Functional Analysis
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Lecture No. 59
Spectrum of an operator - Part 2

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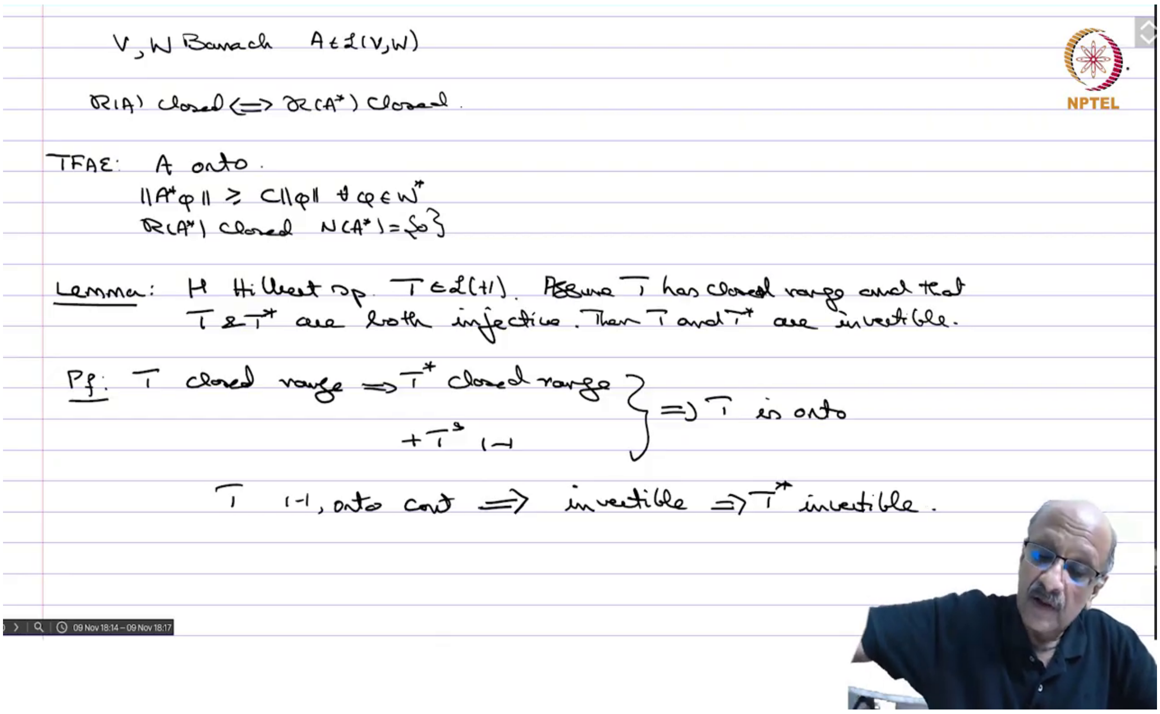

We continue with the study of the spectrum. This time we will talk about operators on a Hilbert space. So, I want to recall once more the theorem which we proved long ago. So, we have that, if V, W Banach and $A \in L(V, W)$, then we have $R(A)$ is closed if and only if $R(A^*)$ is closed. Then we had the following result. Namely, A is onto and then

$||A^*\phi|| \geq C\,||\phi||\ \forall \phi \in W^*$ and $R(A^*)$ is closed, $N(A^*) = 0$.

These are all equivalent to each other and this is a theorem which we had. So, now, we prove a very useful lemma which we will use again and again.

Lemma: H Hilbert space and $T \in L(H)$. So, it is a bounded linear operator from H to itself. Let us assume T has closed range and that T and T^* are both injective. Then, T and T^* are invertible.

Proof. T has closed range $\Rightarrow$ T^* has closed range and T^* is 1-1 is (given) and therefore, these two imply the T is onto, by the theorem which we stated.

So, T is 1-1, onto and continuous, and therefore, by the open mapping theorem, T is invertible. And T invertible means T^* is invertible as we have already seen yesterday. So, T and T^* are both invertible. So, this is a very useful lemma which we will have occasion to use today again and again.

