Functional Analysis
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Lecture No. 51
Duality

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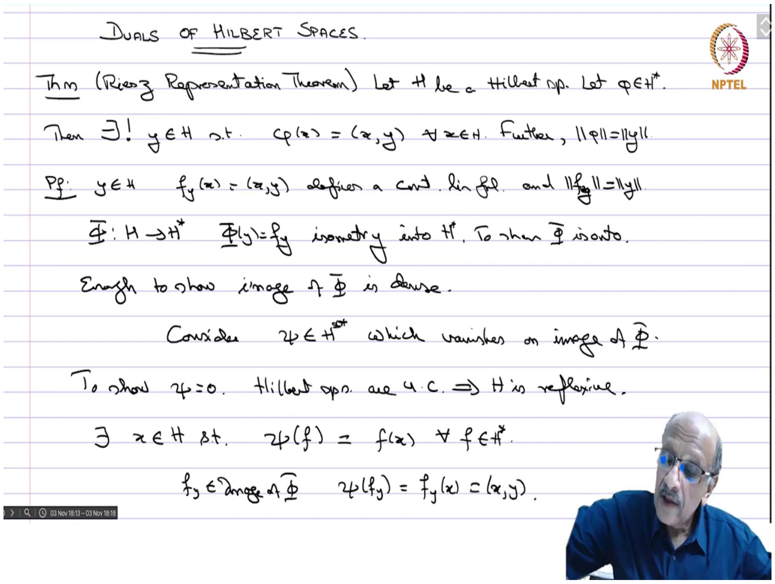

We will now study about Duals of Hilbert Spaces. So, we have the following theorem. Again, it is called the Riesz Representation Theorem because it gives you a representation of the element of the dual.

Theorem (Riesz Representation Theorem): Let H be a Hilbert Space. Let $\phi \in H^*$. Then $\exists$ a unique $y \in H$ such that $\phi(x) = (x, y) \quad \forall x \in H$. Further, $||\phi|| = ||y||$.

We already saw that if you have $y \in H$, then f_y given by (x, y) gives you a linear functional whose norm is equal to the norm of the vector. Now, we say that every linear functional occurs in this fashion.

Proof. If $y \in H$, then $f_y(x) = (x, y)$ defines a continuous linear functional and by the Cauchy-Schwarz inequality, $||f_y|| = ||y||$. This, we have already seen in an example last time. Now, we have to show that every mapping occurs in this way. So, if you take the mapping ϕ: $H \mapsto H^*$ namely $\phi(y) = f_y$. So, this is an isometry into H^*. So, the image of ϕ is closed and all we have to show, to show ϕ is onto, enough to show the image of ϕ is dense. Since the image of ϕ is already closed since you have an isometry, so it is enough to show that the image is dense. So, that will complete the proof. So, you consider a linear functional. Let us say $\psi \in H^{**}$ which vanishes on the image of ϕ. To show, $\psi = 0$. Hilbert spaces are uniformly convex, H Hilbert space $\Rightarrow$ H is reflexive. Therefore, $\exists\, x \in H \quad \forall f \in H^*$. So, if you take any $f_y \in$ image of ϕ, so $\psi(f_y) = f_y(x) = (x, y)$.

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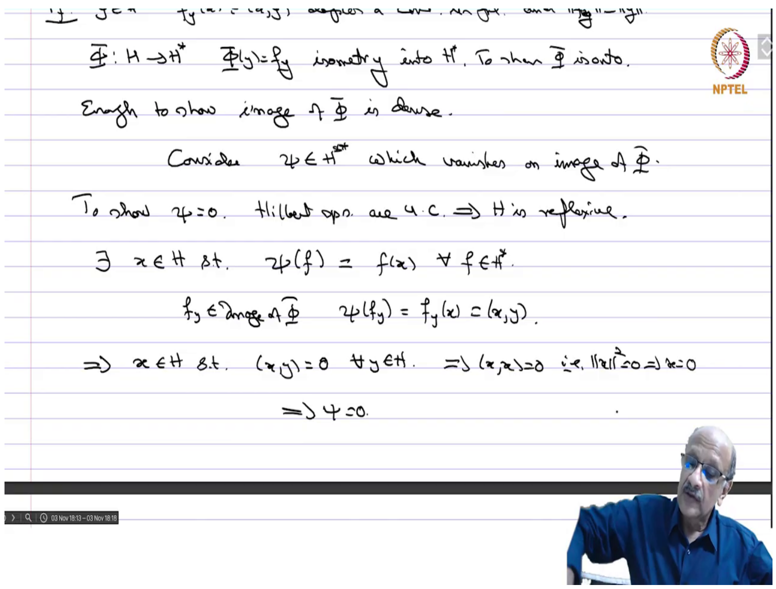

