Functional Analysis
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Lecture No. 55
Orthonormal Bases - Part 1

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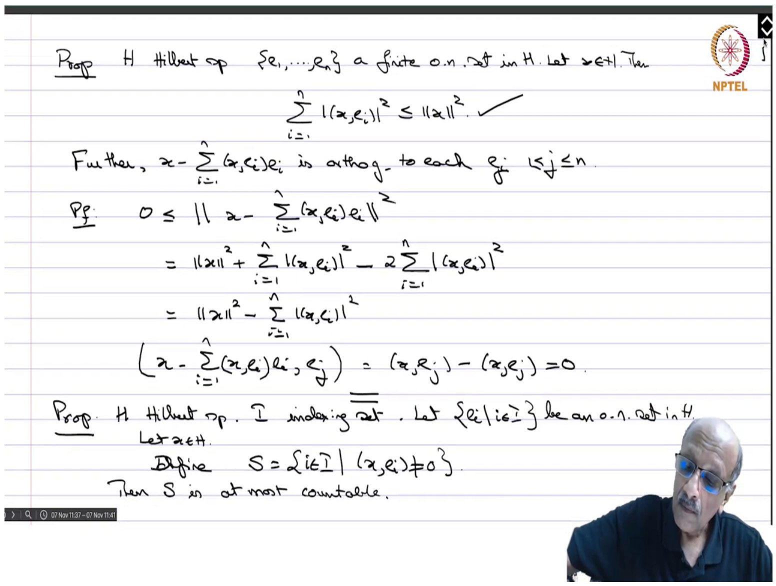

We now investigate properties of orthonormal sets. So, we start with the following proposition.

Proposition. Let H be a Hilbert space and $\{e_1, e_2, \cdots, e_n\}$ be a finite orthonormal set in H. Let $x \in H$ then $\sum_{i=1}^{n} |(x, e_i)|^2 \leq ||x||^2$. Further, $x - \sum_{i=1}^{n} (x, e_i)e_i$ is orthogonal to each e_j, $1 \leq j \leq n$

Proof. Proof is fairly straightforward. So, you have that $0 \leq ||x - \sum_{i=1}^{n} (x, e_i)e_i||^2$. So

$$||x - \sum_{i=1}^{n} (x, e_i)e_i||^2 = ||x||^2 + \sum_{i=1}^{n} |(x, e_i)|^2 - 2\sum_{i=1}^{n} |(x, e_i)|^2 = ||x||^2 - \sum_{i=1}^{n} |(x, e_i)|^2.$$

So, other cross terms will involve (e_i, e_j) but, that will be 0. Because if $i \neq j$ $(e_i, e_j) = 0$ and consequently they all do not appear. Now if, you take

$(x - \sum_{i=1}^{n}(x, e_i)e_i, e_j) = (x, e_j) - (x, e_j) = 0$ and that proves you the second part of the proposition namely that every, $x - \sum_{i=1}^{n}(x, e_i)e_i$ is orthogonal to every e_j.

So, next is an important proposition.

Proposition Let H be a Hilbert space and I indexing set and let $\{e_i / i \in I\}$ be an orthonormal set in H. Let $x \in H$. Define $S = \{i \in I / (x, e_i) \neq 0\}$. Then S is at most countable.

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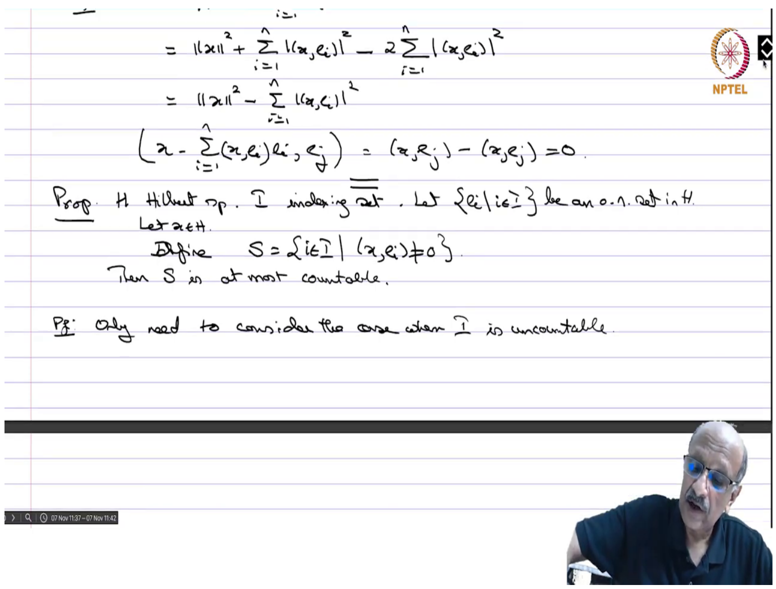

Proof. Only need to consider the case when I is uncountable. If I is finite, empty finite or countable there is nothing to do. S will automatically be countable at most and therefore we need not worry about it. So, there is only one case.

