Functional Analysis
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Lecture No. 53
Applications

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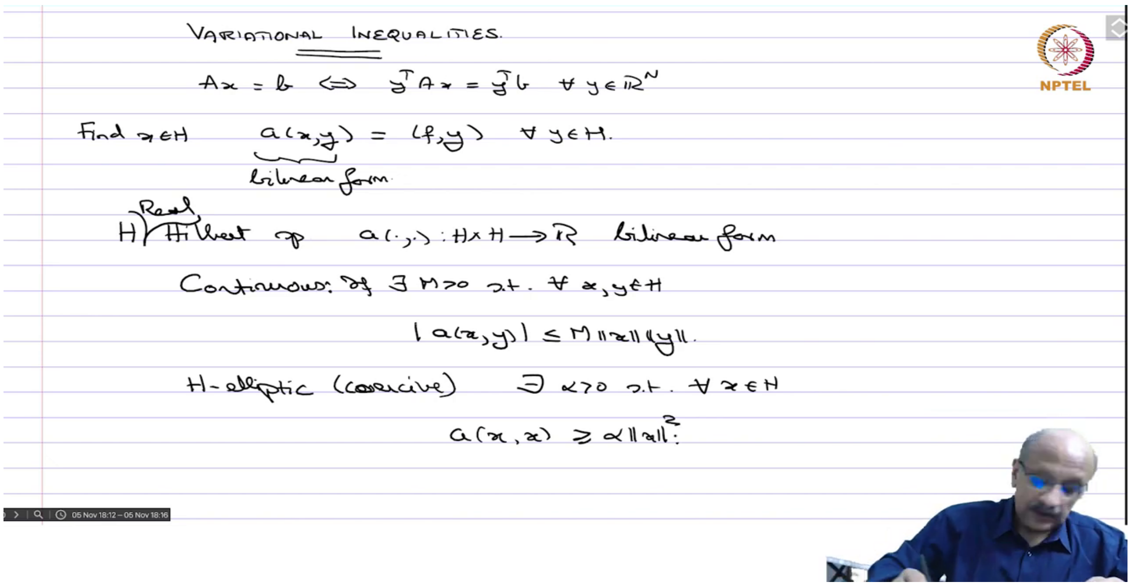

We will study some applications. This is called Variational Inequalities. Suppose we want to solve the system of linear equations $Ax = b$. This is the same as saying $y^T Ax = y^T b \quad \forall\ y \in \mathbb{R}^N$. So, you can write it in this form also. So, we treat x as a column vector, b is a column vector, so transpose is a row vector and this is the inner product.

So, generalization of these two infinite dimensional spaces is to have a bilinear form, let us say in a Hilbert space, and to solve $a(x, y) = (f, y) \quad \forall y \in H$. H is a Hilbert space and you want to find an $x \in H$. And $a(x, y) = (f, y)$ is a bilinear form. Variational inequalities are further generalizations of this and they occur especially in constrained optimization problems. That is why the name variational comes there. And many problems in science and engineering can put in this form like two phase heat transmission problems, melting of ice or the stretching of an elastic membrane over an obstacle, etc. They can all be cast in the language of variational inequalities. So today, we will study an important existence theorem for variational inequalities.

H is a real Hilbert space and you have $a(.\ ,\ .) : H \times H \rightarrow \mathbb{R}$. So, we will deal with a real Hilbert space this time because we are going to do inequalities, you cannot do that in complex numbers and anyway you have to go to the real part and therefore, you might as well deal with a real Hilbert space. So, $a(.,\ .) : H \times H \rightarrow \mathbb{R}$ is a bilinear form. That means, you fix one variable, then it is linear in the other variable. That is what we have, a bilinear, a typical, inner product in the real Hilbert space is a bilinear form. So, and this is continuous if $\exists\ M > 0$ s.t. $\forall\ x,\ y \in H$ we have $|a(x,\ y)| \leq M\,||x||\,||y||$. And then it is called H-elliptic, another word is coercive. This terminology we have seen before and you will know immediately why I am having that. If $\exists\ \alpha > 0$ such that $\forall\ x \in H$, we have $a(x,\ x) \geq \alpha||x||^2$. Then it is called H-elliptic.

