Functional Analysis
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Lecture No. 52
Adjoints

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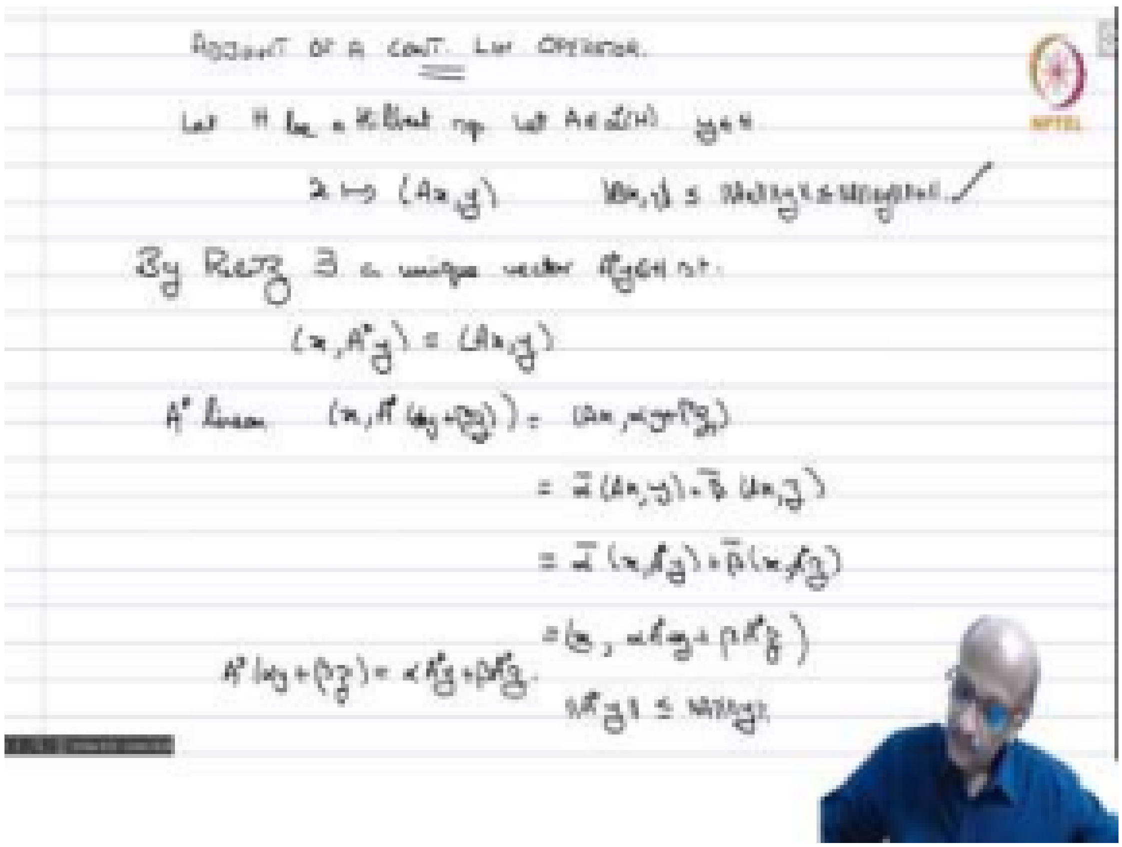

Now we will talk about the adjoint of a continuous linear operator in the context of a Hilbert space. So, we have already discussed adjoints earlier. This is essentially the same except that we are now looking in the view of the Riesz Representation theorem. You do not have to really go to the dual space. You can work on the base space itself. So, that is the idea.

