Functional Analysis
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Lecture No. 58
Spectrum of an operator- Part 1

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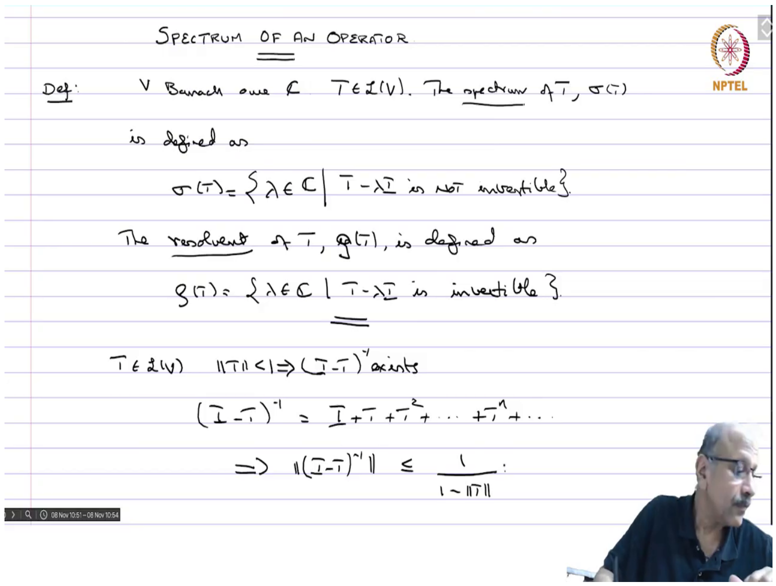

We will now discuss a very important topic in functional analysis namely Spectrum of an Operator. You would have already heard of this word from linear algebra where given a matrix its spectrum is the set of all Eigenvalues. So we generalize this notion. So we will briefly go back to Banach spaces. And now we will work with the complex field as the basis so V Banach over $\mathbb{C}$ and $T \in L(V)$. So it is a continuous linear operator on V.

So we have a definition the spectrum of T $\sigma(T) = \{\lambda \in \mathbb{C} / T - \lambda I$ is not invertible $\}$. The resolvement of T, denoted by $\rho(T)$, is defined as the complement of $\sigma(T)$. So $\rho(T) = \sigma(T) = \{\lambda \in \mathbb{C} / T - \lambda I$ is invertible$\}$. So we want to study the properties of the spectrum of a given operator.

Suppose $T \in L(V)$ and $||T|| < 1$ then we know that, we have shown this in an exercise, $(I - T)^{-1}$ exists. And in fact we have shown that $(I - T)^{-1}$ is nothing but $I + T + T^2 + ... + T^n + ...$ is the infinite series. If you now take the norm of this so you get $1 + ||T|| + ||T||^2 + ... + ||T||^n + ...$That is a geometric series with the $||T|| < 1$ which is convergent and so that immediately tells you that $||(I - T)^{-1}|| \leq \frac{1}{1-||T||}$.

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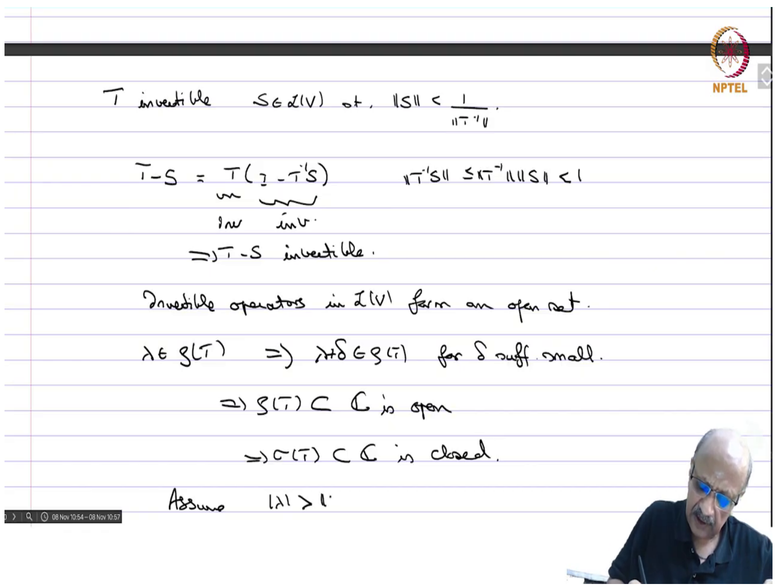

