**Functional Analysis**
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**The Institute of Mathematical Sciences**
**Lecture No. 57**
**Fourier series**

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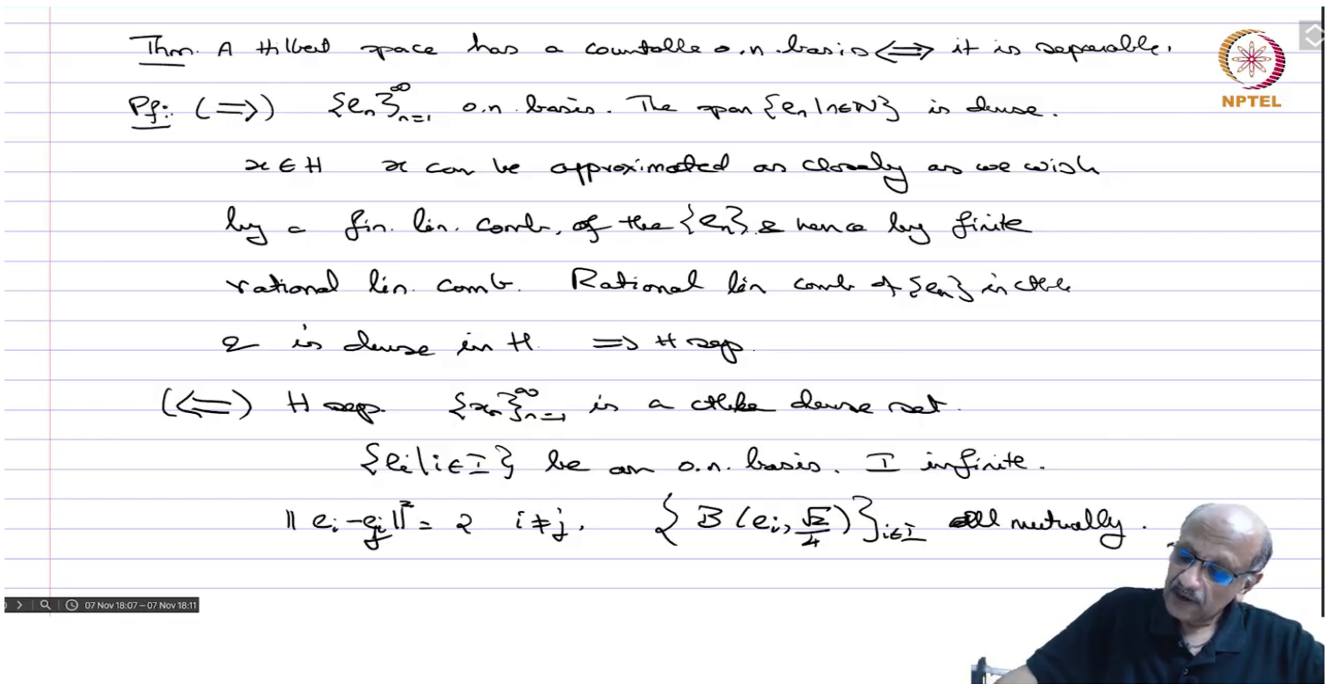


**Theorem**. (Hilbert theorem) A Hilbert space has a countable orthonormal basis if and only if it is separable.

**Proof. (⇒)** Existence of a countable orthonormal basis is equivalent to saying the space is separable though this way separable Hilbert spaces are very important and almost every Hilbert space which we come across in applications is separable. So we will prove this way. So it has a countable orthonormal basis. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis. Therefore, we just proved that the span of $\{e_n \;/\; n \in \mathbb{N}\}$ is dense in the Hilbert space. Now that means every so $x \in H$, then $x$ can be approximated as closely as we wish by finite linear combination of $\{e_n\}$ and hence by finite rational linear combinations that means every $x$ can be approximated as close as possible $\sum \alpha_i e_i$. If we are in a real Hilbert space we can take $\alpha_i$ is to be rational, if it is a complex Hilbert space we can take the real and imaginary parts of the $\alpha_i$ to be rational. And these rational combinations of the $\{e_n\}$ form a countable set. So rational linear combinations of $\{e_n\}$ is countable and is dense in $H$ and therefore $H$ is separable.