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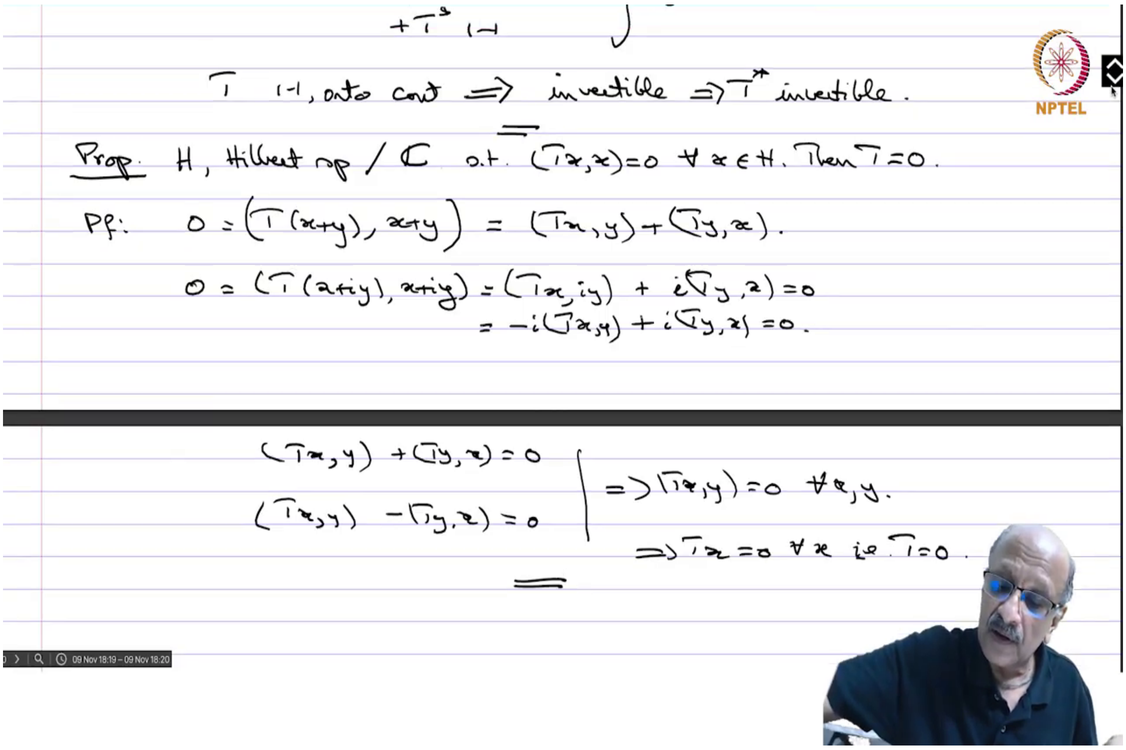

Proposition. H Hilbert space over $\mathbb{C}$, such that $(Tx,\, x) = 0,\ \forall x \in H$. Then $T = 0$.

Proof. $0 = (T(x + y),\, x + y) = (Tx,\, y) + (Ty,\, x)$.

$$0 = (T(x + i\,y),\, x + i\,y) = (T\,x,\, i\,y) + i\,(Ty,\, x) =-\, i\,(T\,x,\, y) + i\,(Ty,\, x)$$

$\Rightarrow (Tx,\, y) + (Ty,\, x) = 0\ \&\ (Tx,\, y) - (Ty,\, x) = 0$

$\Rightarrow (Tx,\, y) = 0,\ \forall x,\, y \Rightarrow Tx = 0,\ \forall x$ i.e. $T = 0$. So, $\mathbb{C}$ is important in this particular result.