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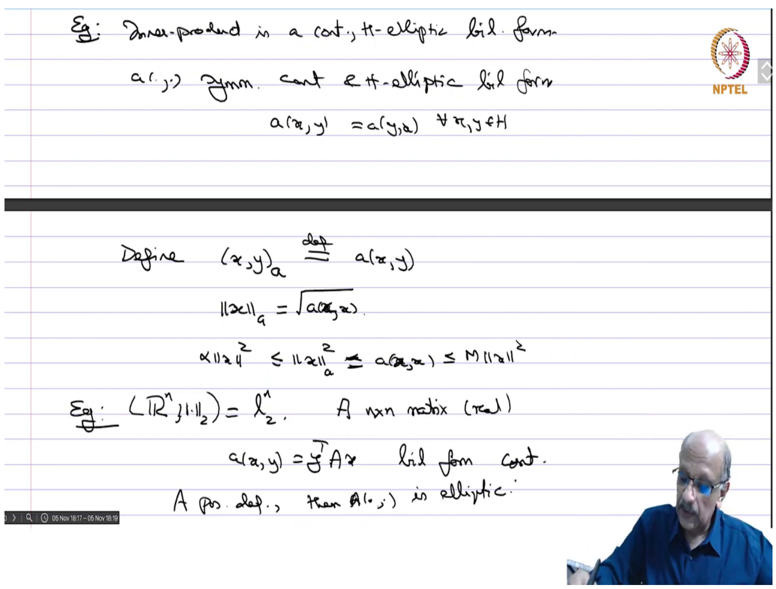

Example. Inner product is a continuous, H-elliptic, bilinear form. It is bilinear and then it is continuous by the Cauchy-Schwarz inequality and H-elliptic, in fact $(x,\ x)$ gives you $||x||^2$ itself. So, $\alpha = 1$. More generally, if you have $a(.,\ .)$ is a symmetric, continuous and H-elliptic bilinear form, so symmetry means what? $a(x,\ y) = a(y,\ x)\ \forall x,\ y \in H$. Then you can Define : $(x,\ y)_a = a(x,\ y)$. Well, it is linear in both variables, it is symmetric, it is given and $a(x,\ x) \geq \alpha||x||^2$, so $||x||_a = \sqrt{a(x,\ x)}$. This gives you a norm and in fact, this norm is equivalent to the original norm because $\alpha||x||^2 \leq ||x||_a^{\ 2} = a(x,\ x) \leq M||x||^2$ (by the continuity). Therefore, you have $||x||_a$ is equivalent to $||x||$. So, you get an equivalent inner norm and you have a new inner product.

Example. So, you have $(\mathbb{R}^N, ||\,.\,||_2) = l_2^N$. And A is $n \times n$ matrix with real entries, then you have $a(x, y) = y^T A\, x$. So, this is a bilinear form and obviously, continuous. So, if A is positive definite, then $a(x, y)$ is elliptic, i.e., $\mathbb{R}^N$ elliptic or l_2^N elliptic, whatever you want to say.

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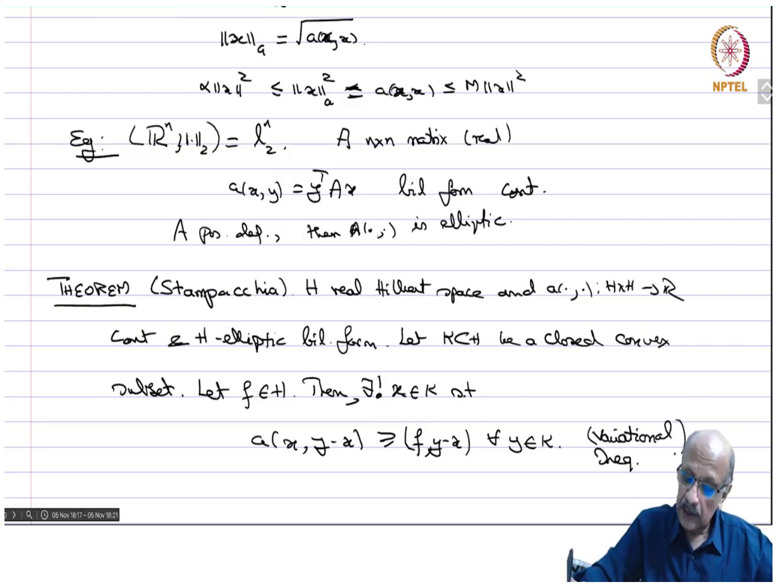