Conversely let us assume that $H$ is separable. So we have $\{x_n\}_{n=1}^{\infty}$ is a countable dense set.

Let $\{e_i \ / \ i \in I\}$ be an orthonormal basis and let us assume $I$ infinite because if it is finite, we have nothing to say. Therefore I use an infinite set. And now we have the $||e_i - e_j||^2 = 2$, if $i \neq j$. And therefore, if you take the ball, $\{B(e_i, \frac{\sqrt{2}}{4})\}_{i \in I}$. Then for $i \in I$ these are all mutually disjoint but then these are all open balls.

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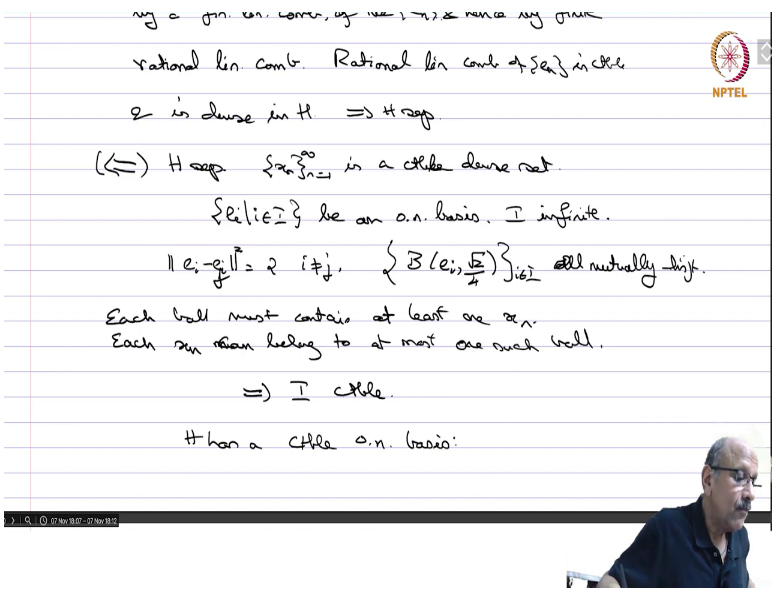


So each ball must contain at least one $x_n$ and each $x_n$ can belong to at most one of such balls. Because the balls are all disjoint. You cannot have a common element in 2 of them. So since $x_n$'s are countable so this means so every ball must contain the only countable number of these. And therefore, $I$ is countable. So countably infinite and therefore $H$ has a countable orthonormal basis.

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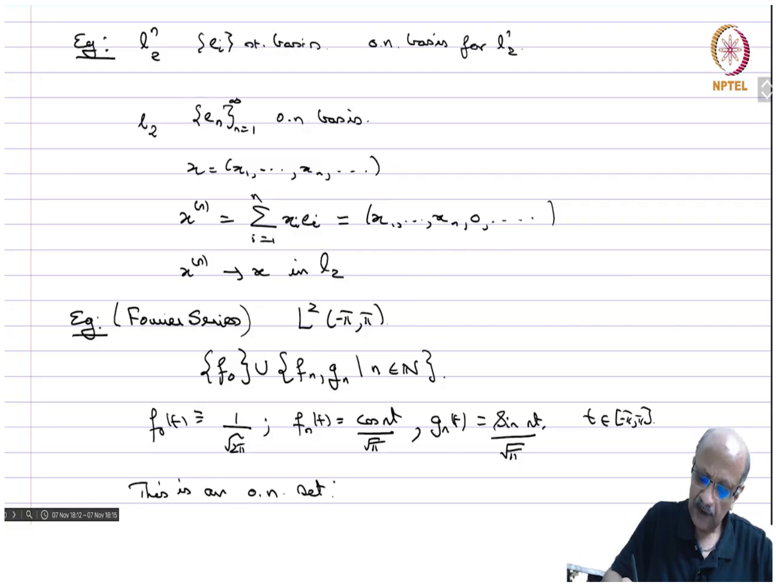


