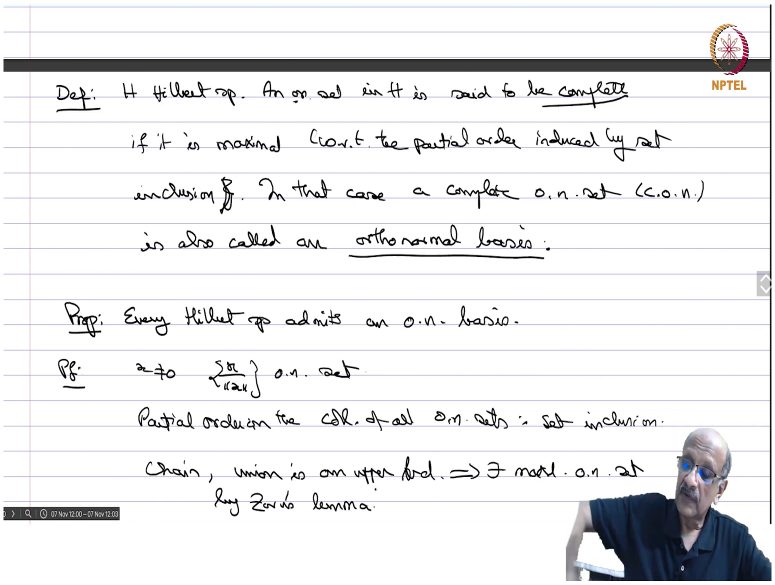
Functional Analysis
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No. 56
Orthonormal Bases - Part 2

(Refer Slide Time: 00:16)



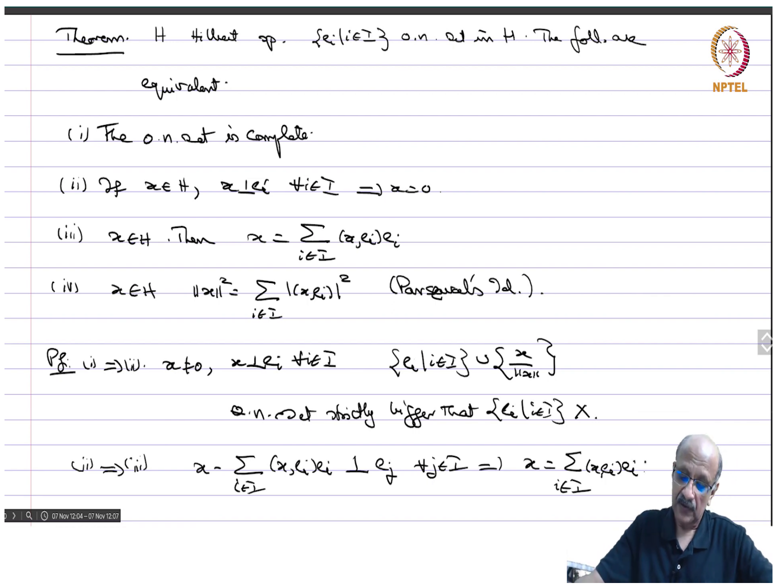
Definition H Hilbert space. An orthonormal set in H is said to be complete if it is maximal with respect to the partial order induced by set inclusion. That means there is no other orthonormal set which contains this one and is strictly bigger than this. So, that is the meaning of the set saying it is maximal. In that case, a complete orthonormal set, sometimes we write c. o. n., is also called an orthonormal basis.

Proposition. Every Hilbert space admits an orthonormal basis.

Proof. So, we do have orthonormal sets in fact if, you have any $x \neq 0$, then you have $\{\frac{x}{||x||}\}$ is an orthonormal set. If you are given any linearly independent thing, the Gram Schmidt process gives you an orthonormal set. So, we have a lot of orthonormal sets and. The collection of all orthonormal sets is non-empty and you put a partial order on the collection of all orthonormal sets, namely set inclusion.

So, $e_1 \le e_2$ if $e_1 \subseteq e_2$. So, now you take any chain then union is an upper bound. So, you take a chain that means any two elements, simply take the union that is an orthonormal set and therefore, that will give you an upper bound implies there exists a maximal orthonormal set by Zorn's lemma. So, it is a very simple application of Zorn's lemma that every Hilbert space has a maximal orthonormal set or an orthonormal basis.

(Refer Slide Time: 03:49)



So, now we have a very important theorem

Theorem. H Hilbert space and $\{e_i / i \in I\}$ is an orthonormal set in H, then the following are equivalent.