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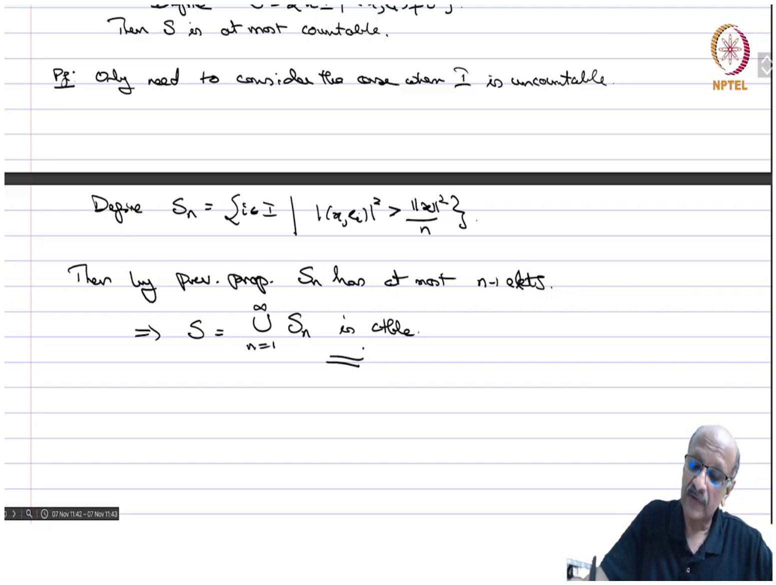

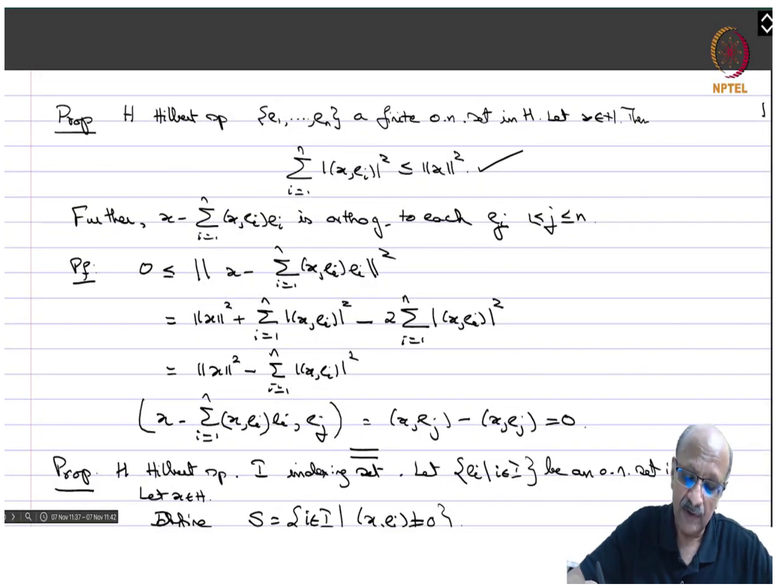

Now, define $S_n = \{i \in I \,/\, |(x, e_i)|^2 > \frac{||x||^2}{n}\}$, then by previous proposition you have that $\sum_{i=1}^{n} |(x, e_i)|^2 \leq ||x||^2$. So, if each one of them is bigger than $\frac{1}{n}$; how many can there be? So, then by previous proposition S_n has at most $n-1$ element this implies $S = \bigcup_{n=1}^{\infty} S_n$ is countable. So, that proves that thing.

So, now this is very important though it is a very simple proposition it is a very important proposition. Because, it tells you how to define.