Let H be a Hilbert space. And let A be a bounded linear operator or continuous linear operator on H, that means it maps H into H. So, you fix $y \in H$ and you look at the map $x \mapsto (Ax, y)$. So, this is the continuous linear functional because $|(Ax, y)| \leq ||Ax|| \, ||y||$

by the Cauchy-Schwarz inequality, i.e. $|(Ax,\ y)| \leq ||A||\ ||y||\ ||x||$. So, this is less than some constant times $||x||$ and therefore this is a continuous linear functional. Consequently, by Riesz Representation theorem, there exists a unique vector, $A^* y \in H$, such that you have $(x,\ A^* y)$ is precisely the linear mapping $(Ax,\ y)$. So, you see this relationship comes in very naturally and now, thanks to the Riesz Representation theorem we can do it. So, it is easy to check that A^* is linear. Let us check that itself. So, since we are always dealing with complex spaces, it is better to check it once and for all.

So, let us take $(x,\ A^* (\alpha y + \beta z)) = (Ax, \alpha y + \beta z) = \bar{\alpha}(Ax,\ y) + \bar{\beta}(Ax,\ z)$

$$= \bar{\alpha}(x,\ A^* y) + \bar{\beta}(x,\ A^* z) = (x,\ \alpha A^* y + \beta A^* z)$$

$\Rightarrow A^* (\alpha y + \beta z) = \alpha A^* y + \beta A^* z$, this is a linear map. It is also continuous because you have $||A^* y|| \leq ||A||\ ||y||$, $A^* \in L(H)$.

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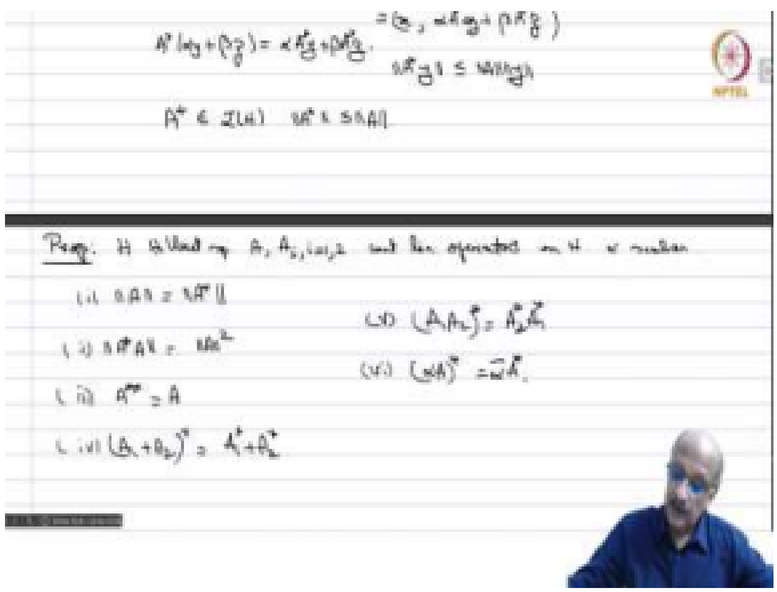


So, this proves that A^* in fact belongs to $L(H)$ and in fact, you have $||A^*|| \leq ||A||$. So, A^* is called the adjoint map. So, we have the following properties of the adjoint.

Proposition. H Hilbert and A and A_i, $i = 1, 2$ are continuous linear operators on H.

And α scalar, generally complex.

(i) $||A|| = ||A^*||$.

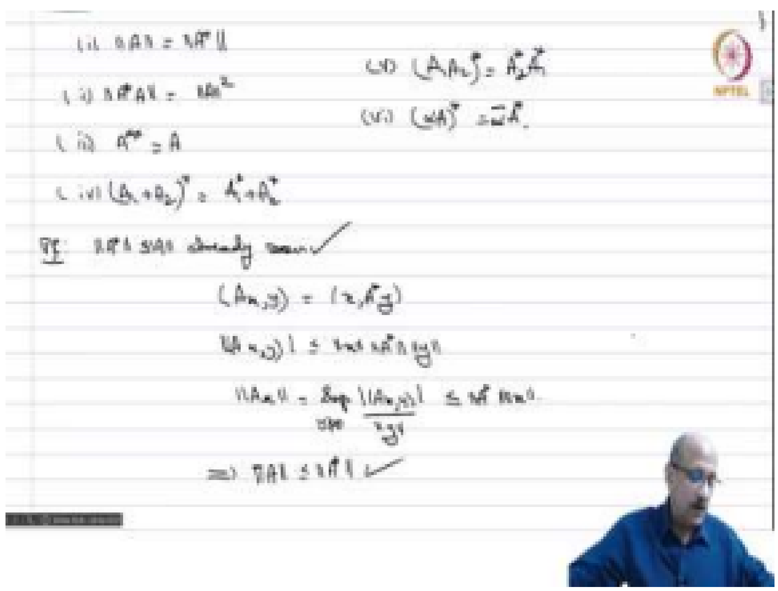
(ii) $||A^* A|| = ||A||^2$.