So you also recall something else which we have done. So let us take T invertible and $S \in L(V)$ such that $||S|| < \frac{1}{||T^{-1}||}$. Then look at $T - S = T(I - T^{-1}S)$, $||T^{-1}S|| \leq ||T^{-1}|| \, ||S|| < 1$. T is invertible and $(I - T^{-1}S)$ is also invertible implies $T - S$ is invertible. So from this we saw that invertible operators in $L(V)$ form an open set. So if $\lambda \in \rho(T)$ that means $(T - \lambda I)$ is invertible then this implies that $\lambda + \delta \in \rho(T)$ for δsufficiently small. Because invertible operators form an open set. So this implies that $\rho(T) \subset \mathbb{C}$ is open implies $\sigma(T) \subset \mathbb{C}$ is closed. So the spectrum is a closed set. Assume $|\lambda| > ||T||$.

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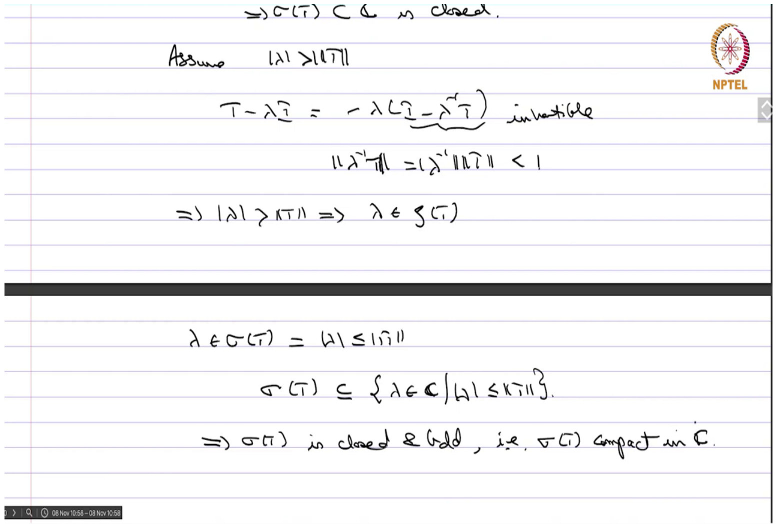

Then $(T - \lambda I)$ this can be written as $-\lambda(I - \lambda^{-1}T)$. And $||\lambda^{-1}T|| = |\lambda^{-1}|\,||T|| < 1$ Because $||T|| < |\lambda|$. So this again becomes invertible. Therefore, $|\lambda| > ||T|| \Rightarrow \lambda \in \rho(T)$. So $\lambda \in \sigma(T) \Rightarrow |\lambda| \leq ||T||$, and therefore $\sigma(T) \subseteq \{\lambda \in \mathbb{C} \,/\, |\lambda| \leq ||T||\}$. So it is a closed set. So $\sigma(T)$ is closed and bounded that is $\sigma(T)$ is compact. So the spectrum is a compact subset of the complex plane.

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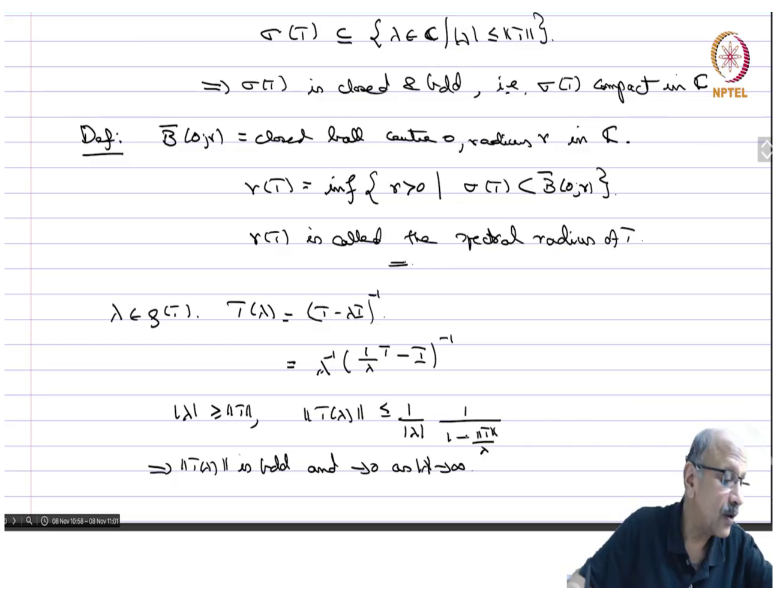