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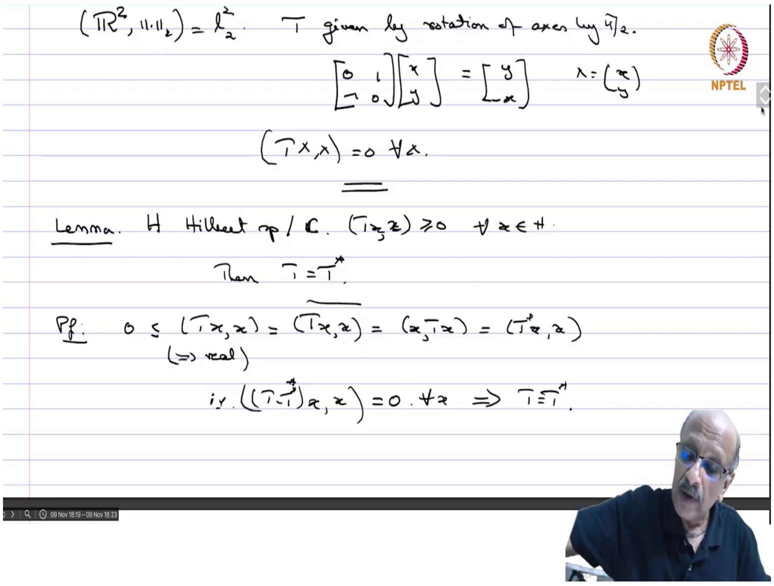

So, if you had a real Hilbert space. For instance, if you take $(\mathbb{R}^2, ||.||_2) = l_2^2$, then you take T given by rotation of axis by $\pi/2$. So, the matrix of T is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

And then if you take $[x,\ y]^T$, you get $T[x,\ y]^T = [y,\ -x]^T$ and therefore, $(Tx,\ x) = 0,\ \forall x$, and then $T \neq 0$ at all. So, it is the presence of the i, which makes a difference there, and therefore, that is important.

Lemma. Let H be a Hilbert space over $\mathbb{C}$. Henceforth we will work over $\mathbb{C}$ and $(Tx,\ x) \geq 0,\ \forall x \in H$. Then T self-adjoint, $T = T^*$. Again, obviously, this does not apply to the real case because the previous example gave you $(Tx,\ x) = 0$ which is a particular case of $(Tx,\ x) \geq 0$ and this matrix is not self-adjoint matrix and therefore, you have that, this is not applicable in the real case, so it depends on the complex case.

Proof. $0 \leq (Tx,\ x) = \overline{(Tx,\ x)} = (x,\ Tx) = (T^*x,\ x)$, i.e., $((T - T^*)x,\ x) = 0,\ \forall x \in H$

($(Tx,\ x)$ is real $\Rightarrow (Tx,\ x) = \overline{(Tx,\ x)}$). By the previous proposition you have the $T = T^*$.

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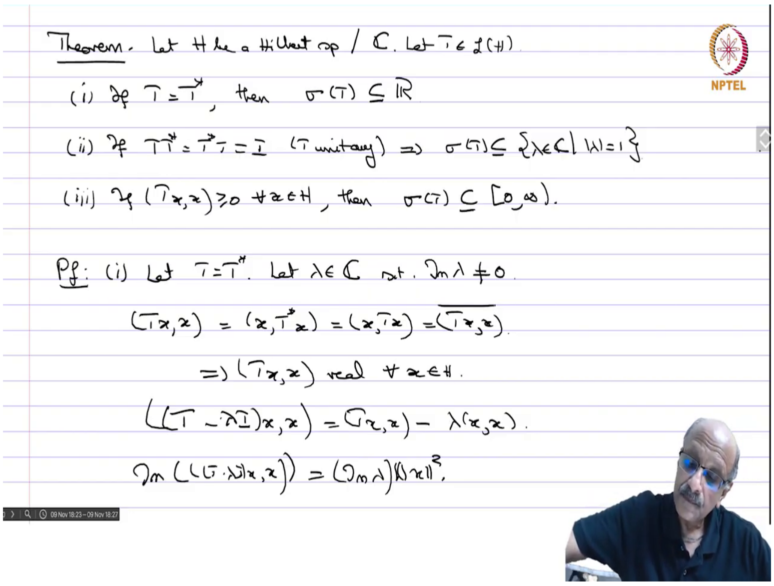

So, now we come to a very nice theorem.