So, now, let us prove the main theorem in this connection.

Theorem. (Stampacchia) H is a real Hilbert space and $a(.\,,\,.) : H \times H \to \mathbb{R}$ is a continuous and H-elliptic bilinear form. Let $K \subset H$ be a closed convex subset. Let $f \in H$. Then, $\exists$ a unique $x \in K$ such that, $a(x, y - x) \geq (f, y - x)$ $\forall\, y \in K$———-(*). These are called variational inequalities.

So, we are proving an existence theorem for this. So, instead of $a(x, y) = f$ which we had here, we are now having an inequality instead and these, as I told you are connected to constrained optimization problems especially if $a(\,.,\,.)$ is symmetric and therefore, we have, it is interesting to prove the existence theorem.

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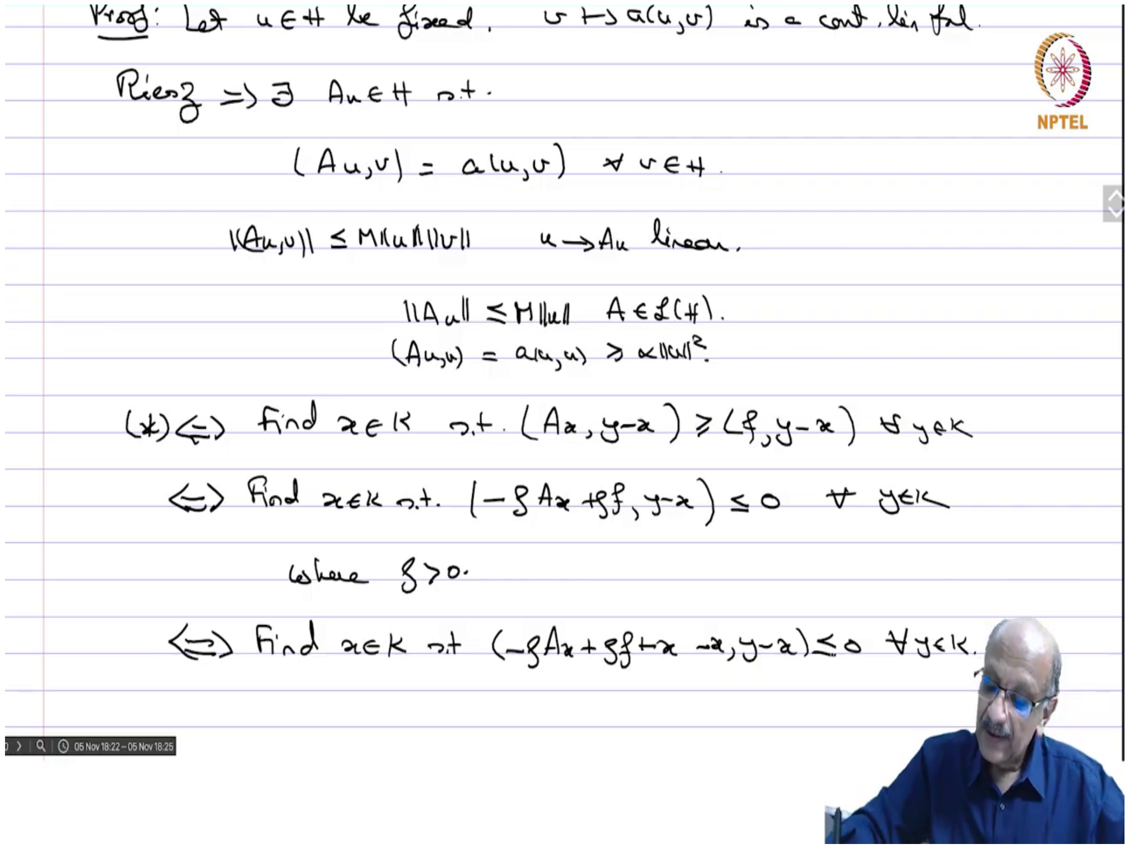