Definition. $\bar{B}(0, r)$ = closed ball center 0 radius, r in $\mathbb{C}$. And then you consider $r(T) = \inf \{r > 0/ \sigma(T) \subset \bar{B}(0, r)\}$. Then $r(T)$ is called the spectral radius of T. And then of course you know that $r(T)$ is less than or equal to $||T||$. Now we want to show that the spectrum is in fact a compact set but it is a non-empty compact set. So the elements always exist in this spectrum.

Let $\lambda \in \rho(T)$ and then you define $T(\lambda) = (T - \lambda I)^{-1}$.Now you can write $T(\lambda) = \lambda^{-1}(\frac{1}{\lambda}T - I)^{-1}$. If $|\lambda| \geq ||T||$, $||T(\lambda)|| \leq \frac{1}{|\lambda|}\frac{1}{1-\frac{||T||}{\lambda}}$, this means that $||T(\lambda)||$ is bounded and tends to 0 as $\lambda \to \infty$. So this is an observation which we are making.

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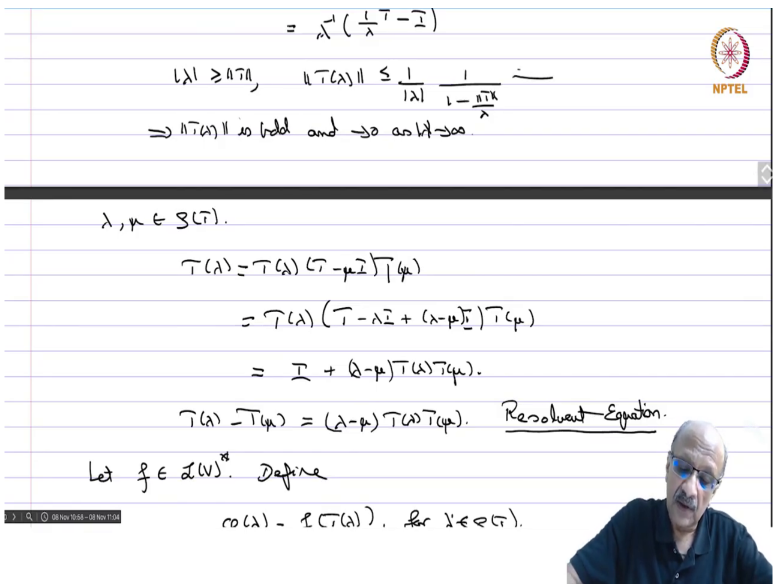

