \sqrt{N}\|x\|_2$. So, these two norms are also equivalent. So, $\|x\|_1$, $\|x\|_2$, $\|x\|_\infty$ are all equivalent norms (because equivalence is of course a transitive condition).

What is the usual topology on R^N? In the usual topology, you take neighborhoods as balls centered at points and this is given by the $\|x\|_2$. So, $\|x\|_2$ gives you the usual topology and therefore $\|x\|_1$, $\|x\|_\infty$ also give the same topology. We know that because inside any ball with respect to one of these norms, you can put any other smaller ball with respect to the other norms at every point. And therefore, the topologies are the same.

So, we now saw that $\|x\|_1$, $\|x\|_\infty, \|x\|_2$ are all equivalent, in fact it turns out that, in a finite dimension and space, all norms are equivalent. So, this is a very beautiful theorem which we will now prove.

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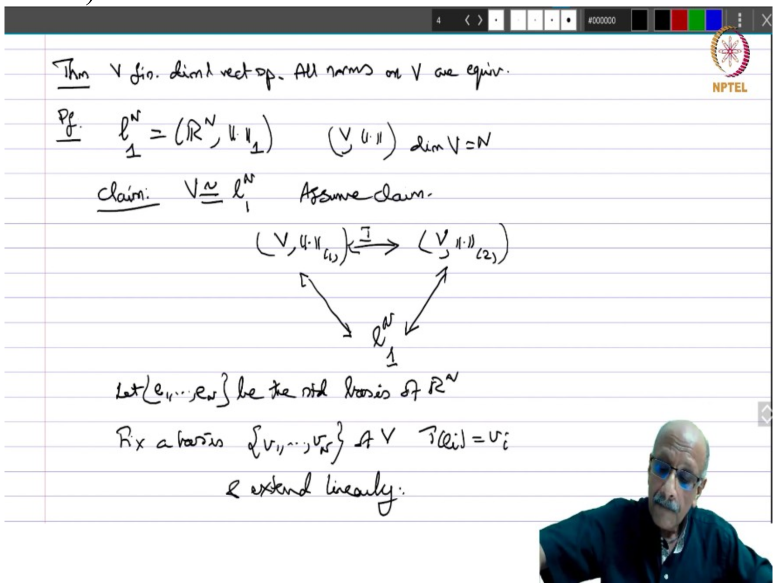

Theorem. Let V be a finite dimensional vector space. Then, all norms on V are equivalent.

Recall that l_N^1 is nothing but the space R^N with $\|.\|_1$. V is a vector space, finite dimensional, say $dim\, V = N$. And given with some norm $\|.\|$. So, we will show that these two are isomorphic.

So, we claim V is isomorphic to l_N^1, whatever be the norm on V. We show that it is going to be isomorphic to l_N^1. Now, assume the claim is true. Take V with two norms, $\|.\|_1$ and $\|.\|_2$ and you also have l_N^1. We have ¿ and the identity map between them and then**,** the same isomorphism T from l_N^1. Then we show that the identity map is an isomorphism. That will ne the idea of our proof.

Proof. Let $e_1, \ldots, e_N$ be the standard basis of R^N. Fix a basis, $v_1, \ldots v_N$ of V and you define $T(e_i) = v_i$ and extend linearly.

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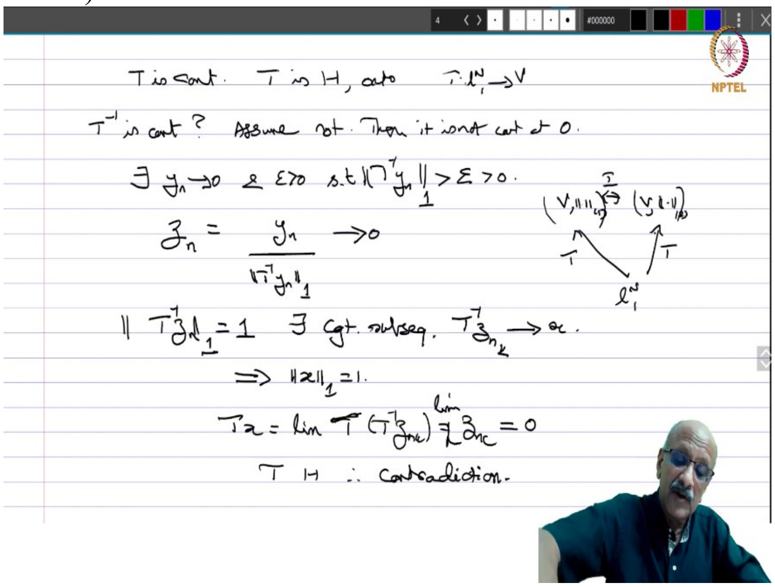