This implies that $x \in H$ such that $(x, y) = 0 \quad \forall y \in H$. In particular, this implies that $(x, x) = 0$, that is, $||x||^2 = 0 \Rightarrow x = 0$. And that implies that $\psi = 0$ and therefore, the image is dense and consequently we have shown that ϕ is onto and therefore the Riesz Representation theorem is completely proved.

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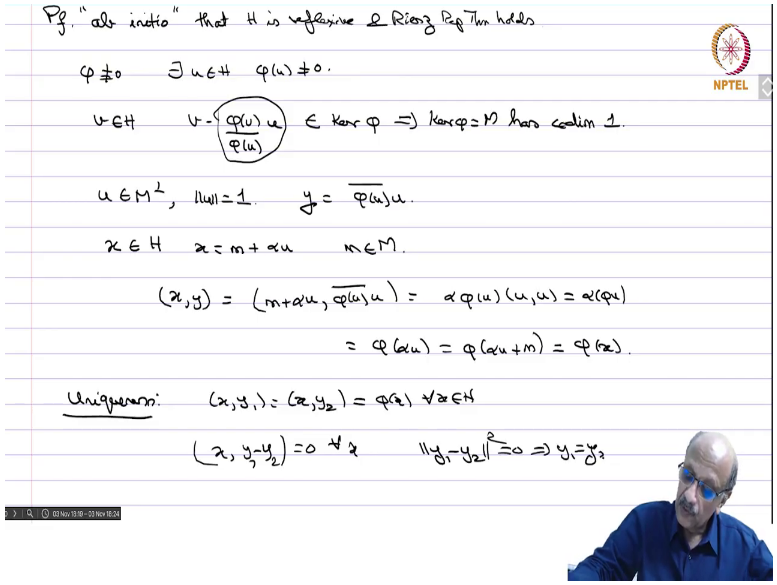

So, now, we can directly prove.

Proof. "ab initio" That means from first principles that H is reflexive and Riesz Representation theorem holds. In fact, we will first prove the Riesz Representation theorem and then from that, we will deduce that H is reflexive. Up till now what have we done? We have used uniform convexity and the fact that uniform convex spaces are reflexive and we prove the Riesz Representation theorem. Here, we will now directly prove the Riesz Representation theorem without using any machinery and then deduce in fact that H is reflexive. It is good to know both the proofs.

Let us assume that ϕ is not identically 0. If it is 0, then the 0 vector is there so we do not need to do anything. So, $\phi \neq 0$. So, $\exists\, u \in H$ such that $\phi(u) \neq 0$. Let us take $v \in H$ and then look at $v - \frac{\phi(v)}{\phi(u)} u$. You can divide because $\phi(u) \neq 0$. Now, $v - \frac{\phi(v)}{\phi(u)} u \in Ker\, \phi$ because $\phi(v - \frac{\phi(v)}{\phi(u)} u) = \phi(v) - \phi(u)\phi(v)/\phi(u) = 0$. This implies that $Ker\, \phi = M$ has codimension 1. Namely, it is, the complement of that is 1 dimensional. It is spanned by a vector u such that $\phi(u) \neq 0$. Therefore, let us take $u \in M^{\perp}$ (the orthogonal complement) and such that $||u|| = 1$. And now, you put $y = \overline{\phi(u)}\, u$. I am doing the complex case always so, up to now, in Banach space theory, I have said, worked with reals and then said the complex case will be the same proof and also in case there was a difference like the Hahn Banach Theorem, I made special mention of that fact. In the case of Hilbert spaces, I will naturally work with complex numbers because there is conjugation which always comes into play and it is better to check everything if that is okay and then if the real case will be obvious because you do not have to put the conjugate at all. So, we will always work with the complex numbers as the base field. Now, if you take any $x \in H$, then x can be written as $x = m + \alpha u,\ \ m \in M$. Now, $(x, y) = (m + \alpha u, \overline{\phi(u)}\, u)$. Now, m and u are orthogonal because m is in M, $u \in M^{\perp}$. Therefore, $(m, \overline{\phi(u)}\, u) = 0$ so you get $(x, y) = (m + \alpha u, \overline{\phi(u)}\, u) = \alpha\, \phi(u)(u, v) = \alpha\, \phi(u) = \phi(\alpha u) = \phi(\alpha\, u + m)$ because m is in the kernel, I can add it and that is precisely $\phi(x)$. The uniqueness, I did not prove. I should have done it even there. So, uniqueness we always have, namely. Suppose you have two elements. You have $(x, y_1) = (x, y_2) = \phi(x)\ \ \forall x \in H$. Then you have $(x, y_1 - y_2) = 0\ \forall x$. So, you put $x = y_1 - y_2$ so you get $||y_1 - y_2||^2 = 0 \Rightarrow y_1 = y_2$. So, you always have a unique vector.