**Example 1.** If you have $l_2^n$ then you take $\{e_i\}$ to be the standard basis. This means 1 in the i-th place and 0 elsewhere. So this is an orthonormal basis for $l_2^n$. Because $e_i$'s are all of norm 1 , $e_i$ and $e_j$ are orthogonal to each other and any $x$ can be written uniquely in terms of the $e_i$ and therefore this forms a basis. And it is therefore a maximal linearly independent set. Therefore, its maximum is not the normal set as well.

$l_2$, $\{e_n\}_{n=1}^{\infty}$ this is an orthonormal basis of $l_2$. Because if you have $x = (x_1, \ldots, x_n, \ldots)$, then

$x^{(n)} = \sum_{i=1}^{n} x_i e_i = (x_1, \ldots, x_n, 0, 0, \ldots)$. And $x^{(n)}$ converges to $x$ in $l_2$. And therefore, all linear combinations are dense. By an earlier corollary we have that $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal set. And therefore it is an orthonormal basis.

**Example 2.** Fourier series. Let us consider $L^2(-\pi, \pi)$. Consider the set $E = \{f_0\} \cup \{f_n, g_n / n \in \mathbb{N}\}$. So what is $f_0(t) = \frac{1}{\sqrt{2\pi}}$, $f_n(t) = \frac{\cos nt}{\sqrt{\pi}}$ and $g_n(t) = \frac{\sin nt}{\sqrt{\pi}}$, $t \in [-\pi, \pi]$. So this is an orthonormal set. It is clear to see, you can very easily check that all these things have norm 1 and that they are all orthogonal to each other. So this is an orthonormal set.

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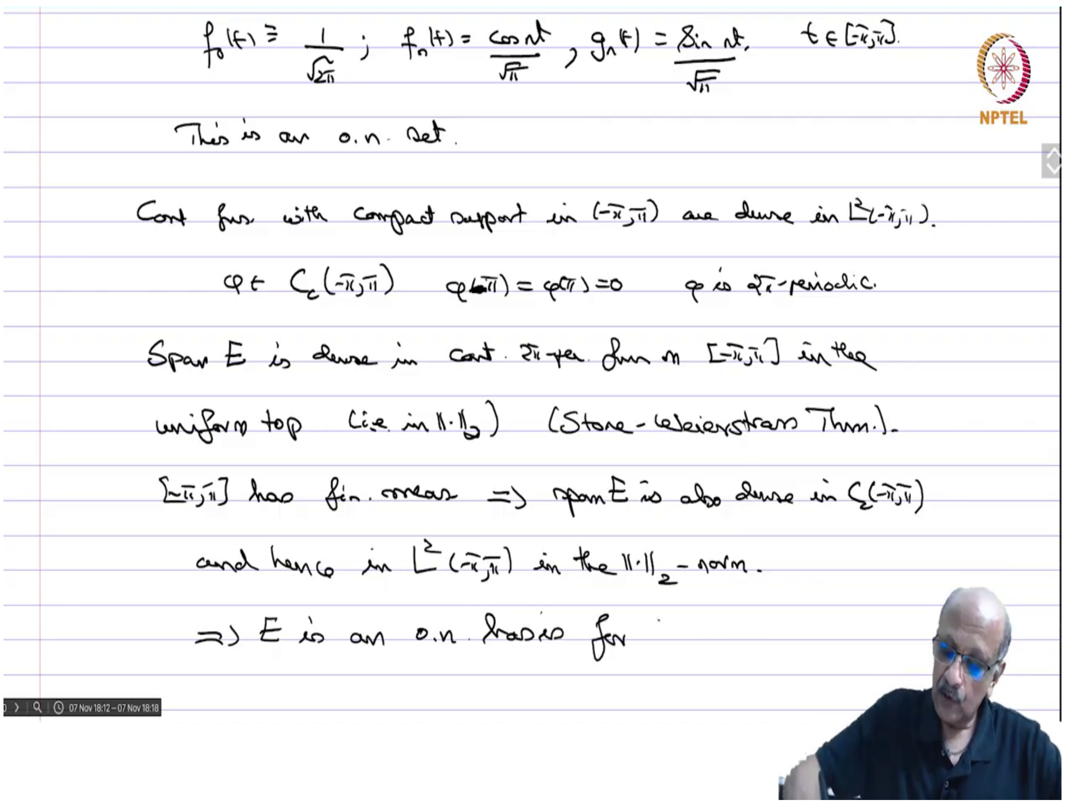