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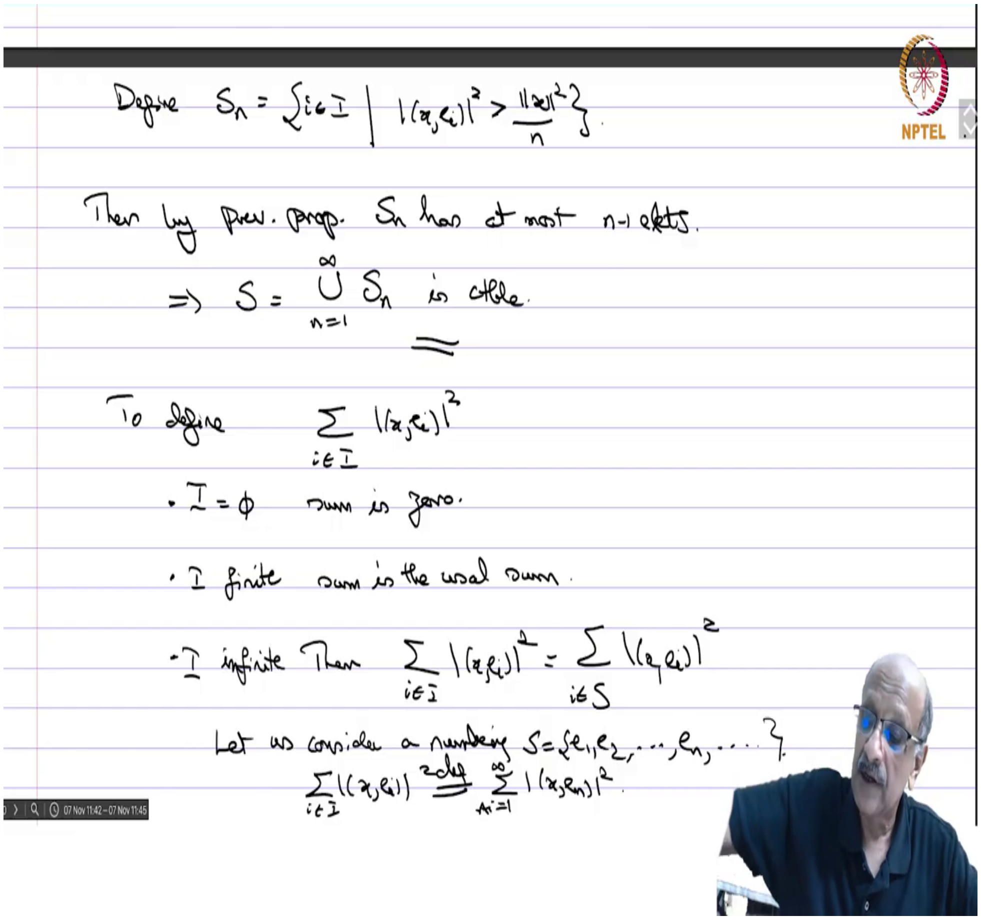

So, to define $\sum_{i \in I} |(x, e_i)|^2$.

- $I = \phi$, sum is 0.
- I is finite, then sum is the usual sum. Because, the finite number. So, you do not have to know.
- I is infinite then $\sum_{i \in I} |(x, e_i)|^2 = \sum_{i \in S} |(x, e_i)|^2$.

S is a countable set. So, let us consider a numbering $S = \{e_1, e_2, \cdots, e_n, \ldots\}$ and then we define $\sum_{i \in I} |(x, e_i)|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$ and that is an infinite series which you know how to define? Is the limit of the partial sums.

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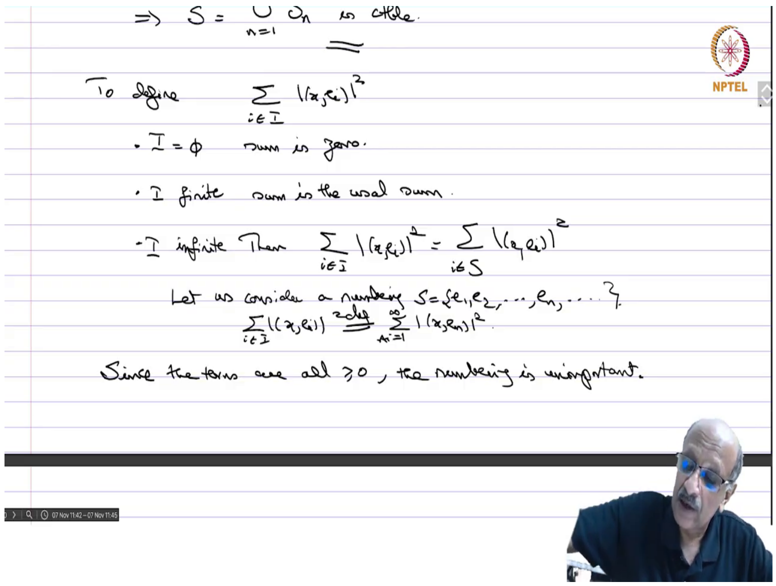

And then, since the terms are all greater than equal to 0, the numbering is unimportant, because if you take a series of non-negative terms then any rearrangement will give you the same sum if, it is one of them is divergent all of them will diverge, if one of them converges they will all converge to the same sum this is a famous theorem infinite series and therefore, when you have a non-negative series then it does not matter how you can number these. You can number them in any way, and it does not matter and that is very important also. Because, we should not worry about how we have to do it?