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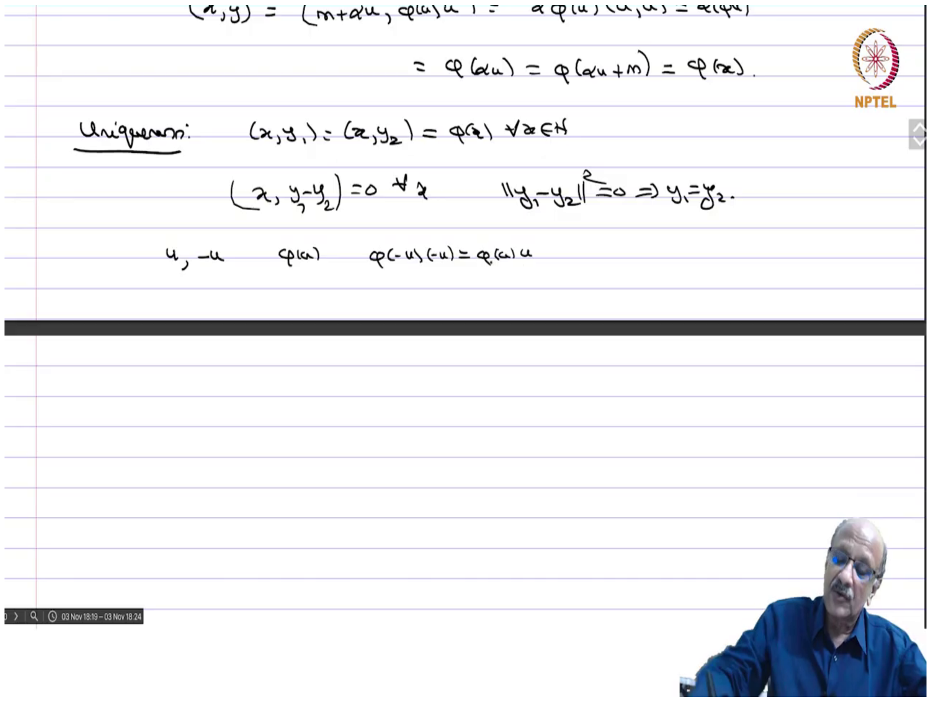

This proves the Riesz Representation theorem. Namely, every continuous linear functional, in fact, so we have even identified what that vector is. So, you have to take a vector, unit vector in the one dimensional orthogonal complement of the kernel and then you put $\phi(u)\, u$. So, you may have some doubt that since it is one dimensional, there will be two vectors. u and $-\, u$ both with norm 1. But then, if I take $\phi(u)\, u$ or $\phi(-\, u)\, (-\, u)$. It is the same. This is equal to $\phi(u)\, u = \phi(-\, u)\, (-\, u)$. So, this vector y will not change. So, the uniqueness will not change. So, the uniqueness is contradicted.