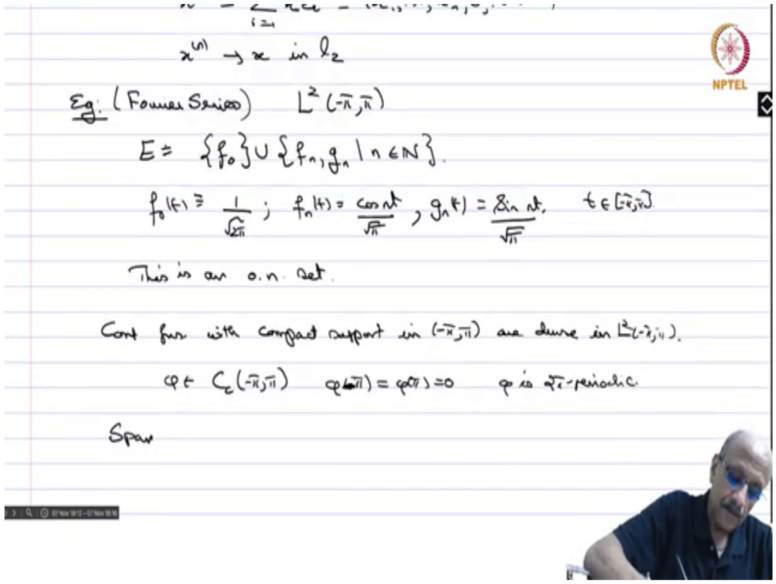


Now continuous functions with compact support in $(-\pi, \pi)$ are dense in $L^2(-\pi, \pi)$. Because they have compact support, you have such functions $\phi \in C_c(-\pi, \pi)$, then $\phi(-\pi) = \phi(\pi) = 0$ so $\phi$ is automatically $2\pi$ periodic. Then span $E$ is dense in continuous $2\pi$ periodic functions on $[-\pi, \pi]$ and in the uniform topology (i.e., $||.||_\infty$). That means every continuous $2\pi$ periodic function can be uniformly approximated by means of a function in the span of $E$. This is nothing but the Stone Weierstrass theorem (this a theorem in analysis). And since we have $[-\pi, \pi]$ has finite measure. So this implies that span of E is also dense in $C_c(-\pi, \pi)$ and hence in $L^2(-\pi, \pi)$ in the $||.||_2$ -norm. Because if it is in the $l^\infty$ norms it is a finite measure $l^\infty$ is continuously embedded in the $l^2$. And therefore it is true in this, therefore the span is dense. So this implies $E$ is an orthonormal basis for $L^2(-\pi, \pi)$. So we have proved the completeness. So this is an example of an orthonormal basis in $l_2$.