Theorem. Let H be a Hilbert space over $\mathbb{C}$, let $T \in L(H)$. Then

1. if $T = T^*$, then $\sigma(T) \subseteq \mathbb{R}$. So, the spectrum contains only real numbers.
2. if $T T^* = T^* T = I$, so T is unitary implies $\sigma(T) \subseteq \{\lambda \in \mathbb{C} / |\lambda| = 1\}$. That means the spectrum is contained in the unit circle of the complex plane.
3. if $(Tx, x) \geq 0, \forall x \in H$, then $\sigma(T) \subseteq [0, \infty)$.

So, real self-adjoint operators have spectrum and the real line, unitary operators are spectrum on the unit circle, and positive operators are spectrum in the positive real line.

This is analogous to the real numbers, this is the normal operators you can see, if you think of them as complex numbers, then * is like conjugation. So, if $T = T^*$ that means it resembles real numbers and that is reflected by the fact, that $\sigma(T)$ is contained in the real numbers. Now, TT^* is like saying $z\bar{z}$. So, this looks like the unit circle. So, that is exactly the spectrum is in the unit circle, $|\lambda| = 1$, means $\lambda\bar{\lambda} = 1$. And similarly, if T is a positive operator, so it looks like positive real numbers then in fact the spectrum is indeed in the positive real numbers.

Proof: We will prove the first one. Let $T = T^*$, let $\lambda \in \mathbb{C}$, such that $\mathrm{Im}(\lambda) \neq 0$.

Now, $(Tx, x) = (x, T^* x) = (x, Tx) = \overline{(Tx, x)} \Rightarrow (Tx, x)$ is real $\forall x \in H$. So, now, when you look at $((T - \lambda I) x, x) = (Tx, x) - \lambda(x, x)$

$\text{Im}\,((T - \lambda I) x, x) = Im(\lambda)\, ||x||^2$

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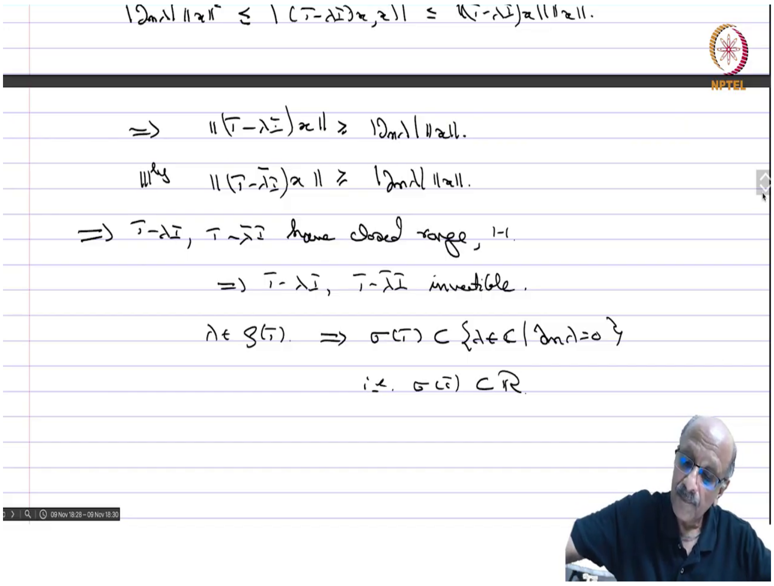

Therefore, $|Im\,(\lambda)|\, ||x||^2 \le |((T - \lambda I) x, x)| \le ||(T - \lambda I)x||\, ||x||$, i.e.

$||(T - \lambda I)x|| \ge |Im\,(\lambda)|\, ||x||$. Similarly, we have $||(T - \bar{\lambda} I)x|| \ge |Im\,(\lambda)|\, ||x||$.