We already saw in the earlier example that T is continuous. And, also we know that T is 1 - 1 and onto because it maps a basis into a basis among spaces of the same dimension. Now we want to show that T^{-1} is continuous. Assume that it is not. Then we saw that continuity is the same as continuity at the origin and therefore, it is not continuous at 0, what does this mean? There exists a sequence (y_n) converging to 0 and $\epsilon > 0$ such that $\|T^{-1}(y_n)\|_1 > \epsilon$. This is the contradiction. I mean contra positive statement of the fact that it is not continuous at the origin.

Now let me define $z_n = \frac{y_n}{\|T^{-1}(y_n)\|_1}$. Now, $\|T^{-1}(y_n)\|_1 > \epsilon$. So, the denominator is staying away from ϵ and therefore (z_n) will also converge to 0 in V. Now, what about $T^{-1}(z_n)$? $\|T^{-1}(z_n)\|_n = 1$ (because I have divided by the correct number). So, $(T^{-1}(z_n))$ is a bounded sequence. Since the l_N^1 norm, the $\|.\|_1$, $\|.\|_2$ are all the same, that is the usual topology in R^N. If you have a bounded sequence then it has a convergent subsequence. So, there exists a convergent subsequence. Let us say $T^{-1}(z_{n_k}) \to x$. Then this implies that $\|x\|_1 = 1$. But what about $T(x)$? $T(x)$ is the limit of $T(T^{-1}(z_{n_k}))$ which is z_{n_k} and $\lim z_{n_k} = 0$. So, $T(x) = 0$ and $\|x\|_1 = 1$, which is impossible since T is 1 -1. Therefore, contradiction. Consequently, T^{-1} is also continuous and therefore as I said, we have ¿ and the identity map between them and then, the same map T from l_N^1 to them, which is an isomorphism. So, the identity is a composition of T^{-1} and T in either ways and therefore, you have the identity map is an isomorphism in both ways. And therefore, the, therefore we have that identity map is an isomorphism and consequently, all the norms are equivalent.

This is not true in infinite dimensions.

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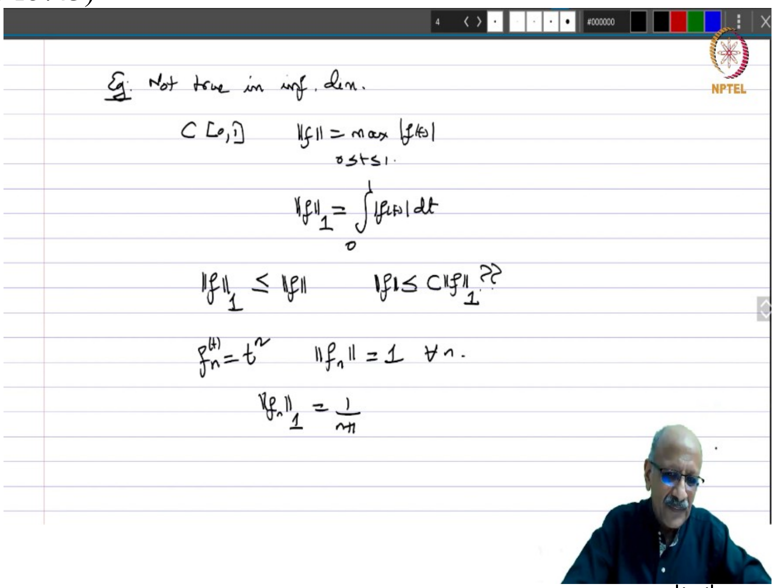