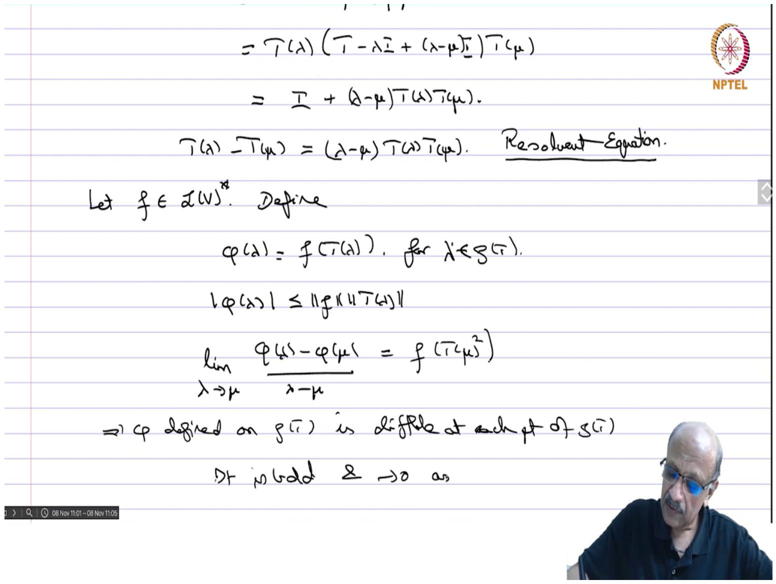

Now let λ and $\mu \in \rho(T)$. So $T(\lambda) = T(\lambda)(T - \mu I)T(\mu) = T(\lambda)(T - \lambda I + (\lambda - \mu)I)T(\mu)$

$$= (I + (\lambda - \mu)\, T(\lambda))\, T(\mu)$$

$\Rightarrow T(\lambda) - T(\mu) = (\lambda - \mu)\, T(\lambda)\, T(\mu)$ This is called the resolvement equation.

Now let f be a continuous functional on $L(V)$ so $f \in L(V)^*$. Define $\phi(\lambda) = f(T(\lambda))\ \ for\ \lambda \in \rho(T)$. So it is defined on an open subset of the complex plane and then $|\phi(\lambda)| \leq ||f||\, ||T(\lambda)||$. We saw that $||T(\lambda)||$ is already is a bounded thing and in fact it goes to 0 as $\lambda \to \infty$ because of $||T(\lambda)|| \leq \frac{1}{|\lambda|}\frac{1}{1-\frac{||T||}{\lambda}}$ relationship. Therefore you have that $\phi(\lambda) = f(T(\lambda))$ is bounded and it goes to 0 as $|\lambda| \to \infty$. Further, if you take $\lim\limits_{\lambda \to \mu} \frac{\phi(\lambda)-\phi(\mu)}{\lambda-\mu}$. What do you get? from the resolvement equation this is nothing but $f(T(\mu)^2)$. $\frac{\phi(T(\lambda))-\phi(T(\mu))}{\lambda-\mu} = \phi(T(\lambda)T(\mu))$, as $\lambda \to \mu$ therefore that goes to $f(T(\mu)^2)$. That means ϕ defined on $\rho(T)$ is differentiable at each point of $\rho(T)$. It is bounded and tends to 0 as $|\lambda| \to \infty$.

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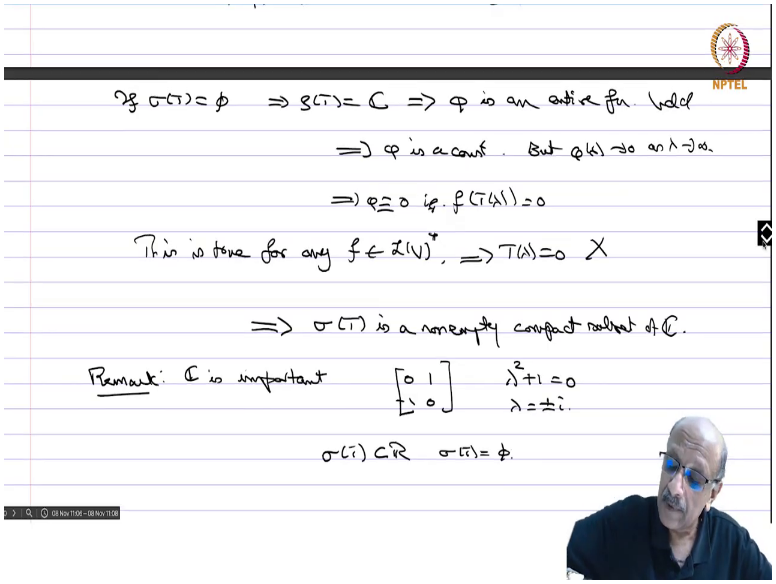

Now we can get a nice contradiction. If $\sigma(T)$ is empty , this will imply that $\rho(T) = \mathbb{C}$ and this will imply that ϕ is an entire function which is bounded and by Luwill’s theorem this means that ϕ is a constant. But $\phi(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. And therefore this implies ϕ is identically zero. But that is absurd because phi is identically 0 and $f(T)\lambda = 0$ is true for any $f \in L(V)^{*}$ but the dual separates points. So this implies that $T\lambda = 0$ and that is a contradiction $T\lambda$ is the inverse of an operator; it cannot be the 0 operator. So this proves so this implies that $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$.