Proof. Let $u \in H$ be fixed. Then, $v \mapsto a(u, v)$ is a continuous linear functional because of the continuity of the bilinear form. Therefore, by the Riesz Representation theorem, there exists $Au \in H$ such that $(Au, v) = a(u, v)$. I should have usually put it in the second coordinate but now we are in the real Hilbert space so I will be fairly carefree in the sense that it is symmetric and therefore, I can put it anywhere here. This is for every $v \in H$. So, by the Riesz Representation theorem, this is. So, $||(Au, v)|| \leq M\,||u||\,||v||$ and therefore you have that, and $u \mapsto Au$ is linear and $||Au|| \leq M\,||u||$, and you have $(Au, u) = a(u, u) \geq \alpha||u||^2$.

(*) $\Leftrightarrow$ find $x \in K$ such that $(Ax, y - x) \geq (f, y - x)$

$\Leftrightarrow$ find $x \in K$ such that you have $(-\rho Ax + \rho f, y - x) \leq 0 \quad \forall y \in K$, where $\rho > 0$.

$\Leftrightarrow$ find $x \in K$ such that $(-\rho Ax + \rho f + x - x, y - x) \leq 0 \quad \forall y \in K$.

(Now, this looks very familiar, this is like the projection inequality which we proved)

$\Leftrightarrow x = P_k(-\rho Ax + \rho f + x) := Sx.$

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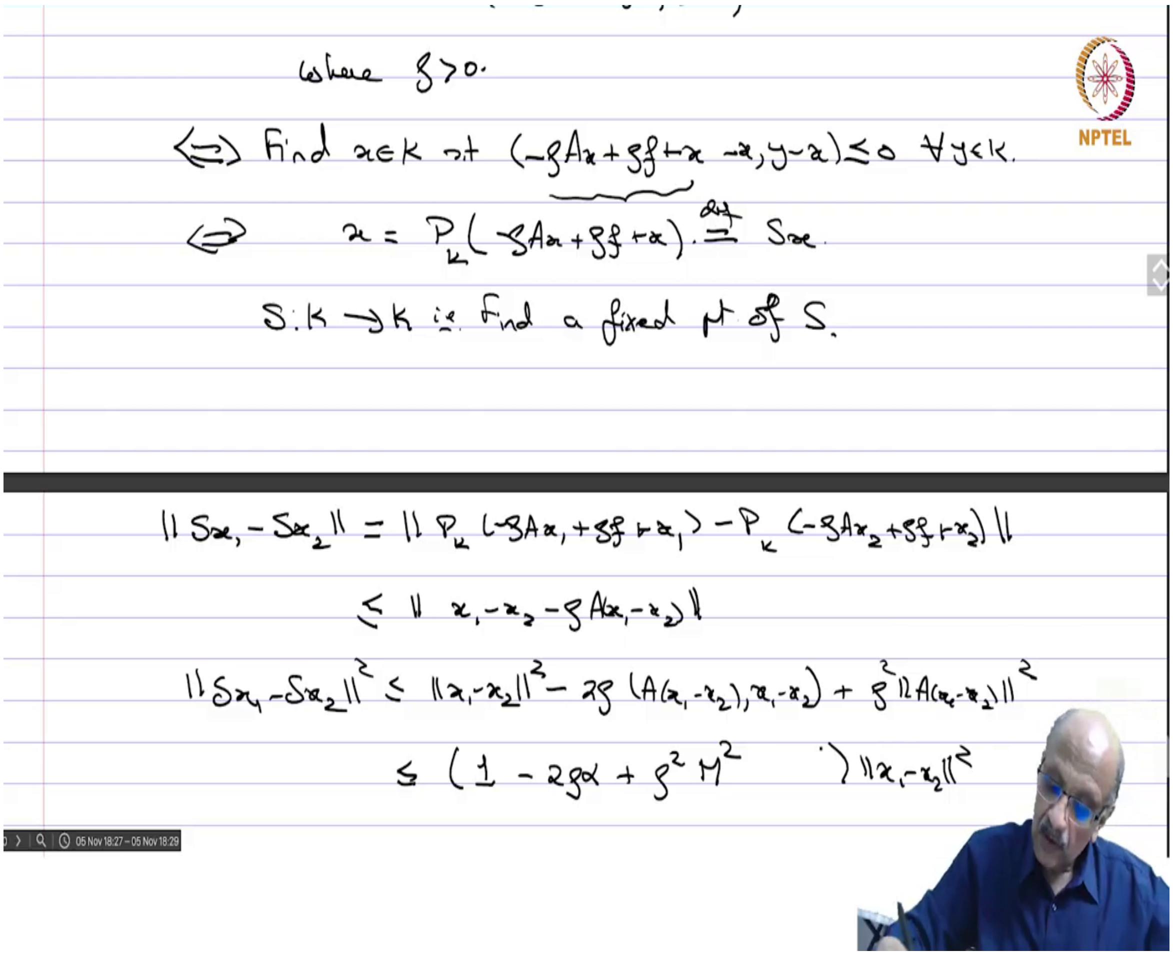