Example. Let us take an example of $C[0,1]$. We have the usual norm $\|f\| = \max_{0\le t\le 1} |f(t)|$. Now, I am going to define $\|f\|_1 = \int_{[0,1]} |f(t)|\,dt.$ You can check that this is a norm. So this is very easy to check, triangle inequality is also trivial and therefore, this defines a nice norm linear space. I claim that these two are not equivalent. Now, first of all, $\|f\|_1 \le \|f\|$. Now, what about the reverse inequality? Can you write $\|f\| \le C\|f\|_1$? Is this possible? Answer is no. So, we can give a counter example, we can construct several, there is one in the book which I mentioned to you and now I will give you another one. So, let us take $f_n(t) = t^n$. Then, what is $\|f_n\|$? $\|f_n\| = 1$ for all n. What is $\|f_n\|_1$? Well, it is a non-negative function. Therefore, it is just an integral which is therefore $\|f_n\|_1 = \frac{1}{n+1}$. So, if you put f_n in $\|f\| \le C\|f\|_1$, you will get 1 on this side and $\frac{1}{n+1}$ on that side, which goes to 0 and therefore this inequality is not possible and Consequently, you cannot have, so you cannot have the equivalence of norms.

So, in infinite dimensional spaces, you do not have the equivalence of norms. Now, the equivalence of norms followed from the main thing which we used in proving the equivalence of norms is that the unit ball or bounded sets, bounded and closed sets were compact. Now, this in fact characterizes finite dimensional spaces. This is a very amazing fact, the dimension is something which is purely algebraic. It says how many, what is the maximal linearly independent set, so that is a completely algebraic statement. On the other hand, the compactness of the unit ball is a topological statement, and these two are connected by this theorem. So, we have the very beautiful result.

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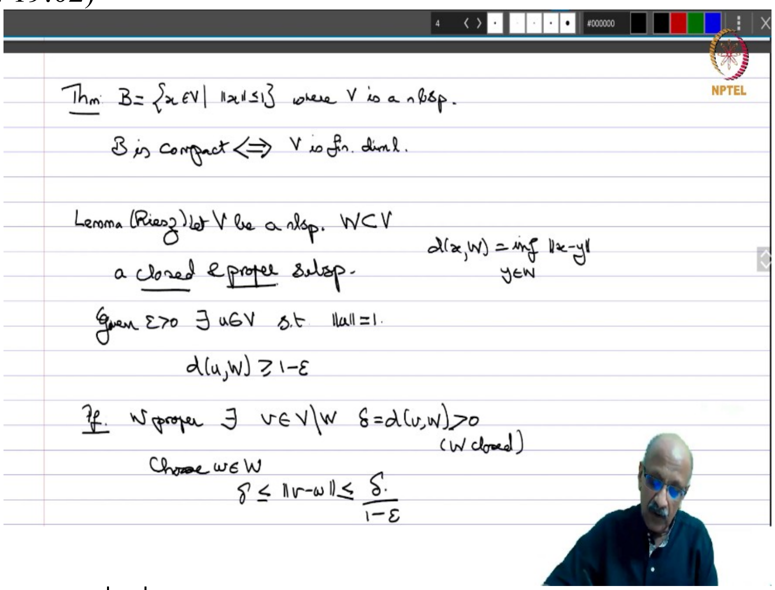

Theorem. Let $B=\{x\in V:\|x\|\le 1\}$, where V is a normed linear space. Then B is compact iff V is finite dimensional.

Before we prove this, we need a Lemma. So, we have a

Lemma (Riesz). Let V be a norm linear space and $W\subset V$ (closed and proper subspace). Given $\epsilon>0$, there exists a $u\in V$ such that $\|u\|=1$ and $d(u,W)\ge 1-\epsilon$.

What is the distance $d(x,W)\ge$? $d(u,W)=inf\{\|x-y\|:y\in W\}$. So it is the shortest distance in which you can come from x to W. So, Reisz Lemma says that you have a closed and proper subspace W, then you can find a vector u which is a unit vector, such that $d(u,W)\ge 1-\epsilon$..