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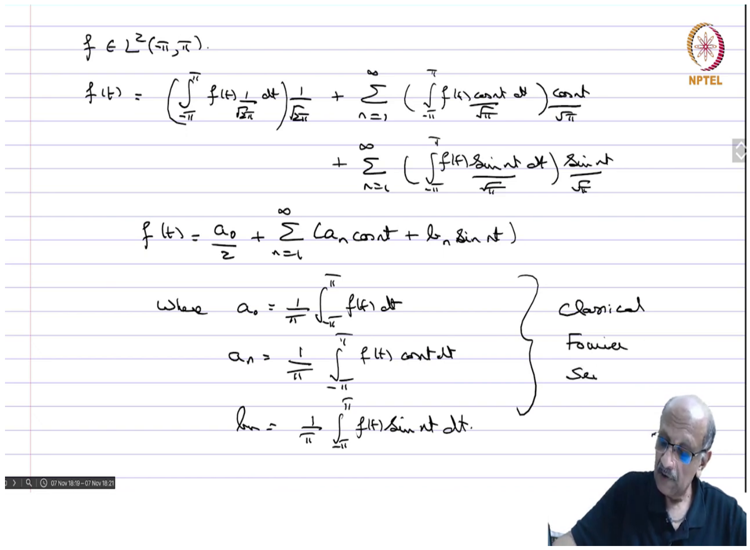


So that means what do you have, if $f \in L^2(-\pi, \pi)$. You can write

$$f(t) = \left(\int_{-\pi}^{\pi} f(t)\frac{1}{\sqrt{2\pi}}dt\right)\frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(t)\frac{\cos nt}{\sqrt{\pi}}dt\right)\frac{\cos nt}{\sqrt{\pi}} + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(t)\frac{\sin nt}{\sqrt{\pi}}dt\right)\frac{\sin nt}{\sqrt{\pi}}$$

So, we can rewrite this as $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$,

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, dt$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt\, dt$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin nt\, dt$.

This is the classical Fourier series, you have already seen in the connection with the uniform boundedness principle.

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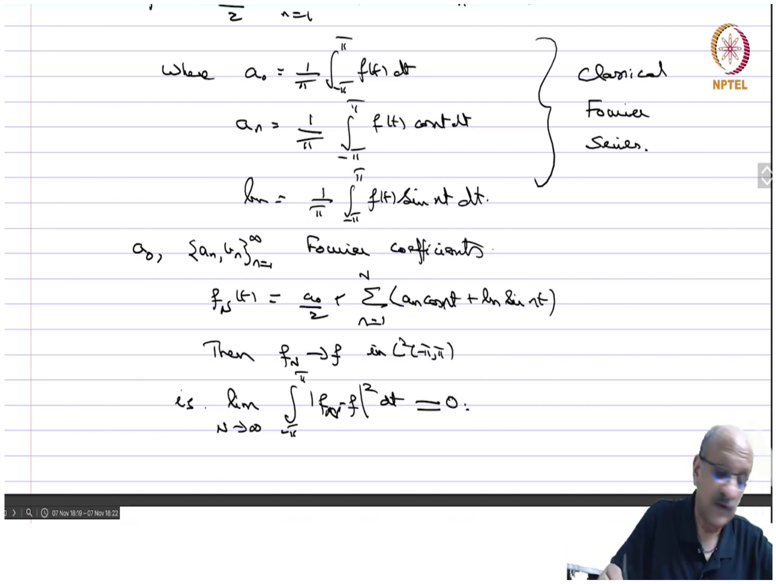


So we call $a_0$, $\{a_n, \ b_n\}_{n=1}^{\infty}$ as the Fourier coefficients. What does it mean if you have

$f_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nt \ + \ b_n \sin nt)$. Then $f_N \to f$ in $L^2(-\pi, \pi)$.

i.e., $\lim_{N \to \infty} \int_{-\pi}^{\pi} |f_N - f|^2 dt \ = \ 0$. So in this sense it converges. So, we have seen the uniform bounded principle that point wise convergence even for continuous functions is not guaranteed. Whereas in the $l_2$ norm the Fourier series always converges to the corresponding function $L^2(-\pi, \pi)$.