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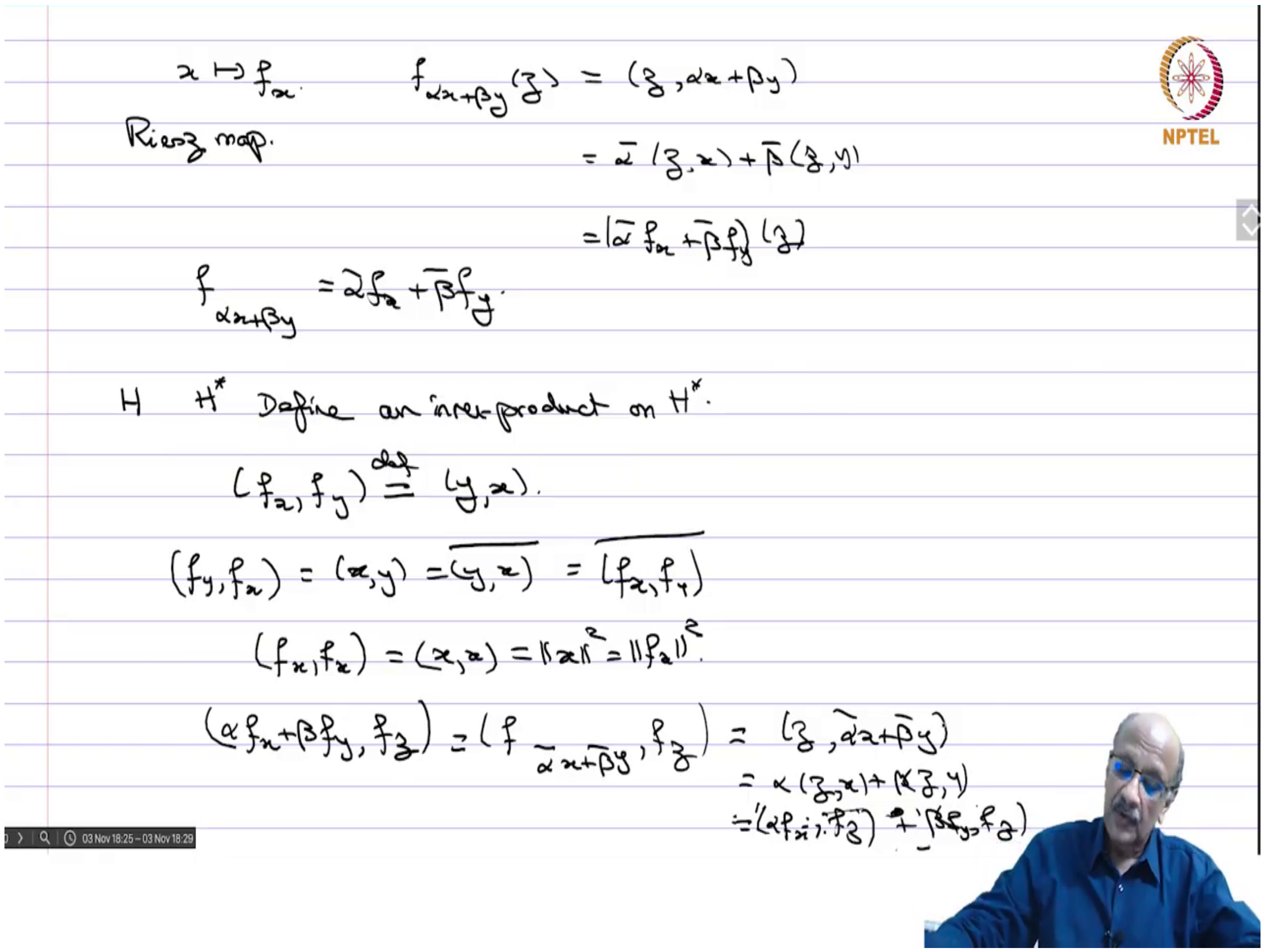

So, now, let us prove the reflexivity of this thing. Before that, let me look at the map $x \mapsto f_x$. If I take $f_{\alpha x+\beta y}(z) = (z,\ \alpha x\ +\ \beta y) = \bar{\alpha}(z, x)\ +\ \bar{\beta}\ (z, y)\ =\ (\bar{\alpha}\ f_x\ +\ \bar{\beta}\ f_y)\ (z)$

so, $f_{\alpha x+\beta y} = \bar{\alpha}\ f_x + \bar{\beta}\ f_y$. So, this is a conjugate linear map, it is not just a linear map. So, we have H, then we have H^*. Now, we are going to give a natural inner product on H^*. So, define an inner product on H^*. So, any element of H^* is of the form f_x ,so we define $(f_x,\ f_y) = (y, x)$. We want to check that this definition takes all the property of inner products. Let us take $(f_y,\ f_x) = (x,\ y) = \overline{(y,\ x)} = \overline{(f_x,\ f_y)}$ and therefore that property is true.