Proof. So, W is proper, so there exists a $v\in V\setminus W$ and let us take $\delta=d(v,W)$. Note that W is closed. So δ has to be strictly positive. Okay, but what is δ? The distance is nothing but the infimum of something, so choose $w\in W$ such that

$$\delta\le\|v-w\|\le\frac{\delta}{1-\epsilon}.$$

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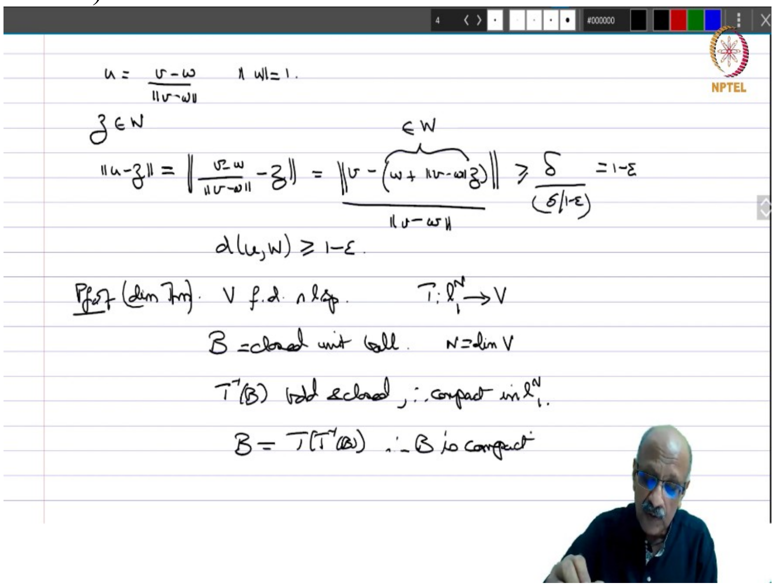

Now put $u = \frac{v-w}{\|v-w\|}$. Then $\|u\| = 1$. So, let us check that the other property is true. Let z be an arbitrary element in W. So what is $\|u-z\|$?

$$\|u-z\| = \left\| \frac{v-w}{\|v-w\|} - z \right\| = \frac{\|v-(w+\|v-w\|z)\|}{\|v-w\|} \geq \frac{\delta}{\left(\frac{\delta}{1-\epsilon}\right)} = 1-\epsilon.$$

Therefore, $\|u-z\| \geq 1-\epsilon$ for any $z \in W$. Therefore $d(u,W) \geq 1-\epsilon$. So, this is our proof.

Now we will prove the dimension theorem. Namely,

Theorem. Unit ball is compact in a normed linear space if and only if the space is finite dimensional.

Proof. Let us assume V be a finite dimensional norm linear space. So, let B be the closed unit ball. Now, recall the mapping T which we defined early, $T: l_N^1 \mapsto V$ where N is the dimension of V. We saw that this is an isomorphism. Since B is closed unit ball, so $T^{-1}(B)$ is bounded and closed, therefore compact in l_N^1. Since continuous image of a compact set is compact, so $T(T^{-1}(B))$ is compact. So, this shows that if you have a finite dimensional space, the closed unit ball is always compact. This is of course expected.

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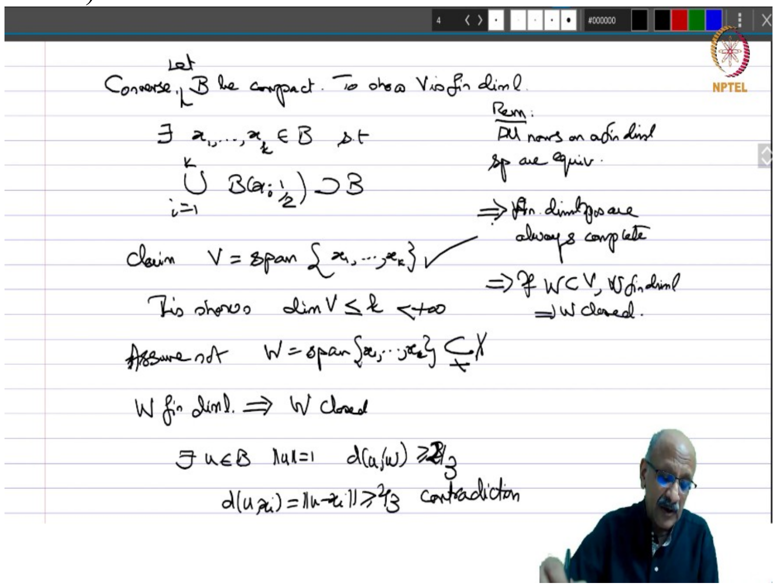