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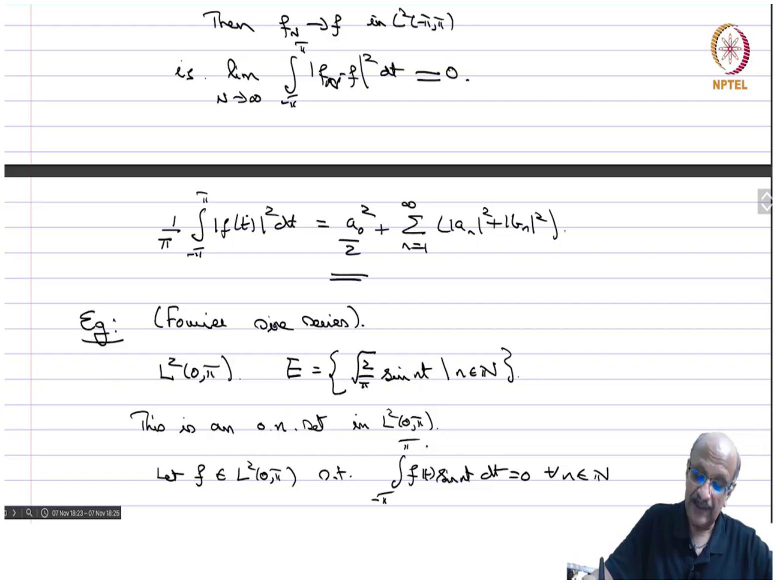


And if you take Parseval's identity $\frac{1}{\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$. So that is given by the sum of the squares of the inner product with the Fourier coefficients. So that is Parseval's identity. So this is how it will work out for the Fourier series. So you have in fact $a_n$ $b_n$ should all go to 0 because of this condition.

**Example** ( Fourier sine series). So let us take now $L^2(0, \pi)$, so this is our Hilbert space.

Then you look at the set $E = \left\{ \sqrt{\frac{2}{\pi}} \sin nt \ / \ n \in \mathbb{N} \right\}$. It is easy to check this is an orthonormal set in $L^2(0, \pi)$. So each one has norm 1 and they are all orthogonal to each other. So now we want to show that this is in fact complete. Let $f \in L^2(0, \pi)$ such that $\int_{-\pi}^{\pi} f(t) \sin nt \, dt = 0, \ \forall n \in \mathbb{N}$. That means the inner product with functions in $E$ are all 0. We will then show that $f$ is identically 0. And therefore, by one of the characterizations we know that this is a complete orthonormal set.

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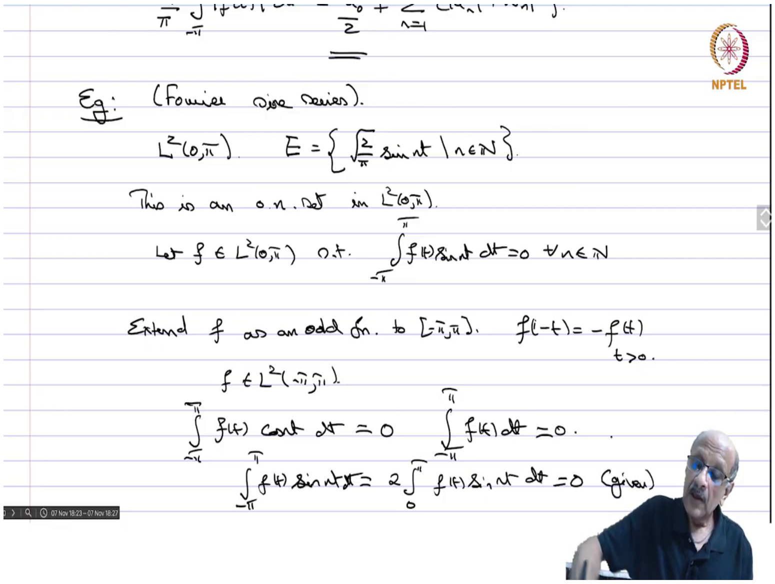