Now, what about linearity in the first variable? So, what about f_x? $(f_x, f_x) = (x, x) = ||x||^2 = ||f_x||^2$. We know, the Riesz map, so this is called, $x \mapsto f_x$ is called the Riesz map. And that is an isometry and therefore you have this. So, we now check about the linearity in the first variable. So, it will be conjugate linear in the second variable automatically because of the conjugacy condition here and therefore, we have to take $(\alpha f_x + \beta f_z, f_z)$ But this, we just saw, nothing but $(f_{\bar{\alpha} x + \bar{\beta} y}, f_z) = (z, \bar{\alpha} x + \bar{\beta} y) = \alpha(z, x) + \beta(z, y) = \alpha(f_x, f_z) + \beta(f_y, f_z)$ Therefore, the distributive law, linearity in the first variable is well defined. So, therefore, this defines the genuine inner product on the space.

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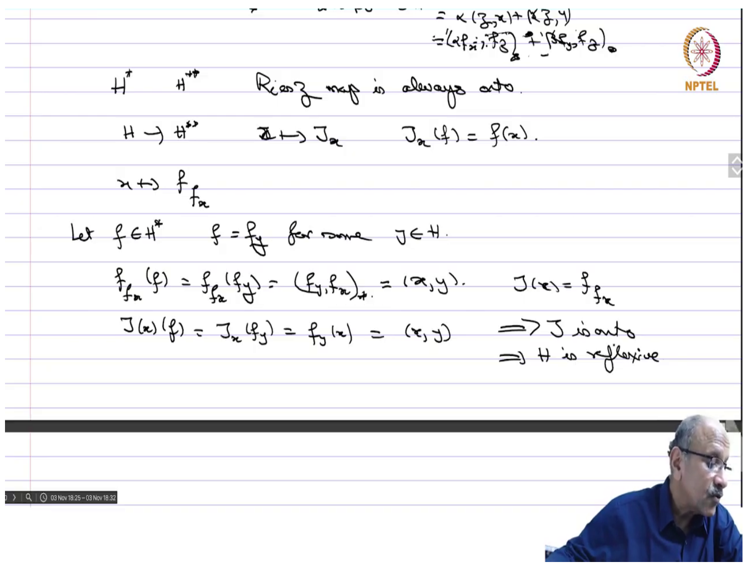

So, now again, we can define. So, H^*, is a Hilbert space and therefore you can define an inner product on H^{**} in the same way using the Riesz map. Now, Riesz map is always onto. So, now, we have two maps going from H to H^{**}. One is a canonical map $x \mapsto J_x$. What is $J\,x$? $J_x(f) = f(x)$. This is the evaluation map. So, this is one map.

The other map from H to H^{**} is $x \mapsto f_{f_x}$. f_{f_x} is a Riesz map from H to H^{**}. So, we want to show that these are one and the same. So, let $f \in H^{*}$. Then $f = f_y$ for some $y \in H$. So, $f_{f_x}(f) = f_{f_x}(f_y) = (f_y, f_x)_* = (x, y)$. Now, $J(x)(f) = J_x(f_y) = f_y(x) = (x, y)$. So, these two are the same. So, $J(x)$ is nothing but f_{f_x}. But we know that the Riesz map is onto. So, this means that J is onto and therefore H is reflexive. So, starting from first principles, we have shown that Hilbert space is reflexive and that the Riesz representation theorem is true.