Remark $\mathbb{C}$ is important, for instance, if you took only $\mathbb{R}$ then look at

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then what are the eigenvalues? So the characteristic polynomial is $\lambda^2 + 1 = 0$. So $\lambda = \pm\, i$ are the only eigenvalues and they are not real. And therefore $\sigma(T) \subset \mathbb{R}$ is empty. So the spectrum could be empty if you are stuck to the real field and that is why the spectrum to be non-empty, you need to work with the complex sample. So that is the important thing. So now we want to see what the spectrum looks like.

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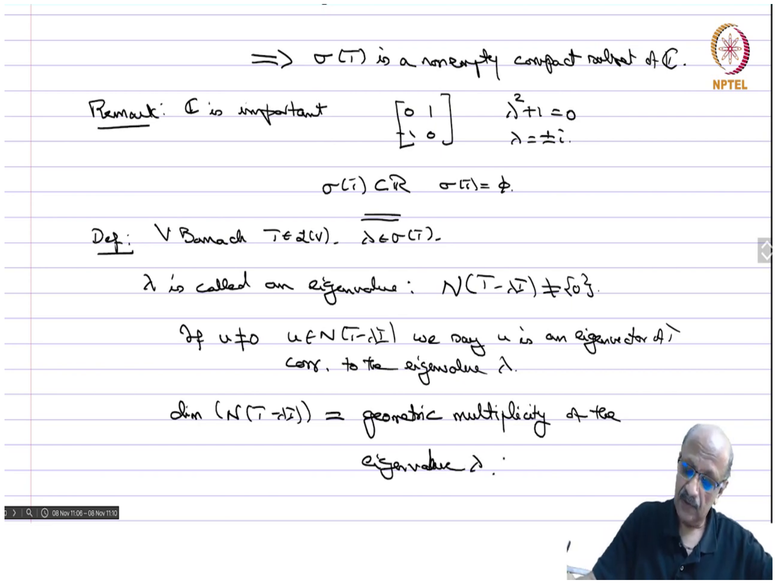

Definition V Banach and $T \in L(V)$ and $\lambda \in \sigma(T)$. So λ is called an eigenvalue if the null space of $(T - \lambda I)$ is not equal to $\{0\}$. That means and if $u \neq 0, u \in N(T - \lambda I)$ we say u is an eigenvector of T corresponding to the eigenvalue λ and the dimension of $N(T - \lambda I)$ is called the geometric multiplicity of the eigenvalue λ. There is something else called algebraic multiplicity which we can discuss.

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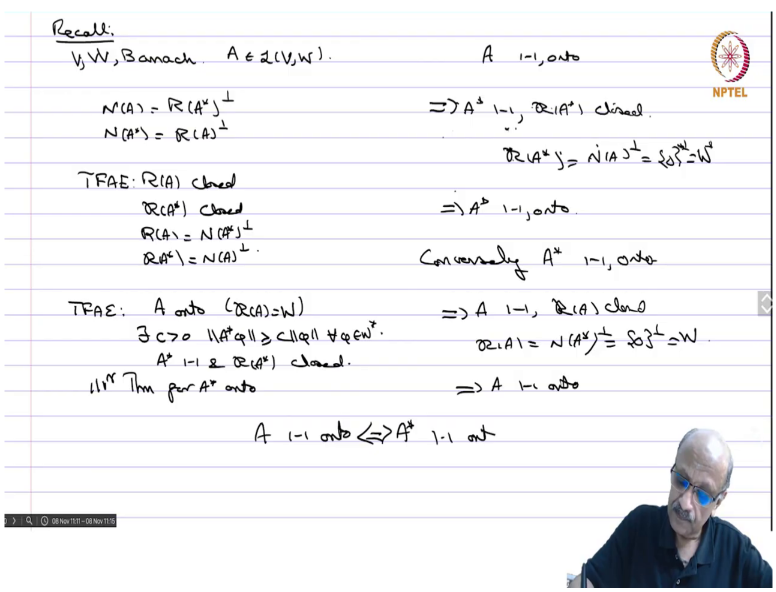

So now let us briefly recall a theorem which we have done before.