So, we are having a map S: $K \rightarrow K$ $x \in K$ and then I am defining $P_k(-\rho Ax + \rho f + x)$ and that comes back to K and I want to find a fixed point. That is, find a fixed point of S.

Let us look at $||Sx_1 - Sx_2|| = ||P_K(-\rho Ax_1 + \rho f + x_1) - P_K(-\rho Ax_2 + \rho f + x_2)||$

$$\leq ||x_1 - x_2 - \rho A(x_1 - x_2)||$$

(As we know $||P_K(x) - P_K(y)|| < ||x - y||$ (we have shown this).

So now, we square this and develop it.

$$||Sx_1 - Sx_2||^2 \leq ||x_1 - x_2||^2 - 2\rho\,(A(x_1 - x_2),\, x_1 - x_2) + \rho^2||A(x_1 - x_2)||^2$$

$$\leq \; ||x_1 - x_2||^2 \,(1 - 2\rho\,\alpha + \rho^2 M^2) \quad [\text{using}, A(z, z) \geq \alpha||z||^2 \;\&\; ||Ax|| \; \leq M||x||]$$

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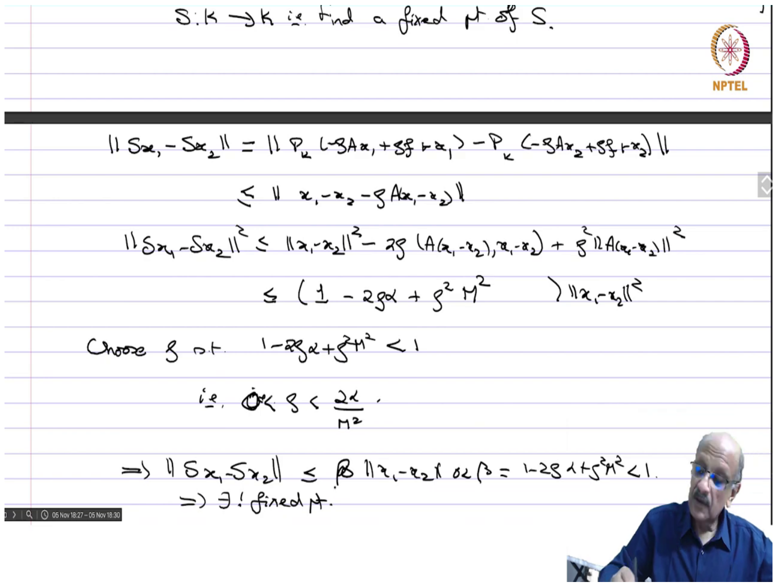