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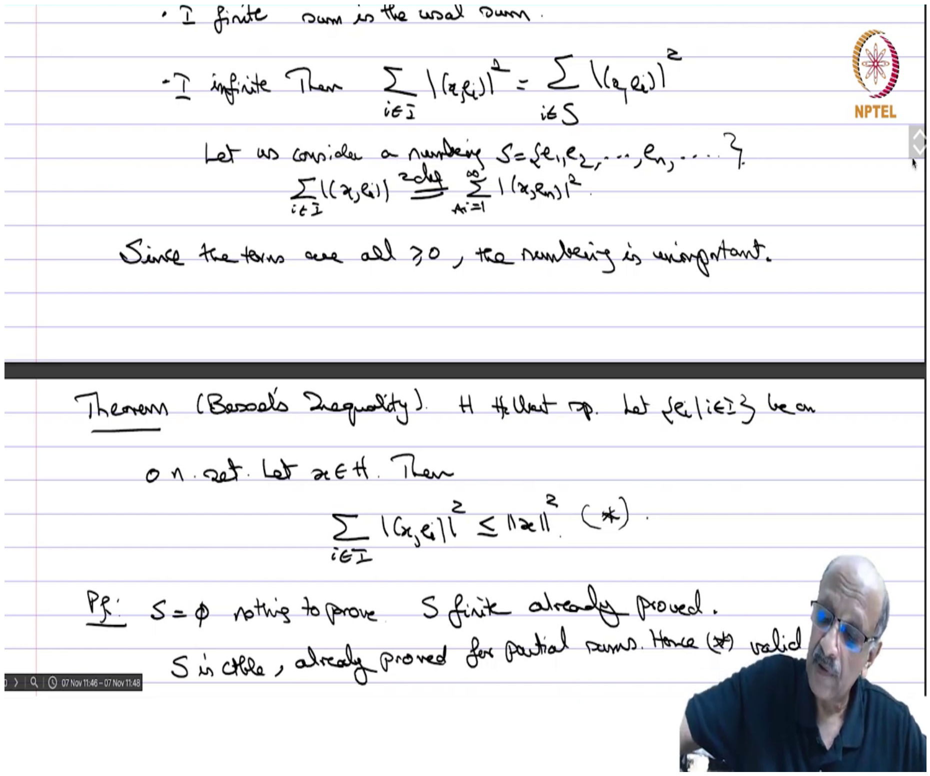

So now, we can generalize the proposition which we proved earlier.

Theorem (Bessel's inequality). H Hilbert space. Let $\{e_i \,/\, i \in I\}$ be an orthonormal set. Let $x \in H$ then the Bessel's inequality says $\sum_{i \in I} |(x, e_i)|^2 \leq ||x||^2$ -------- (*).

Proof. If $S = \phi$ nothing to prove, this sum is 0. If, S is finite already proved. If S is countable, that is the only possibility. Because, that is how you define the infinite sum? And then, already proved for partial sums. Because, that is the finite case which we have done hence (*) is true. Therefore we have proved Bessel's inequality completely.

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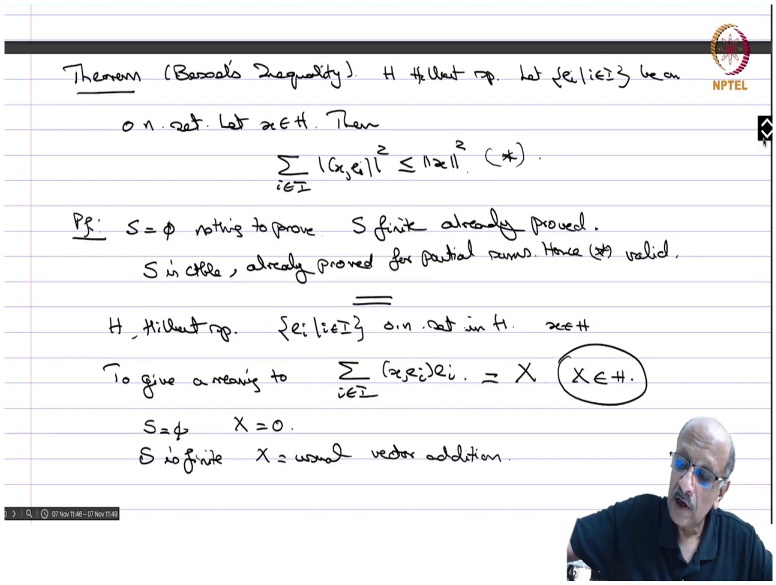