Recall V, W Banach and $A \in L(V, W)$. $N(A) = R(A^*)$, $N(A^*) = R(A)^\perp$. Then we also showed that the following are equivalent,

- $R(A)$ is closed,
- $R(A^*)$ is closed
- $R(A) = N(A^*)^\perp$
- $R(A^*) = N(A)^\perp$

We also showed the following theorem that the following are equivalent,

- A onto that means $R(A) = W$.
- Then there exists a $c > 0$ such that $||A^* \phi|| \geq c||\phi||$, $\forall \phi \in W^*$
- A^* is 1-1 and $R(A^*)$ is closed.

Similar theorem for A^*, with A^* replacing A, A replacing A^*, you have an identical theorem that the left is an exercise to this. So we approved all these things.

Now let us say A is 1-1, onto and of course it is continuous and therefore A is an isomorphism. So then A is 1-1 and $R(A)$ closed this implies A^* 1-1, $R(A^*)$ is closed. And $R(A^*)$ is nothing but $N(A)^\perp N(A) = \{0\}$ that is $\{0\}^\perp = W^*$. So A^* is onto. So A is onto therefore you have that so this implies that A^* 1-1, onto and continuous of course. And therefore it is also an isomorphism. Conversely A^* 1-1 onto. So this will imply that A^* is onto therefore is 1-1 and $R(A)$ closed. And you have$R(A) = N(A^*)^\perp = \{0\}^\perp = W$. So this implies A is 1-1, onto if and only A^* is 1-1, onto. So this is a thing which we want to remember.

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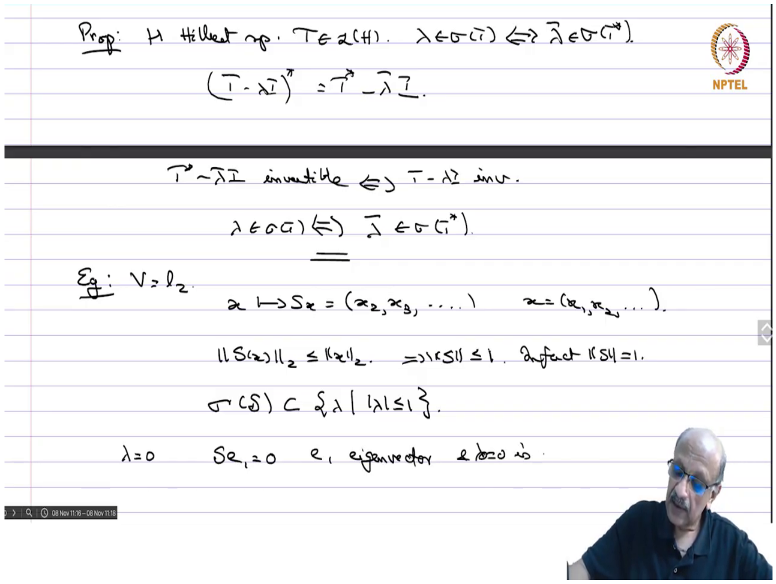

Proposition. H Hilbert space so we are now in the Hilbert space $T \in L(H)$. And $\lambda \in \sigma(T)$ if and only if $\bar{\lambda} \in \sigma(T^*)$. So $(T - \lambda I)^* = (T^* - \bar{\lambda} I)$. So $(T^* - \bar{\lambda} I)$ is invertible if and only if $(T - \lambda I)$ is invertible. And therefore $\lambda \in \sigma(T)$ if and only if $\bar{\lambda} \in \sigma(T^*)$. $\rho(T)$ and $\sigma(T)$ are the complements of each other.

So let us take an example.