Now extend $f$ as an odd function to $[-\pi, \pi]$, i.e., $f(-t) = -f(t),\ t > 0$. So in the negative range you define it is $-f(t)$. So then this is an odd function. Of course this new function $f \in L^2(-\pi, \pi)$. Now what about $\int_{-\pi}^{\pi} f(t) \cos nt \, dt$. $f(t) \cos nt$ is an odd function as cos is an even function, the product is an odd function and an odd function on the symmetric interval will have integral 0 and so $\int_{-\pi}^{\pi} f(t) \cos nt \, dt = 0$. And then $\int_{-\pi}^{\pi} f(t) \, dt = 0$ as $f$ is an odd function. Now what about $\int_{-\pi}^{\pi} f(t) \sin nt \, dt$. Now because both of sin and $f(t)$ are odd so the product is an even function. So, $\int_{-\pi}^{\pi} f(t) \sin nt \, dt = 2\int_{0}^{\pi} f(t) \sin nt \, dt = 0$

given.

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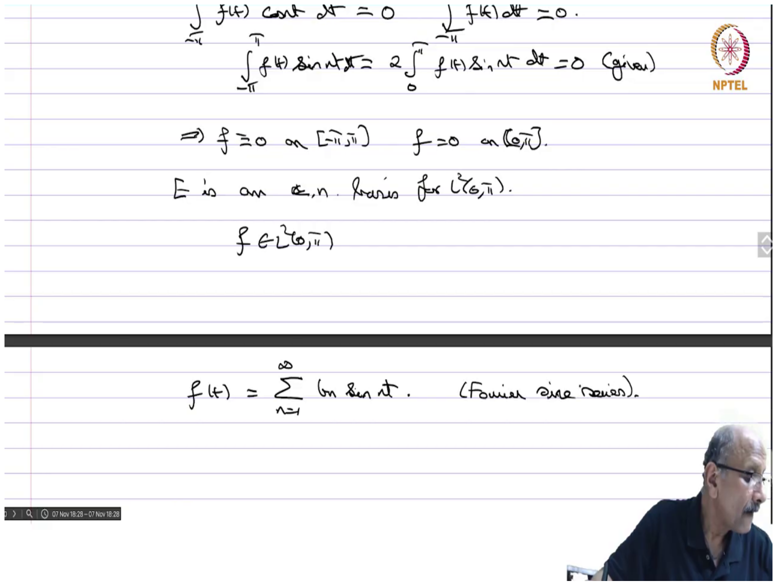


So the inner product of $f$ with all the sin $nt$, cos $nt$ and the constant function is 0. So since we know that these functions form a complete orthonormal set on $[-\pi, \pi]$. So this implies that $f = 0$ on $[-\pi, \pi]$. Therefore, $f = 0$ on $[0, \pi]$ also. And therefore $E$ is an complete orthonormal basis for $L^2(0, \pi)$. That means any $f \in L^2(0, \pi)$ then you can write $f(t) = \sum_{n=1}^{\infty} b_n \sin nt$. And this will converge in the sense of $l_2$ and this is called the Fourier sine series of any function in space.