H is Hilbert space and then $\{e_i / i \in I\}$ is orthonormal set in H. So, we want to give an $x \in H$.

To give a meaning to $\sum_{i \in I} (x, e_i)e_i = X, \; X \in H$. We want to define it as a vector in H. So, this infinite series should be effective. So, if S is empty then X=0. If S is finite then capital $X =$ usual vector addition.

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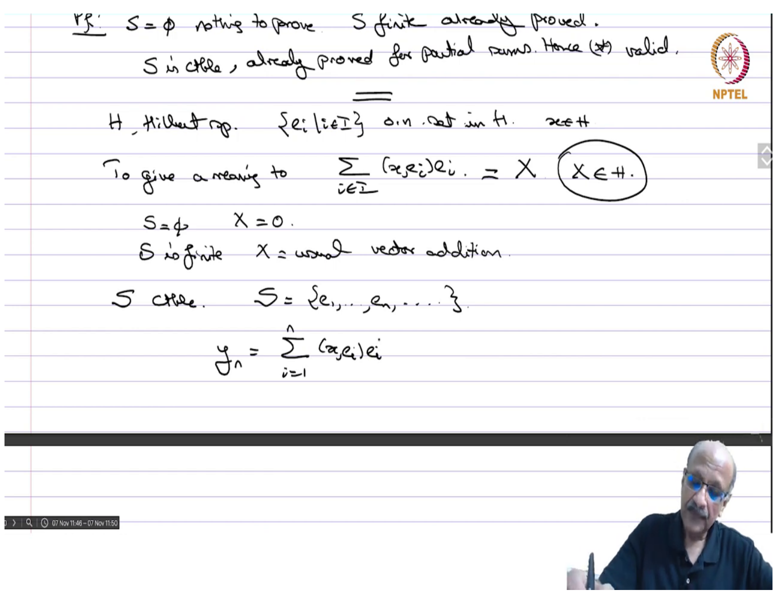