(iii) $A^{**} = A$.

(iv) $(A_1 + A_2)^* = A_1^* + A_2^*$.

(v) $(A_1 A_2)^* = A_2^* A_1^*$

(v) $(\alpha A)^* = \bar{\alpha} A^*$, if you have a real scalar, then the conjugate will not appear. It will just be $(\alpha A)^* = \alpha A^*$. (Refer Slide Time: 06:38)



Proof. So, we already know that $||A^*|| \leq ||A||$. That is, just above. So, we have proved that. Now, we go back to the relationship which define A^*, so we get $(Ax, y) = (x, A^* y)$. So, $|(Ax, y)| \leq ||x|| \, ||A^*|| \, ||y||$. So, if you take $||Ax|| = \sup_{y \neq 0} \frac{|(Ax, y)|}{||y||} \leq ||A^*|| \, ||x||$ and therefore, this implies $\frac{||Ax||}{||x||} \leq ||A^*||$, and this implies $||A|| \leq ||A^*||$. So, we have both the inequalities and therefore, the first one is proved.

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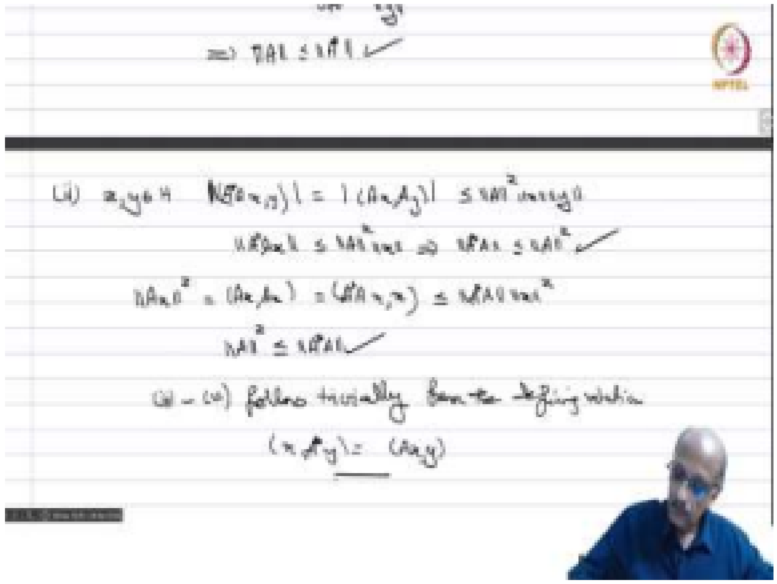

If x and y are arbitrary elements of H, you have $|(A A^* x,\ y)| = |(Ax,\ Ay)| \leq ||A||^2 ||x||\ ||y||$. So, $||A^* Ax||$ is again, by the supremum rule, less than equal to $||A||^2\ ||x||$ and this implies $||A A^*|| \leq ||A||^2$. On the other hand, $||Ax||^2 = (Ax,\ Ax) = (A^* Ax,\ x) \leq ||A^* A||\ ||x||^2$. So, $||A||^2 \leq ||A^* A||$ using the supremum. And therefore, once again, you have both inequalities. You have $||A^* A\ || = ||A\ ||^2$. Now, the remaining are all very trivial consequences. $A^{**} = A$ just comes from the relation $(x,\ A^* y) = (Ax,\ y)$. From the symmetry in this relationship, you immediately see that $A^{**} = A$. And the fourth one, which is for the sum, again, and all these are immediately from the defining relationship. So, (iii) to (vi) follow trivially from the defining relation. What is that? $(x,\ A^* y) = (Ax,\ y)$. So, this, really nothing to prove there. This is a very simple exercise.