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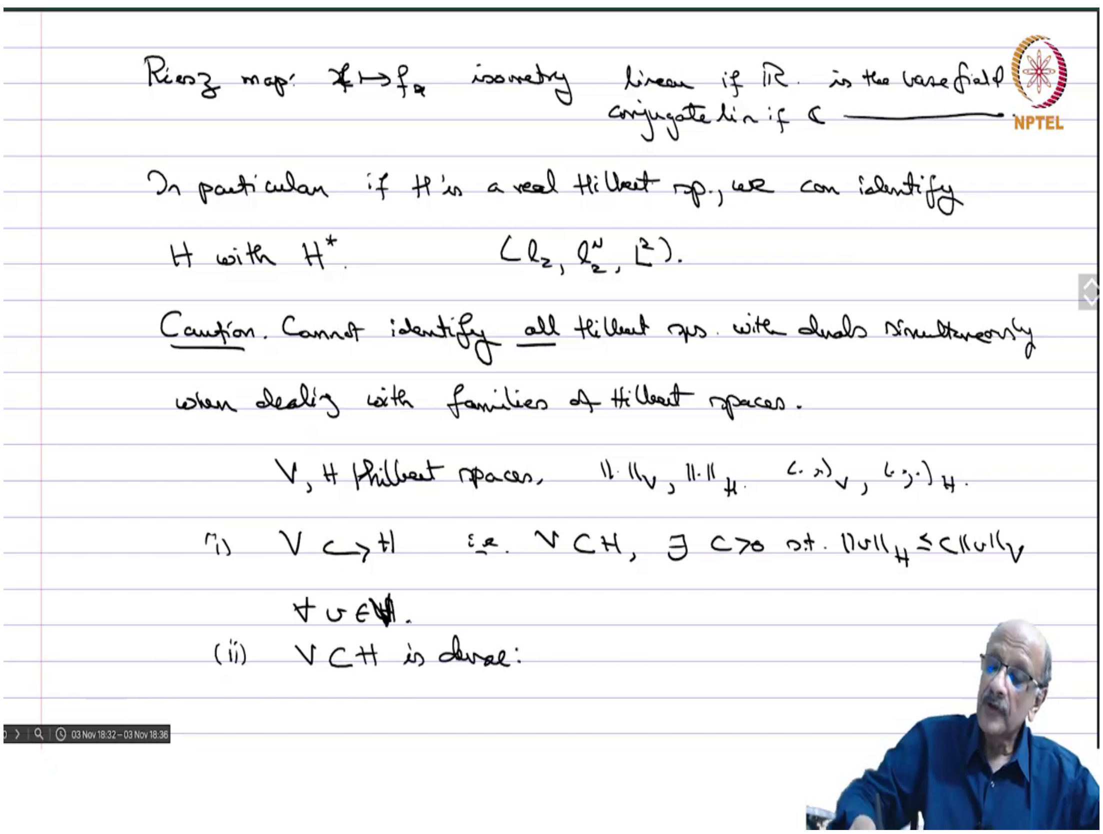

Now, if you look at the map, Riesz map, f, $x \mapsto f_x$, this is isometry and linear, if you are working with $\mathbb{R}$, and conjugate linear if $\mathbb{C}$ is the base field. So, it is not a big deal if you have a conjugate linear map but, in particular if H is a real Hilbert space, we can identify H with H^{*}. Because we have an isometric isomorphism, we have been doing this repeatedly. In fact, we have already done it in case of l_2, l_2^n etc and the space L^2. All these spaces, we have actually identified the dual. We have often used it like that.

So, now, the question is can we always do it or can we exercise some caution? So, there are some situations where you should not blindly do it. A typical situation is the following.

Caution. Cannot identify all Hilbert spaces with dual simultaneously when dealing with families of Hilbert Spaces. So, let me give you a typical example.

Let V and H be Hilbert spaces. And so, we have $||.||_V$ and $||.||_H$ as the norms and then you have the inner product $(. , .)_V$ and the inner product $(. , .)_H$ such that (i) V is included in H. That means that V is a subspace of H as an algebraic set and the inclusion is continuous. That is, V is contained in H and there exists a constant $C > 0$ such that norm $||v||_V \leq C\,||v||_H\ \forall\, v\ \in V$. (ii) $V \subset H$ is dense. So, it is a dense and continuous inclusion.

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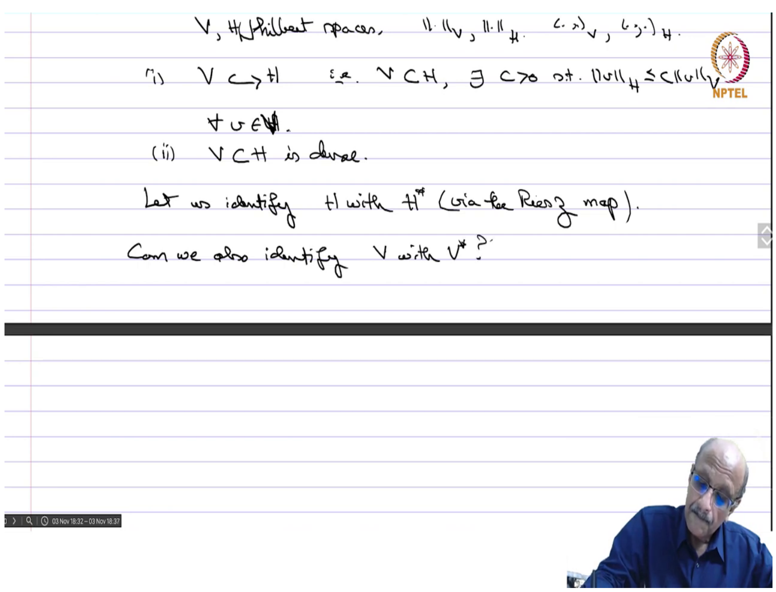