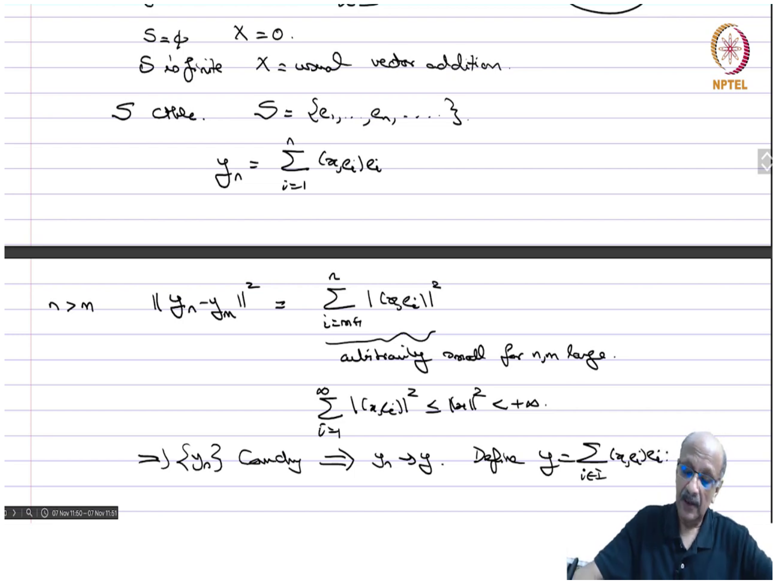

So, the third possibility is when S is countable. Let us write $E = \{e_1, e_2,\ldots\ldots, e_n,\ldots\}$ and then let us take $y_n = \sum_{i=1}^{n} (x, e_i)e_i$, which is well defined. Now let us look at $||y_n - y_m||^2$, $n > m$. $||y_n - y_m||^2 = \sum_{i=m+1}^{n} |(x, e_i)|^2$, because all other things get cancelled. Now, this can be made arbitrarily small for n, m large, why? Because, $\sum_{i \in I} |(x, e_i)|^2 \leq ||x||^2 < \infty$ by Bessel's inequality. $\sum_{i=m+1}^{n} |(x, e_i)|^2$ is part of the tail of a convergence series and therefore the Cauchy criterion tells you can make this arbitrarily small. So, this implies that y_n is Cauchy and H is complete and therefore $y_n \to y = \sum_{i \in I} (x, e_i)e_i$. So, we have done this. But then, we have to be careful because now we have to really check if you numbered the vectors in a different way then you may get a different vector. So that we should not that will be an unpleasant thing to happen and therefore we must avoid that.

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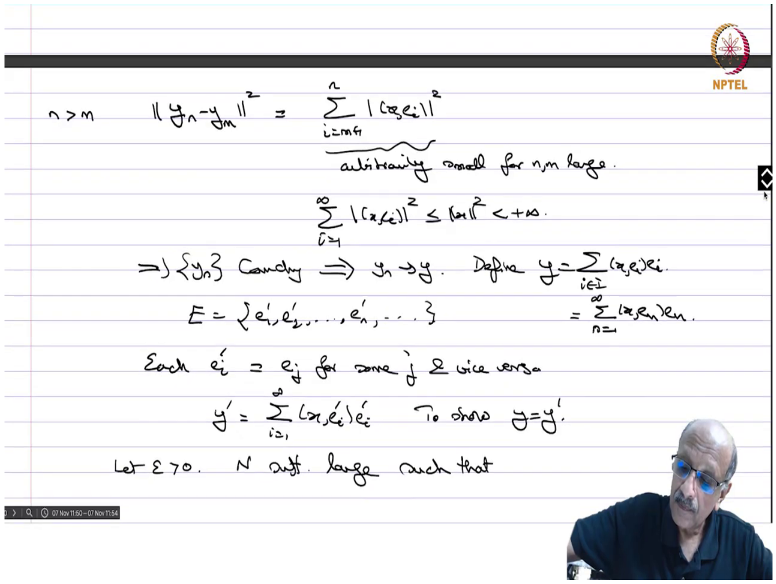

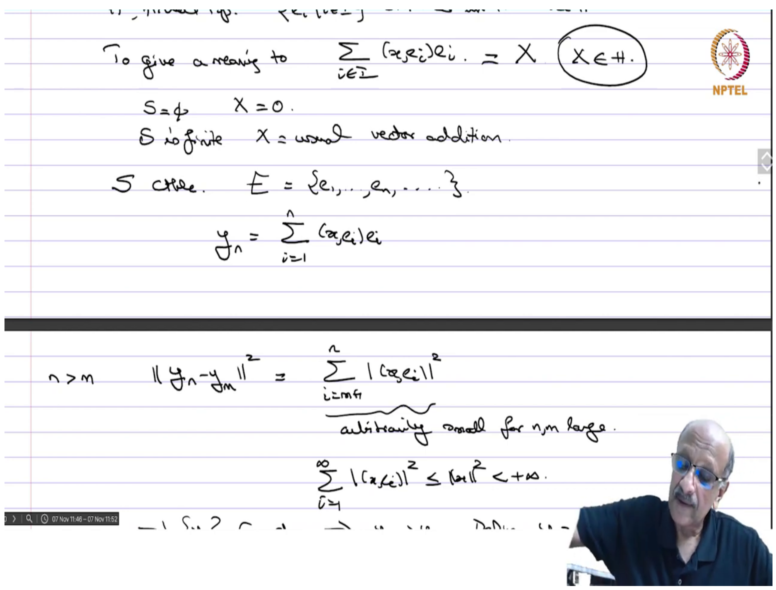

Let us take E to be the set of the E a, $S = \{i \in I / (x, e_i) = 0\}$. So, S is a subset of the indexing set. So, I want this here. So, now let us say we have renumbered E. So, let us say E is now written as some $\{e'_1, e'_2, \cdots, e'_n\}$. So, each $e'_i = e_j$ for some j and vice versa. So, it is the same set which we have renumbered and therefore we call them $\{e'_1, e'_2, \cdots, e'_n\}$ because it is written in a different order than the name set as the original set.