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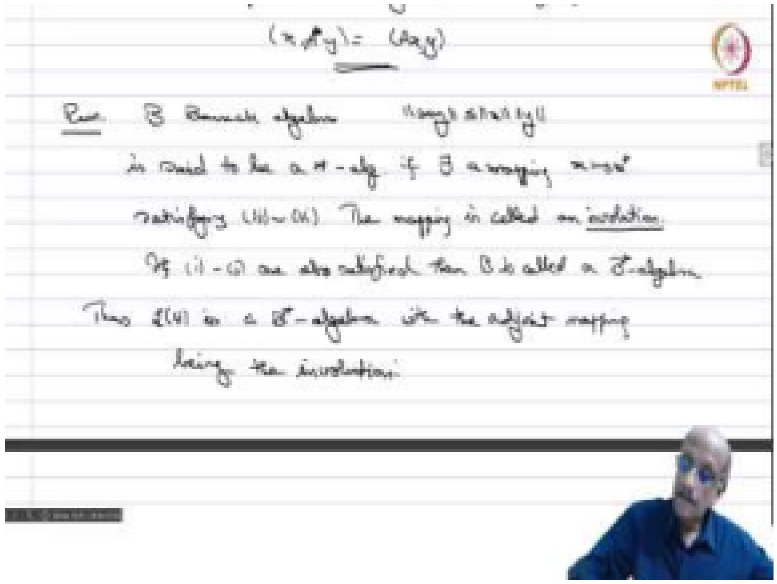

If you have a Banach algebra,

Remark. What is a Banach algebra? We have seen this. This is a Banach space in which you can multiply vectors so, and you have $||x\, y|| \leq ||x||\, ||y||$. So, $L(V)$ for V normed linear space, is typically a Banach algebra because, you all have addition, scalar multiplication and composition which is like multiplication of vectors and you know if you have two operators T and S, then $||T\, S||$ can be defined. If T and S are continuous linear operators, $||T\, S|| \leq ||T||\, ||S||$. We know this.

So, a Banach algebra is said to be a $\star$-algebra if there exists a mapping $x \mapsto x^*$ satisfying (iii) to (vi). And the mapping is called an involution. If (i) and (ii) are also satisfied, then B is called a B^*-algebra. Thus, $L(H)$ is a B^* algebra with the adjoint mapping being the involution.

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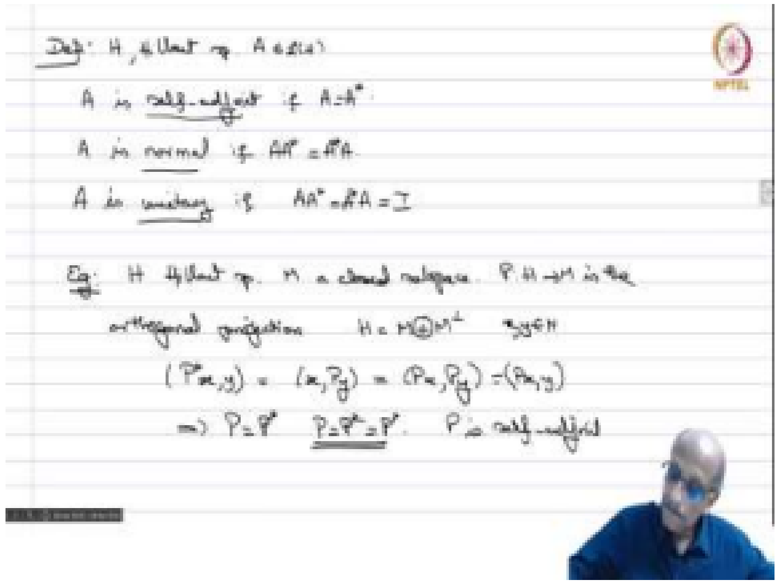

So, now, some terminologies.