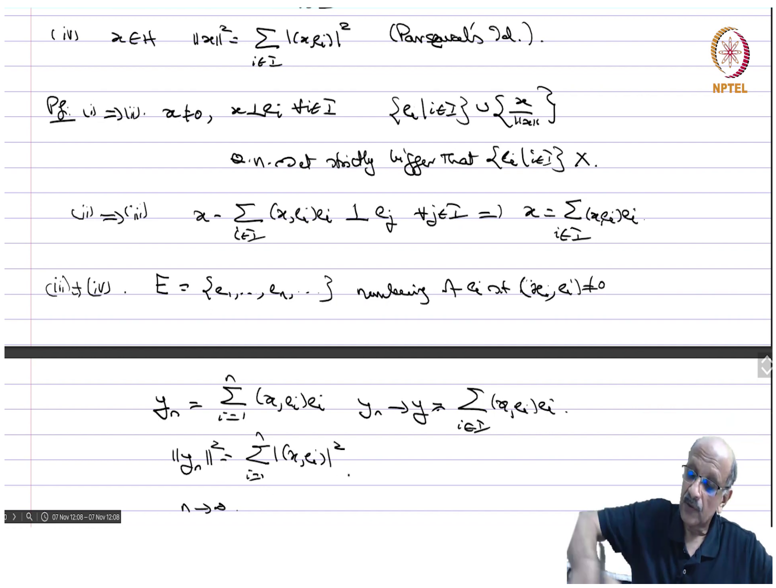
1. The orthonormal set is complete,
2. $x \in H,\ x$ orthogonal to $e_i\ \forall\, i \in I \ \Rightarrow\ x = 0$.
3. $x \in H, x = \sum_{i \in I} (x,\ e_i) e_i$, we know how to define this sum and if, you take the sum it has to be x. So, then is equivalent to saying that the space is complete.
4. $x \in H$ then norm $||x||^2 = \sum_{i \in I} |(x,\ e_i)|^2$. This is called Parseval's identity.

Proof. $1 \Rightarrow 2$. If $x \neq 0$ but x orthogonal to e_i $\forall\, i \in I$ then you take $\{e_i / i \in I\} \cup \{\frac{x}{||x||}\}$. So, $\frac{x}{||x||}$ is again a unit vector and it is orthogonal to all of the other elements. So, this is an orthonormal set strictly bigger than $\{e_i / i \in I\}$ and that is a contradiction. Because, we know that $\{e_i / i \in I\}$ is complete and therefore this is maximal. So, you cannot have something bigger.

$2 \Rightarrow 3$. We already saw that $x - \sum_{i \in I} (x, e_i) e_i$ is orthogonal to e_j for all $\forall\, j \in I$ and (1) implies

$x - \sum_{i \in I} (x, e_i) e_i = 0 \Rightarrow x = \sum_{i \in I} (x, e_i) e_i$.

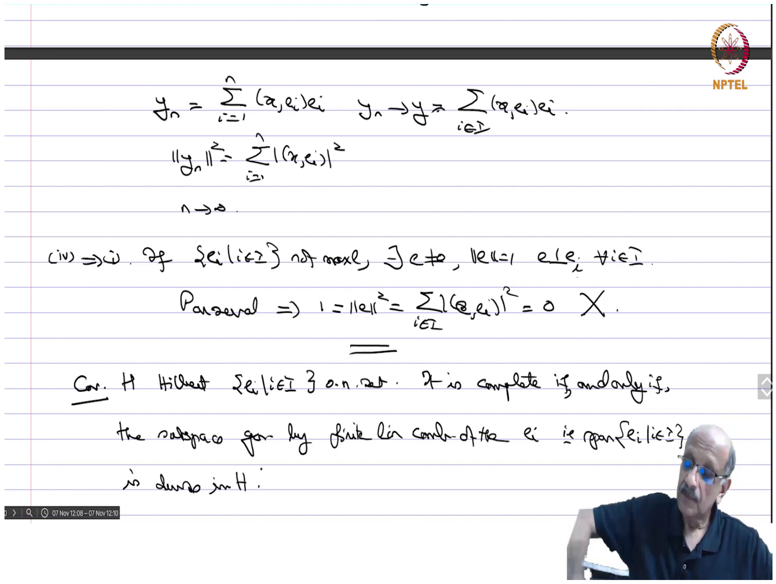
(Refer Slide Time: 07:10)



$3 \Rightarrow 4$. Set E to be numbering. So, $E = \{e_1, \ldots, e_n, \ldots\}$ is a numbering of e_i such that $(x, e_i) \neq 0$, and you take $y_n = \sum_{i=1}^{n} (x, e_i) e_i$, then $y_n \rightarrow y = \sum_{i \in I} (x, e_i) e_i$ in H.

So, $||y_n||^2 = \sum_{i=1}^{n} |(x, e_i)|^2$. So, now let $n \rightarrow \infty$ and we get (4). So, this proves Parseval's identity.