Let $V = l_2$. So you consider the map $x \mapsto Sx = (x_2, x_3, \ldots)$. This is the shift operator we have already seen. Where $x = (x_1, x_2, \ldots)$. So $||S\,x||_2 \le ||x||_2 \Rightarrow ||S|| \le 1$. But in fact if you take e_2 or e_3, for instance, and then the shift will be e_1 if it is e_2. And therefore $||S\,x|| = ||x||$. Therefore, $||S|| = 1$. So this means $\sigma(S) \subset \{\lambda \,/\, |\lambda| \le 1\}$. Now you take $\lambda = 0$ then $S(e_1) = 0$, e_1 is an eigenvector and $\lambda = 0$ is an eigenvalue.

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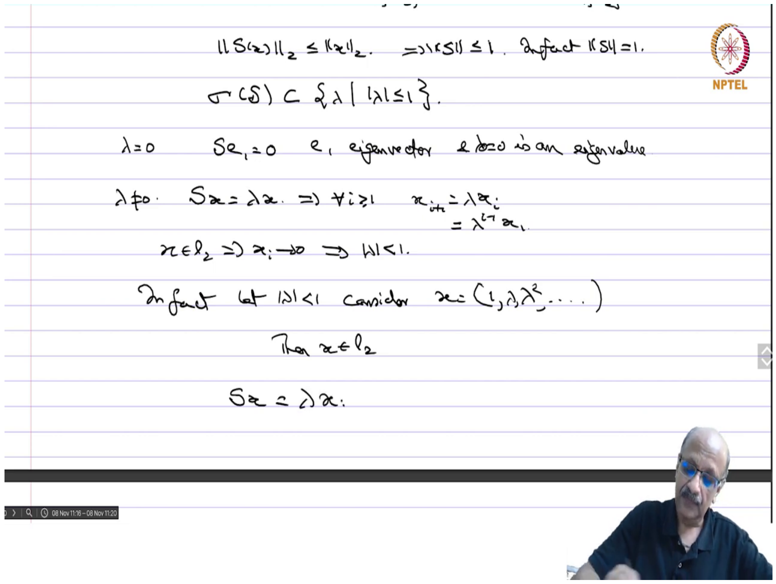

Now let $\lambda \neq 0$ and then you consider Sx if possible let us see if we can find $Sx = \lambda x$ this implies for all $i \geq 1$ $x_{i+1} = \lambda x_i$, so inductively this will give you $\lambda^{i-1} x_1$. But $x \in l_2$ that means $x_i \to 0$. Therefore, this implies that $|\lambda|$ has to be strictly less than 1. In fact, let $|\lambda| < 1$ and consider $x = (1, \lambda, \lambda^2, . . .)$. Now then $x \in l_2$ because $|\lambda| < 1$ you have a geometric series for the norm. So that is fine. So $x \in l_2$ and if you take $Sx = \lambda x$.

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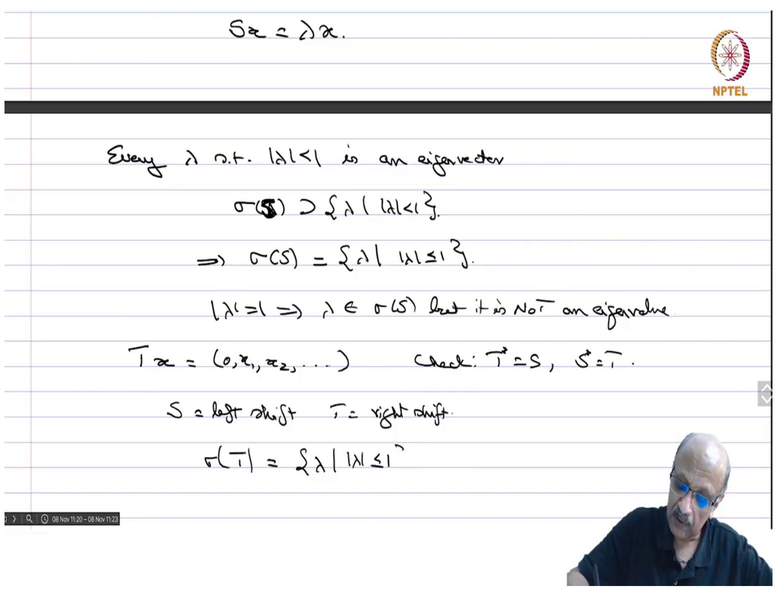