Definition. So, H Hilbert space and $A \in L(H)$. So, A is self-adjoint if $A = A^*$.

A is normal if $A A^* = A^* A$. So, self adjoint is in particular normal.

A is unitary if $A A^* = A^* A = I$.

Example. H Hilbert and M is a closed subspace. And P: $H \rightarrow M$ is the orthogonal projection. That means H is written as $M \oplus M^{\perp}$ and then P: $H \rightarrow M$ is the orthogonal projection. So, now, if you have $(P^* x, y) = (x, Py)$, if $x, y \in H$. Now, $Py \in M$. So, what is the characterization? So, this is nothing but (Px, Py) because that is how the projection is defined. They are characterized by this. So now, $(Px, Py) = (Px, y)$. Because again by the repeated application of the definition of the orthogonal projection. So, this means that for all x and y, $(P^* x, y) = (Px, y)$ is true so $P = P^*$ and of course P is a projection, therefore $P = P^2 = P^*$. So, P is self adjoint.

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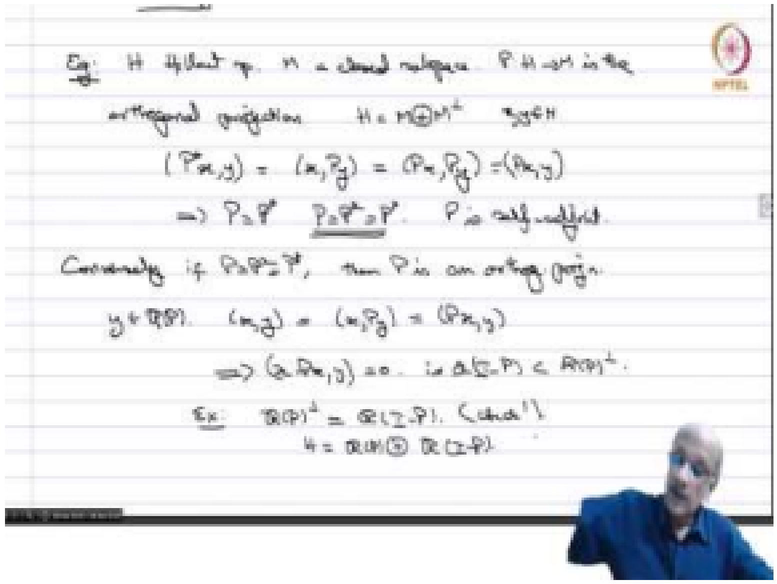

Conversely, if $P = P^2 = P^*$, then P is an orthogonal projection. So, let us take $y \in R(P)$. So, $(x, y) = (x, Py)$, because $P = P^*$ and $(x, y) = (x, Py) = (Px, y)$ because $P^2 = P$ so, (x, y) is the same as (Px, y). Therefore, you have that $(x - Px, y) = 0$, that is, $R(I - P) \subset R(P)^{\perp}$.

Converse part: Exercise If you have $R(P)^{\perp} = R(I - P)$. So, therefore, you have $H = R(P) \oplus R(I - P)$. So, check. And therefore, it is an orthogonal projection.