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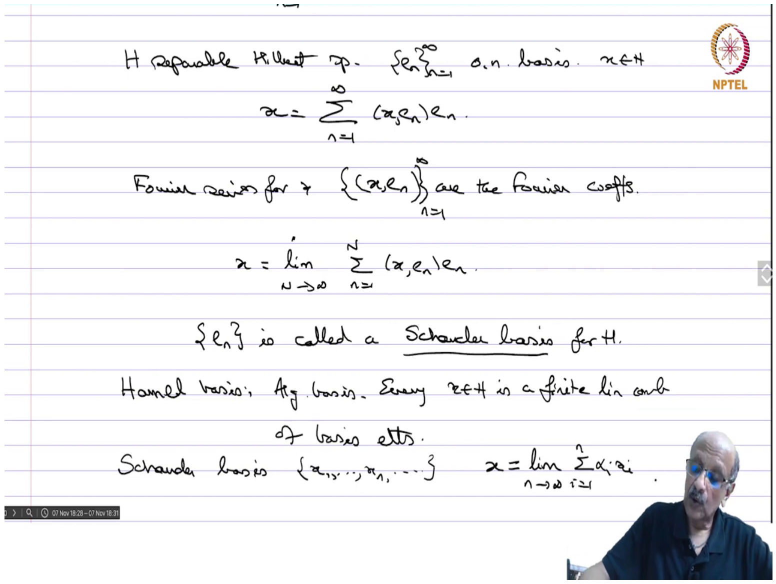


$H$ is separable Hilbert space and $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis or complete orthonormal set. Then we know any $x \in H$ can be written as $x = \sum_{n=1}^{\infty} (x, e_n) e_n$. This is called the Fourier series for $x$ and you call $(x, e_n)$ are the Fourier coefficients. Now since we can write like this so $x$ is in fact $x = \lim_{N \to \infty} \sum_{n=1}^{N} (x, e_n) e_n$. The set $\{e_n\}_{n=1}^{\infty}$ is called a Schauder basis. So, we have two kinds of basis, we have the Hamel basis. This is a algebraic basis that means every $x \in H$ is a finite linear combination of basis elements. So this is the basis as we know it. A Schauder basis is a sequence $\{x_2, x_2, \ldots, x_n, \ldots\}$. a countable set and given any $x$ can be written as $x = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i x_i$. If you can write it like this then we say it is Schauder basis.

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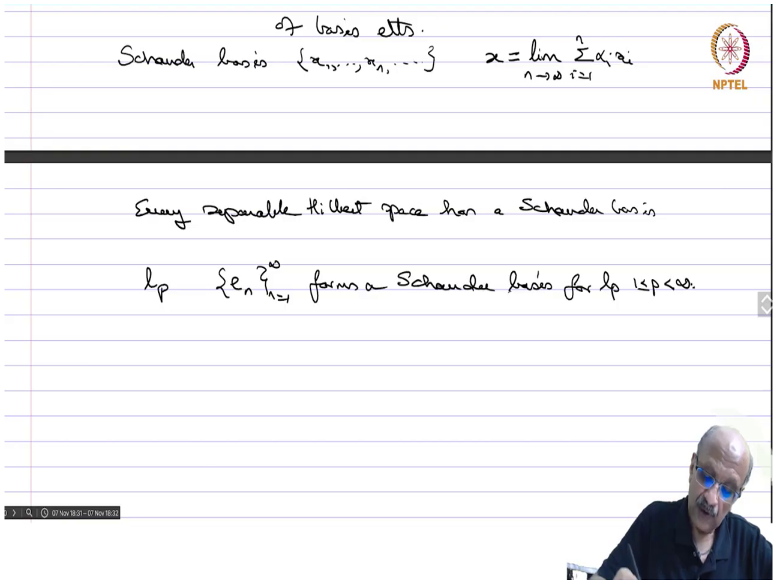


So, this definition holds in any Banach space and so every separable Hilbert space. So every separable Hilbert space has a Schauder basis in the orthonormal basis. Now if you look at $l_p$. So again if you took a look at the sequences $\{e_n\}_{n=1}^{\infty}$. These are the sequences with 1 in n-th place and 0 in all other corners. So this forms a Schauder basis for $l_p$, $1 \le p < \infty$ it is not true in $l_\infty$ . So it also forms a Schauder basis in $C_0$ , for instance, and we have used these properties before.