Now, I am going to choose a ρ. So, choose ρ such that $1 - 2\rho\,\alpha + \rho^2 M^2 < 1$. That means, $0 < \rho < \frac{2\alpha}{M^2}$. So, choose ρ such as this. So, this implies $||Sx_1 - Sx_2|| < \beta\,||x_1 - x_2||$, where $\beta = 1 - 2\rho\,\alpha + \rho^2 M^2 < 1$. So then, this is a contraction implies there exists a unique fixed point and therefore that proves the theorem of Stampacchia. So, we have found a unique fixed point of f and that is precisely the solution of the variational inequality.

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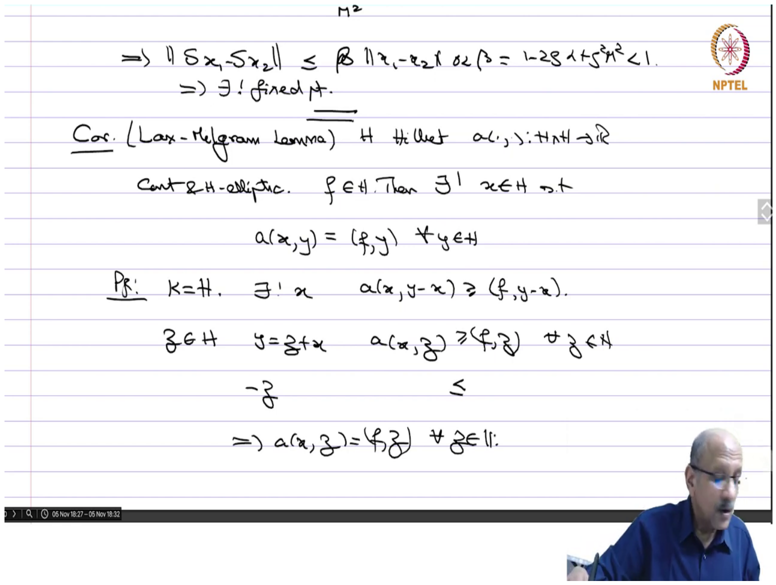

Corollary. (Lax-Milgram Lemma) H Hilbert and $a(.\,,\,.)\colon H \times H \to \mathbb{R}$ continuous and H-elliptic, $f \in H$ is given. Then, $\exists$ a unique $x \in H$ such that $a(x,\, y) = (f,\, y) \;\; \forall y \in H$. So, this is exactly the infinite dimensional version of solving n linear equations in (())(19:09) which I started this lecture with.

Proof. $K = H$. So, this is, we can apply the Stampacchia theorem. So, $\exists$ a unique x such that $a(x,\, y - x) \geq (f,\, y - x) \;\; \forall y \in H$. Now, take any $z \in H$. So, you write $y = z + x \in H$. Then you get $a(x,\, z) \geq (f,\, z) \;\; \forall z \in H$. Now, you apply it for $-z \in H$, then, you will get the opposite inequality. You will get less than equal to and therefore $a(x,\, z) = (f,\, z) \;\; \forall z \in H$. and that proves the Lax-Milgram Lemma.