So, when you have an operator which is bounded below, we have seen that it has closed range because if you take any sequence in the image which converges, then this will be Cauchy, which implies that $\{x_n\}$ is also Cauchy, so $\{x_n\}$ will converge and so the image limit will be in the image. So, any operator which is bounded below has always closed range. This implies that $(T - \lambda I)$, $(T - \bar{\lambda} I)$ have closed range and both 1-1. So, then we have the $(T - \lambda I)$, $(T - \bar{\lambda} I)$ are both invertible, by the lemma which I just proved. And therefore, $\lambda \in \rho(T)$. So, $Im\,(\lambda) \ne 0$, this implies that $\sigma(T) \subseteq \{\lambda \in \mathbb{C} \,/\, Im(\lambda) = 0\}$, i.e. $\sigma(T) \subseteq \mathbb{R}$.

So, that proves the first relationship.

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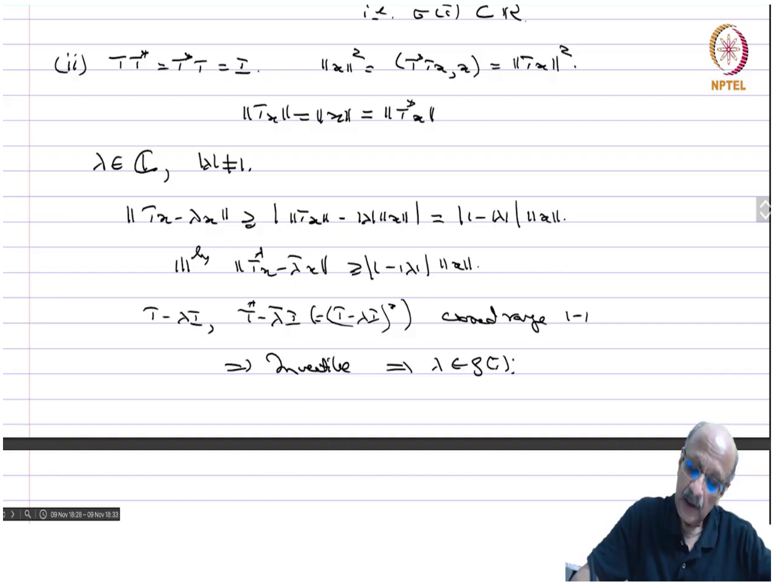

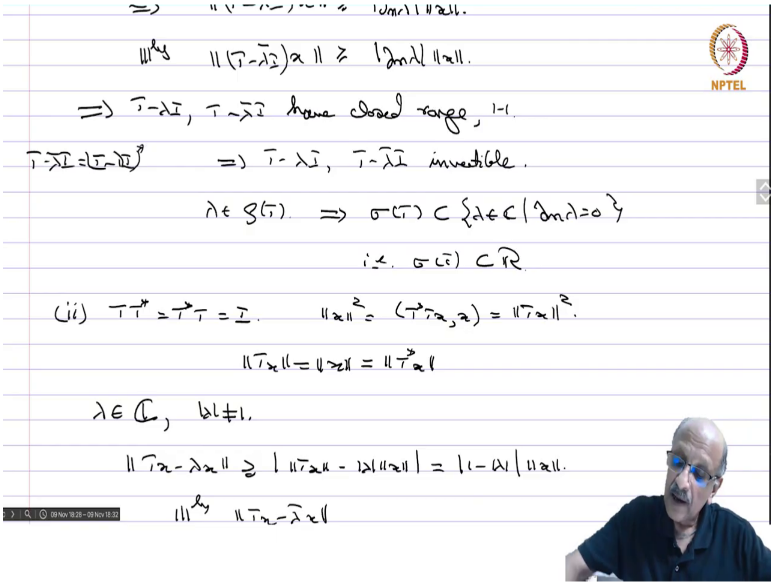

2. Now we have $T\,T^* = T^*\,T = I$. So, if you take $||x||^2 = (T^*\,T\,x,\ x) = ||T\,x||^2$. So, these are all isometries, any unitary operator gives you an isometry namely $||T\,x|| = ||x||$,

and also $||T\,x|| = ||x|| = ||T^*\,x||$, for the same reason. Instead of $T\,T^*$ you took T^*T, you would have got the same thing. so you have .