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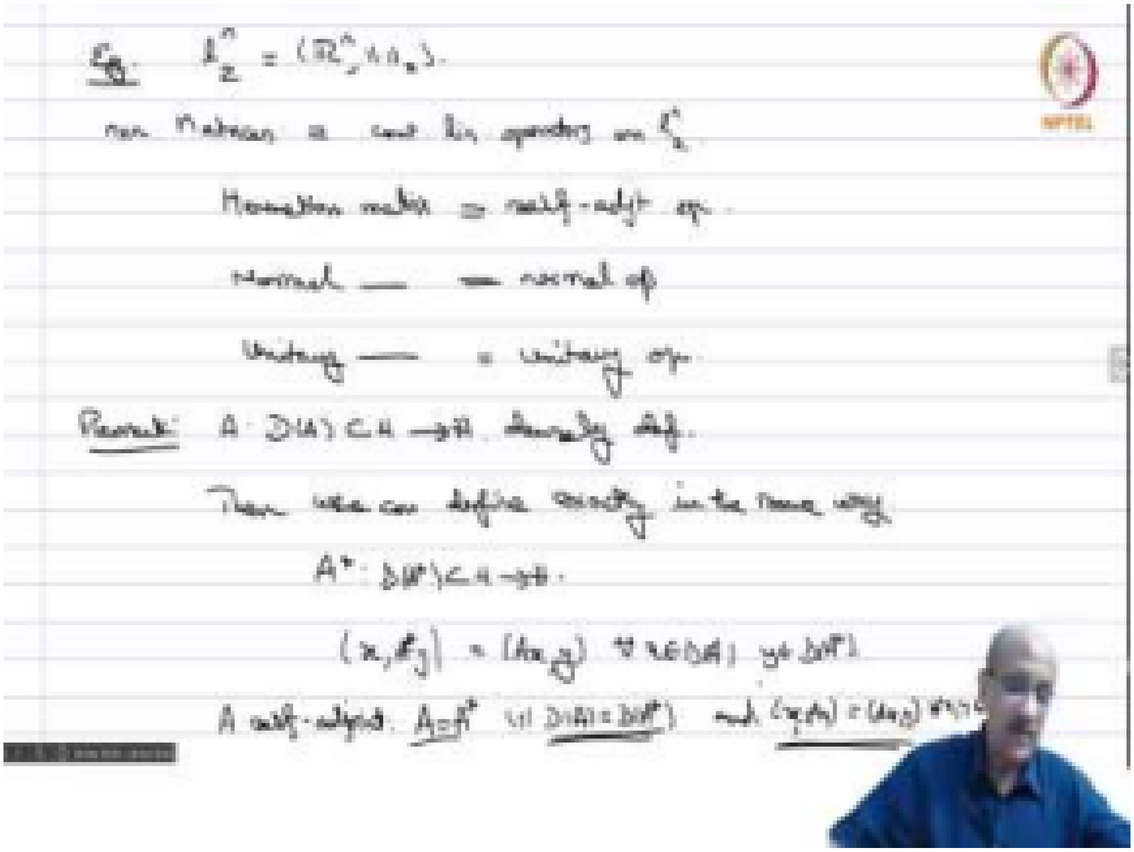

Example. Let us take $l_2^n = (\mathbb{R}^n, ||.||_2)$. So if you have matrices then, matrices are the same as continuous linear transformation, $n \times n$ matrices on l_2^n. And therefore you have that Hermitian matrix, that means A equals the conjugate transpose. So this gives a self adjoint operator. And then, the operator defined by a normal matrix, gives you a normal operator. And if you have a unitary matrix, then you get the unitary operator. So, we are just retaining the terminology of these things.

Remark. Suppose you have an unbounded operator. So, $D(A) \subset H \to H$ which is densely defined. Then, we can define, exactly in the same way, A^* from $D(A^*) \subset H \to H$ and again, you have $(x, A^* y) = (Ax, y)\ \forall x \in D(A)\ \&\ y \in D(A^*)$. So, I leave it to you to check that. And then, all the properties of the adjoint which we have proved earlier are all true. So, if you want, A is self adjoint, that means $A = A^*$. So, this should mean, you have to prove two things. (i) $D(A) = D(A^*)$, and (ii) $A = A^*$. That means

$(x, A\ y) = (Ax, y)\ \forall x, y \in D(A)$. So, it is what is called a symmetric operator. So, symmetric and domain coincides, then you say that it is a self adjoint operator. So, when you check self adjoint for unbounded operators, you have to be a little careful. Just checking the symmetry relationship is not enough. We have to also check that the domain of A is the same as the domain of A^*.