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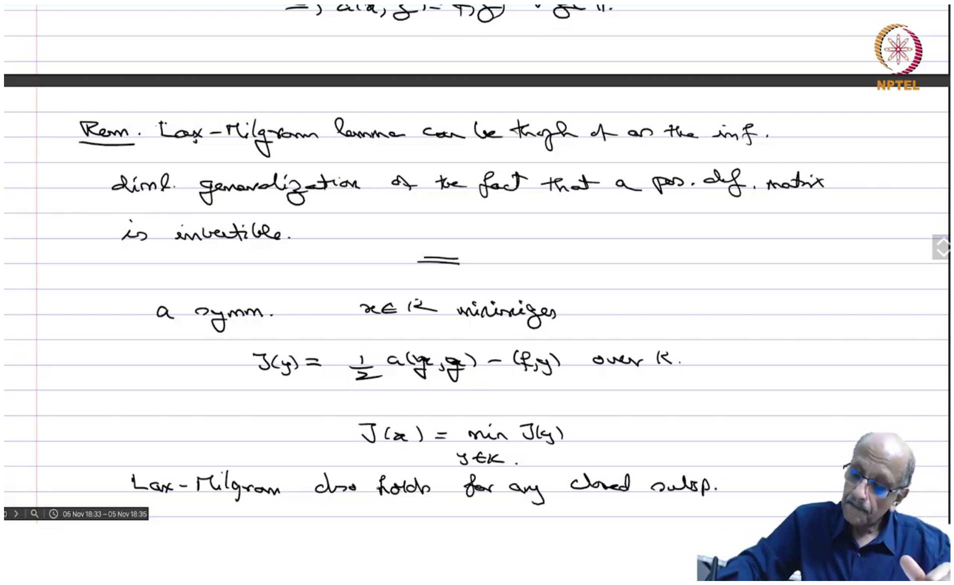

So, the Lax-Milgram Lemma is, as I said, the infinite dimensional.

Remark. Lax-Milgram Lemma can be thought of as the infinite dimensional generalization of the fact that a positive definite matrix is invertible. If you have a positive definite matrix, it is invertible. It is deterministically positive and you can have, you can uniquely solve $Ax = b$ and this is exactly the generalization.

And positive definite matrix is $(Ax, x) = x^T Ax \geq \alpha$ and therefore, you will get precisely the generalization of this. And this is the cornerstone of the existence theory for elliptic partial differential equations and, and it is also for numerical methods like the finite elements method. The Lax-Milgram Lemma is in fact a very useful tool. And in the symmetric case, A is symmetric, then x can be thought of as a minimizer. So, $x \in K$ minimizes $J(y) = \frac{1}{2}a(y, y) - (f, y)$ over K. $J(x) = \min_{y \in K} J(y)$. If $a(. , .)$ is symmetric, you can interpret the variational inequality as this. And the variational inequality becomes equality by namely the Euler-Lagrange Equations when you have this, it is the whole space and therefore that is the Lax-Milgram Lemma. Then, the Lax-Milgram Lemma also holds for any closed subspace.

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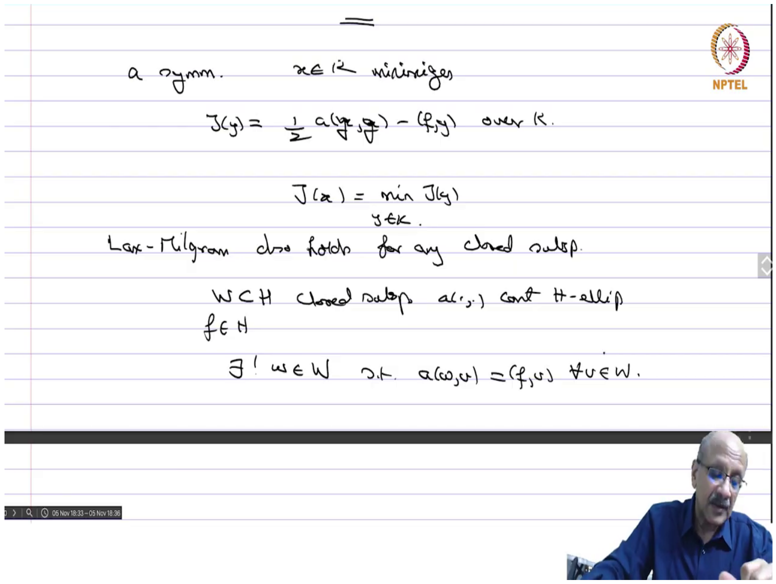

That means if $W \subset H$ is a closed subspace and $a(. \, , .)$ is continuous and H-elliptic and then $f \in H$, then $\exists$ a unique $w \in W$ such that $a(w \, , \, v) = (f, \, v) \, \forall v \in W$. This can also be very easy to prove. Just again, it is exactly the same proof which I used for the Lax-Milgram Lemma and therefore, this can be proved.