Now, let us take $|\lambda| \neq 1$. Then you take $||T\,x - \lambda\,x|| \geq |\,||T\,x|| - |\lambda|\,||x||\,|$ but $||T\,x|| = ||x||$. So, $||T\,x - \lambda\,x|| \geq |\,||T\,x|| - |\lambda|\,||x||\,| = |1 - |\lambda|\,|\,||x||$.

And similarly, $||T\,x - \bar{\lambda}\,x|| \geq |1 - |\lambda|\,|\,||x||$.

Remark. $T^{*} - \bar{\lambda} I$, is nothing but $(T - \lambda I)^{*}$. So, once again $(T - \lambda I)$, $(T - \lambda I)^{*}$ have closed range and 1-1, so implies invertible, implise, $\lambda \in \rho(T)$.

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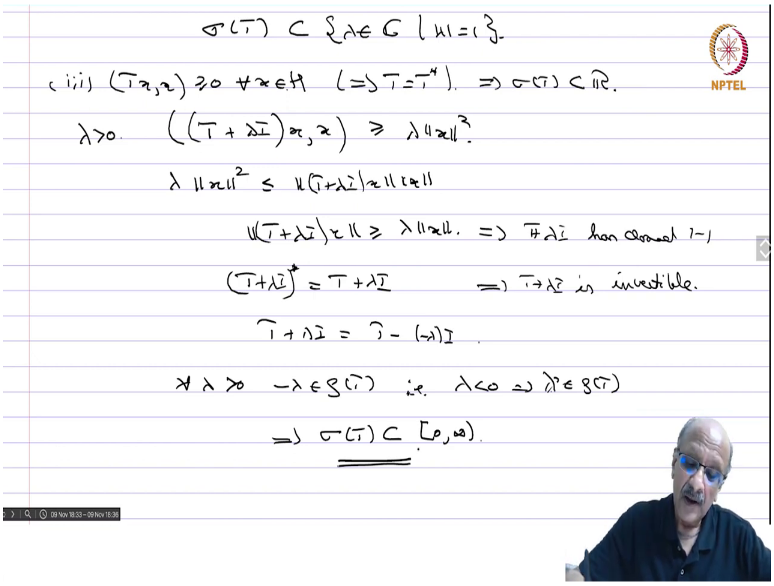

And therefore, you have $\sigma(T) \subseteq \{\lambda \in \mathbb{C} / |\lambda| = 1\}$

3. Given that $(Tx, x) \geq 0$, $\forall x \in H$. And of course, we have seen that this implies $T = T^{*}$, therefore $\sigma(T) \subseteq \mathbb{R}$. Let us take $\lambda > 0$. Then $((T + \lambda I)x, x) \geq \lambda ||x||^{2}$. Cauchy Schwarz inequality $\Rightarrow \lambda ||x||^{2} \leq ||(T + \lambda I)x|| \, ||x||$. Therefore,

$||(T + \lambda I)x|| \geq \lambda ||x||$. $(T + \lambda I)^{*} = (T + \lambda I)$ because λ is real, and T is self-adjoint. Therefore, this implies that T has a closed range. So, $(T + \lambda I)$ has closed range and it is 1-1. So, this implies $(T + \lambda I)$ is invertible. $(T + \lambda I)$ is nothing but $(T - (-\lambda) I)$. And therefore, for every λ positive, $-\lambda \in \rho(T)$, i.e., λ negative implies $\lambda \in \rho(T)$ and therefore, this tells you that $\sigma(T) \subset [0, \infty)$. So, with this, we will wind up this chapter and then do some exercises next.