So every λ such that $|\lambda| < 1$ whether it is zero or nonzero is an eigenvector. Therefore, $\sigma(S) \supset \{\lambda \,/\, |\lambda| < 1\}$ and $\sigma(S) \subset \{\lambda \,/\, |\lambda| \leq 1\}$ and therefore this implies that $\sigma(S) \subset \{\lambda \,/\, |\lambda| = 1\}$. And $|\lambda| = 1 \Rightarrow \lambda \in \sigma(S)$ but it is not an eigenvalue. So the spectrum contains a whole continuum of points unlike the finite dimensional case where the spectrum consists only of Eigenvalues. And they were n discrete depending on the dimension. Whereas here you have a spectrum of this operator which is a whole continuum of points in the complex plane the entire disk close disk and in that you have uncountably many eigenvalues and uncountably many members of the spectrum which are not eigenvalues. So every member in the interior of the disk is an eigenvalue, every member in the boundary is not an eigenvalue. Now you define $Tx = (0,\, x_1,\, x_2,\, .\,.\,.\,)$. Then we have already seen this before so check again $T^* = S$ and $S^* = T$. You can check this. So S is called the left shift and T is called the right shift. So now by a previous proposition if λ is in the spectrum of 1 of them $\bar{\lambda}$ bar should be in the spectrum of other.

Now closed unit ball in the complex plane is symmetric with respect to conjugation. Therefore, $\sigma(T)$ also has to be set $\{\lambda \,/\, |\lambda| \leq 1\}$. So we just deduce it directly from the previous. We do not have to do anything.

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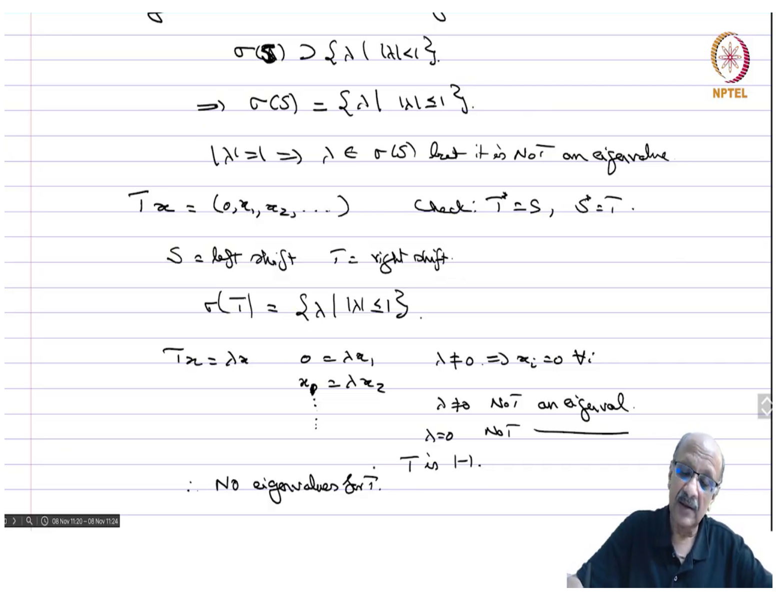

Now if you take $Tx = \lambda x$. So then what happens? $0 = \lambda x_1$, $x_1 = \lambda x_2$ and so on. If $\lambda \neq 0$ this implies $x_i = 0, \ \forall i$. Therefore, $\lambda \neq 0$ is not an eigenvalue. Again, $\lambda = 0$ is also not an eigenvalue because T is an injective map. Therefore, it cannot have any nonzero vector going to 0. And therefore no eigenvalues for T. So, you have a spectrum which is a fully closed unit ball but not a single eigenvalue for this one. So, we have that the spectrum can behave really strangely because in the finite dimensions 1-1 if and only if onto and that is equivalent to invertibility. In infinite dimensions invertibility may fail in many ways, it may be 1-1, it may not be 1-1, it may not be onto etc. So that is why you have the spectrum interesting. So, we will continue with this.