Now, let us do the converse. Let B be compact. To show V is finite dimensional. Since B is compact there exists $x_1, \ldots, x_k \in B$ such that, $\cup_{i=1,\ldots,k} B\left(x_i, \frac{1}{2}\right) \supset B$. I claim V is nothing but the span of $x_1, \ldots, x_k$. So, this will prove, this shows that the dimension of V is less than or equal to k, which is finite. We cannot say it is equal to k because we do not know if these are linearly independent or not, but we do not need that.

So, let us assume not, so $W = span\{x_1, \ldots, x_k\} \subset V$. So, W is finite dimensional. What does it mean? In every finite dimensional space, all norms are same and therefore every finite dimensional space is complete and since it is a complete subspace of a norm linear space, it is close, implies W is closed. (Remark - all norms on a finite dimensional space are equivalent.)

So, W is closed and it is proper subspace of V, therefore by Reisz Lemma, there exists a $u \in B$ i.e., $\|u\| = 1$ such that $d(u, W) \geq \frac{2}{3}$. But this is a contradiction. In particular, $d(u, x_i) = \|u - x_i\| \geq \frac{2}{3}$. But the entire B is contained in the union of the balls of radius half with center x_i, so this is a contradiction. Therefore, the claim is established and we have shown that this is true.

So, finite dimensional spaces characterized by the compactness of the ball. So, in infinite dimensional spaces, the unit ball or a closed bounded set is never compact.

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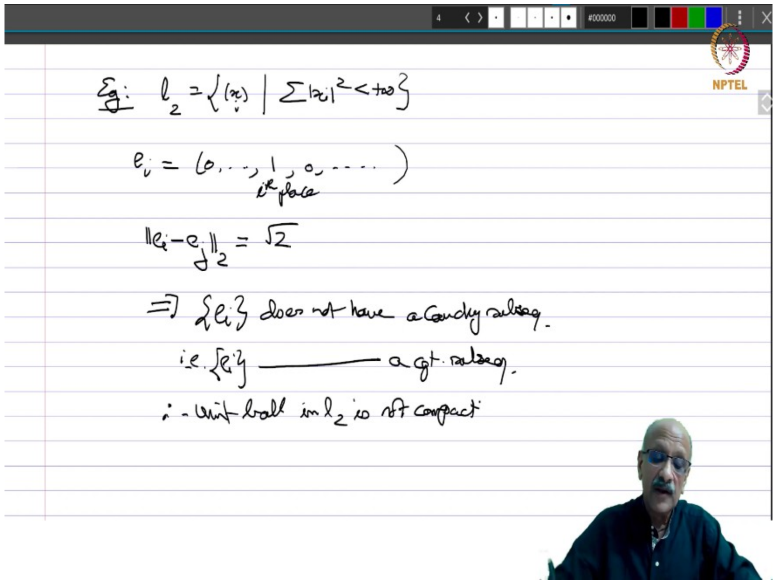

Example. Let us take $l_2=\{(x_i): \sum_{i=1,\ldots,\infty} |x_i|^2<\infty\}$. So, let me take $e_i=(0,\ldots,1(ith),\ldots 0)$. Then $\|e_i-e_j\|=\sqrt{2}$. So, this means each element is at equal distance from all other elements. Therefore, (e_i) does not have a Cauchy subsequence, that is, (e_i) does not have a convergence subsequence. Therefore, unit ball in l_2 is not compact, because if it were compact, then every bounded sequence has to have a convergence subsequence. So this is an example where this is not true. So, l_2 is not finite dimensional.