So, now again if you write $y' = \sum_{i=1}^{\infty} (x, e'_i)e'_i$. To show $y = \sum_{i=1}^{\infty} (x, e_n)e_n$ is the same as y'. So, once again we are clear that we can number the vectors in a countable set of vectors in any way we like and the answer will not be the same. So, we have to show that. So, let ϵ be positive and let N be sufficiently large such that the following happen.

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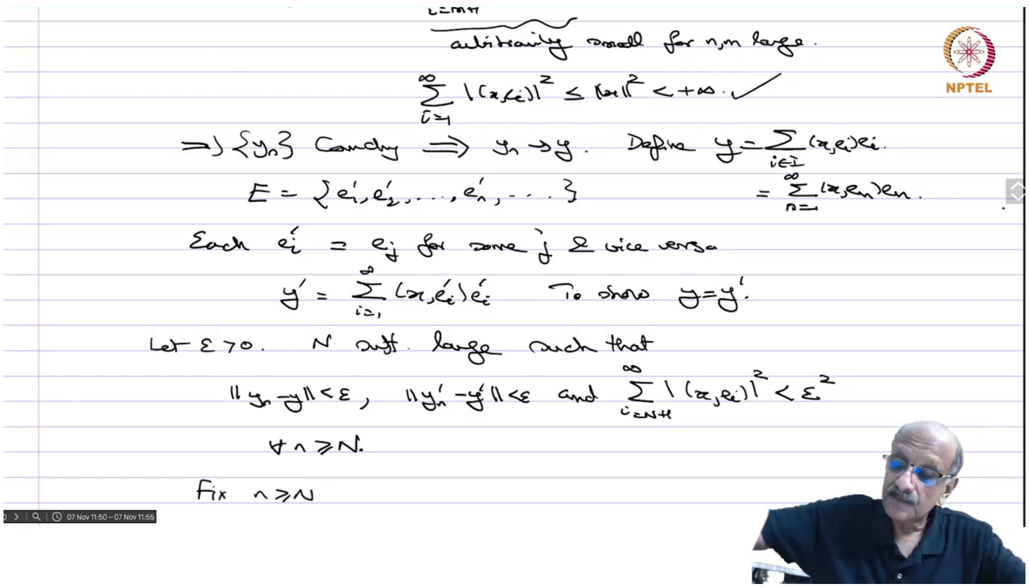

$||y_n - y|| < \epsilon$, $||y'_n - y'|| < \epsilon$ and $\sum_{i=N+1}^{\infty} |(x, e_i)|^2 < \epsilon^2 \ \forall n \geq N$. Now, this is possible y_n convergence to y and therefore for n sufficiently these are y'_n dash converges to y' again for n sufficiently that. So, you take n the bigger of the two and then $\sum |x_n|^2$ is a convergent series as you know $\sum_{i=N+1}^{\infty} |(x, e_i)|^2 < \epsilon^2$. Therefore you could always find an N such that this the tail of the convergence series, i.e., $\sum_{i=N+1}^{\infty} |(x, e_i)|^2$ can be made as small as you can. So, we can find such an N. So, now fix $n \geq N$.

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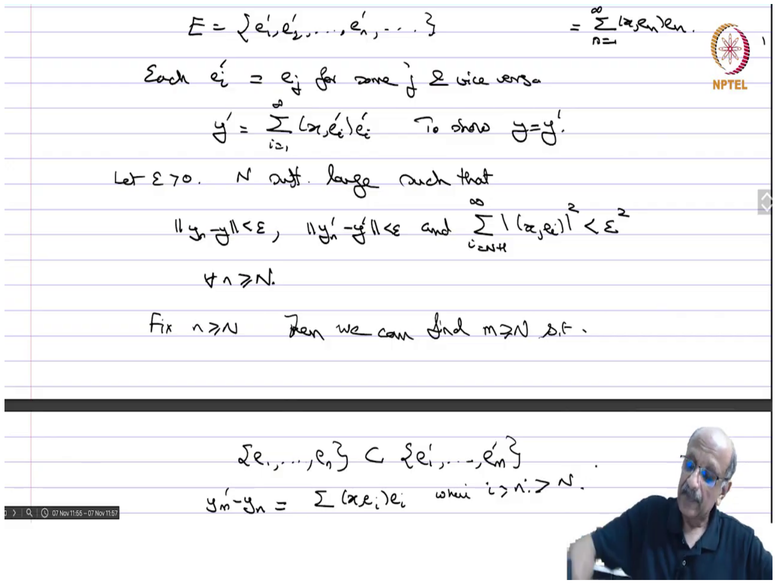