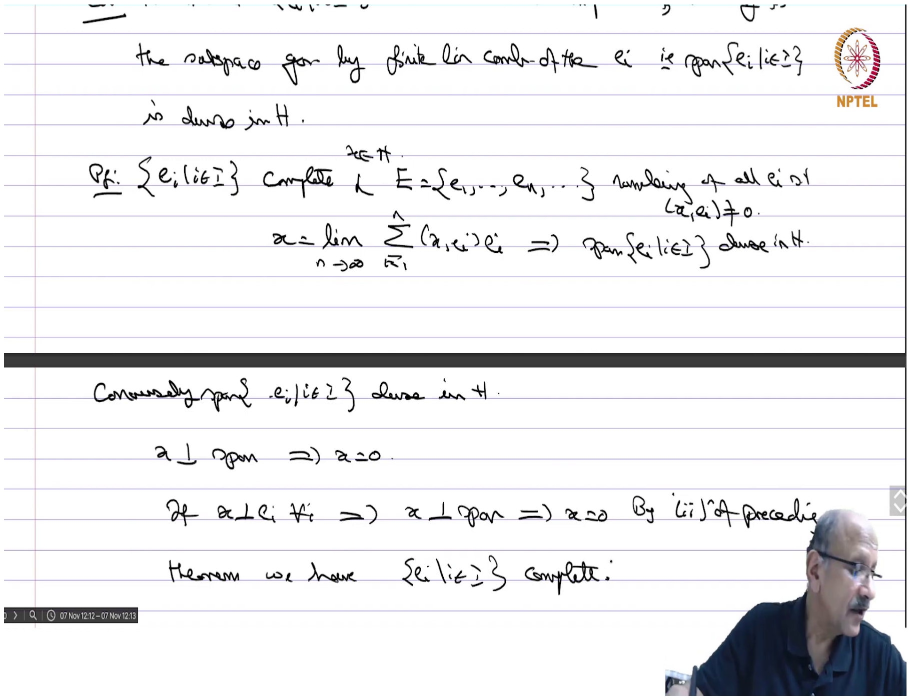
(Refer Slide Time: 08:24)



$4 \Rightarrow 1$. If $\{e_i / \, i \in I\}$ not maximal, then $\exists \, e \neq 0, \; ||e|| = 1$ and $e \perp e_i, \; \forall i \in I$. So, that is the contradiction of the maximality but, then by Parseval's implies $1 = ||e||^2 = \sum_{i \in I} |(e, \, e_i)|^2 = 0$ and that is a contradiction. So, this proves the theorem completely.

Corollary. H Hilbert and $\{e_i / \, i \in I\}$ is an orthonormal set then, it is complete if and only if the subspace generated by finite linear combinations of the e_i that is span of the $\{e_i / \, i \in I\}$ is dense in H.

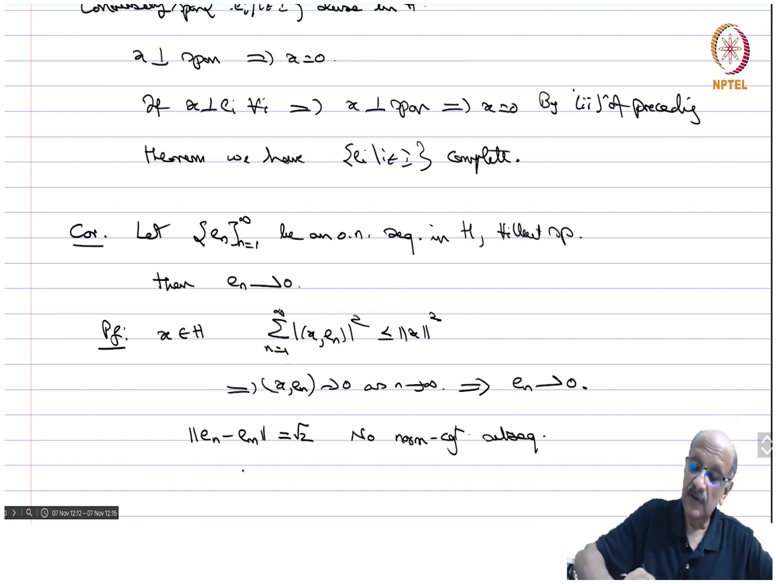
(Refer Slide Time: 10:35)



Proof. If $\{e_i / i \in I\}$ is complete then clearly you take a numbering. So, let us say $E = \{e_1, e_2, ... e_n, ..\}$ is a numbering of all e_i such that, $(x, e_i) \neq 0$. Let $x \in H$ and then we know that $x = \lim_{n \to \infty} \sum_{i=1}^{n} (x, e_i) e_i$ and therefore implies span $\{e_i / i \in I\}$ e i is dense in H every element can be approximated from a finite linear combination of these.

Conversely Span of $\{e_i / i \in I\}$ dense H. So, if $x \perp span \Rightarrow x = 0$. Now if, $x \perp e_i \forall i$ then automatically x is orthogonal to Span of $\{e_i / i \in I\}$ and therefore $x = 0$ and therefore, by (2) of preceding theorem we have $\{e_i / i \in I\}$ is complete and therefore this is true.

(Refer Slide Time: 13:39)



Corollary, Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in H which is a Hilbert space then e_n weekly converges to 0.

Proof. If, $x \in H$, then we have $\sum_{n=1}^{\infty} |(x, e_n)|^2 \leq ||x||^2$, this is the Bessel's inequality, therefore this is a convergence series this implies that $(x, e_n) \to 0 \; as \; n \to \infty$. Now, every continuous linear functional by this representation theorem comes as an inner product with respect to x and therefore this simply implies that e_n goes weekly to 0. Because, you have $||e_n - e_m|| = \sqrt{2}$ if you have an orthonormal thing. So, non-convergent subsequence and but this sequence converges weakly to 0 and therefore the weak and norm convergences are strictly different in the case of infinite dimensional Hilbert space. So, we will continue with this.