Then we can find $m \geq N$ such that the set $\{e_1, e_2, \cdots, e_n\}$ is covered by $\{e'_1, e'_2, \cdots, e'_m\}$. So, the n vectors here I am finding a bigger set because this is a finite set and this is nothing but a renumbering of the same set and therefore if I go far enough I should be able to cover all of them. Now, you look at $y'_m - y_n = \sum (x, e_i)e_i$ where $i > n > N$. $y_n = \sum_{i=1}^{n} (x, e_i)e_i$ and $y'_m = \sum_{i=1}^{m} (x, e'_i)e'_i$, but, all the e_i's are also contained in $\sum_{i=1}^{m} (x, e'_i)e'_i$.

So, they will get cancelled.

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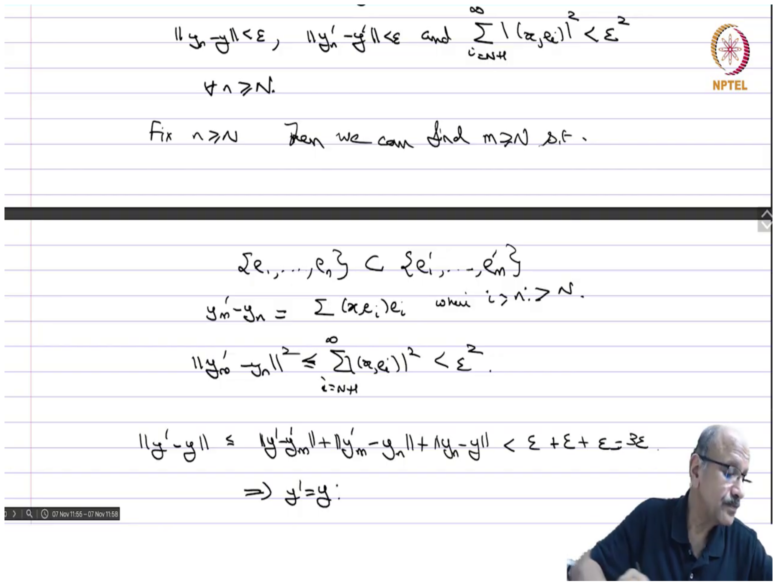

So, what do you get for $||\, y'_{m} - y_{n}||^{2} \leq \sum_{i=N+1}^{\infty} |(x,\, e_{i})|^{2} < \epsilon^{2}$, therefore,

$$||y' - y|| \leq ||y' - y'_{m}|| + ||\, y'_{m} - y_{n}|| + ||\, y_{n} - y\, || < \epsilon + \epsilon + \epsilon = 3\epsilon .$$

Note, the first one, is anyway less than ϵ, the second one we just saw is less than ϵ and the third one is also less than ϵ. Therefore, $||y' - y|| < 3\epsilon$ which is arbitrarily small and this implies that $y' = y$.

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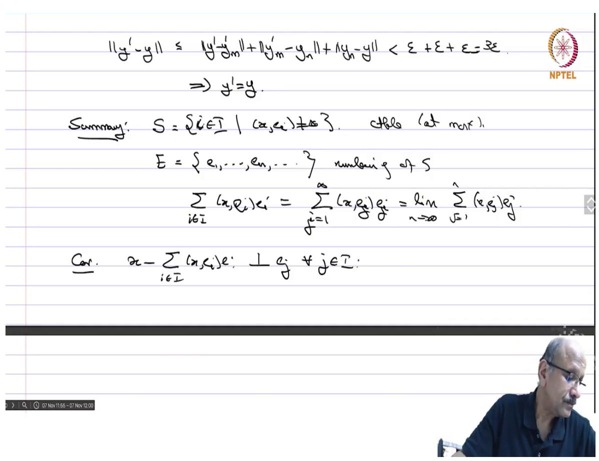

Summary. Let $S = \{i \in I / (x, e_i) \neq 0\}$. Then you say, $E = \{e_1, e_2, \cdots, e_n, . . .\}$ is numbering of S. So, this is a countable set at most. Define $\sum_{i \in I} (x, e_i)e_i = \sum_{j=1}^{\infty} (x, e_j)e_j = \lim_{n \to \infty} \sum_{j=1}^{n} (x, e_j)e_j$.

So, it does not matter how you numbered the elements in E. And therefore, you always get the same vector and so on.

Corollary: $x - \sum_{i \in I} (x, e_i)e_i$ is orthogonal to e_j $\forall j \in I$.