Now, let us identify, H with H^* via the Riesz map. So, we can do that, that is no harm. So, the question is, can I also identify V, so can we also identify V with V^*? So, the answer is no because of the following reason.

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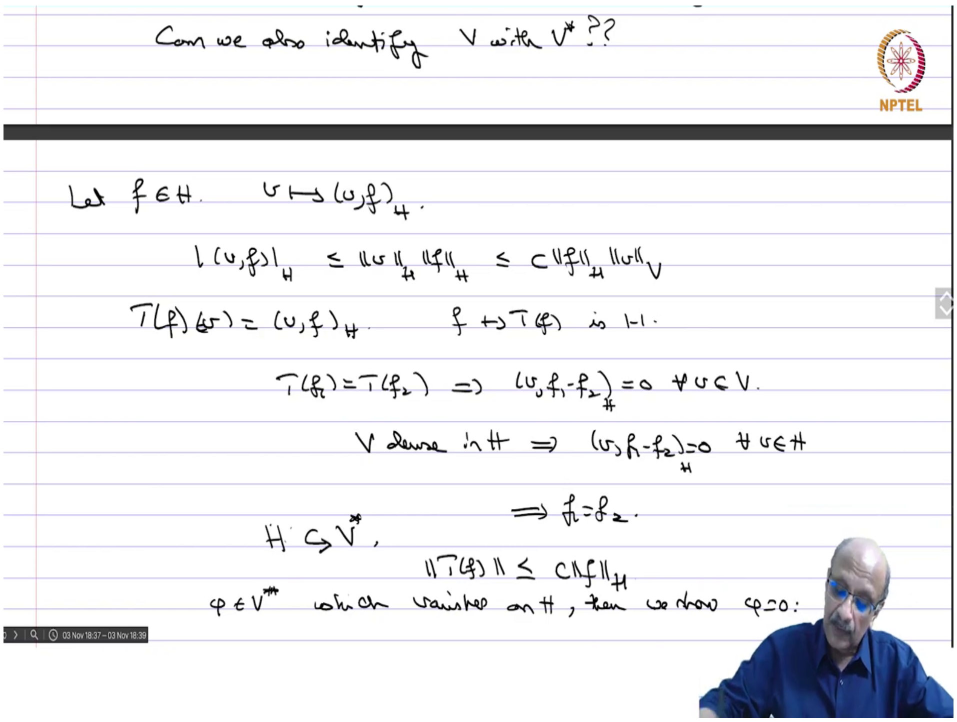

Let us take any $f \in H$. Then consider $v \mapsto (v, f)_H$. So, what is $|(v, f)|_H$? By the Cauchy-Schwarz, $|(v, f)|_H \leq ||v||_H \, ||f||_H \leq C \, ||f||_H \, ||v||_V$. So, this is a continuous linear functional in V. So, that means $v \mapsto (v, f)_H$ generates a continuous linear functional, an element of V^*. So, let us call it $T(f)$. So, $T(f)(v) = (v, f)_H$. Then, $f \mapsto T(f)$ is 1-1, Why is it true? Because if you have $T(f_1) = T(f_2)$, this means $(v, f_1 - f_2)_* = 0, \quad \forall v \in V$, but V is dense in H implies $(v, f_1 - f_2)_H = 0 \ \ \forall \, v \in H$. If you put $v = f_1 - f_2$, you will get $f_1 = f_2$. So, this is a one to one map. So, we can consider, H can be thought of as the same. And $||T(f)|| \leq C \, ||f||_H$, we have seen that and therefore this is in fact a continuous inclusion in this.

Now, we claim that this is also dense. So, we want to show that if you have a continuous linear functional $\phi \in V^{**}$ which vanishes on H, then we show $\phi = 0$. So, that will mean H is also dense.

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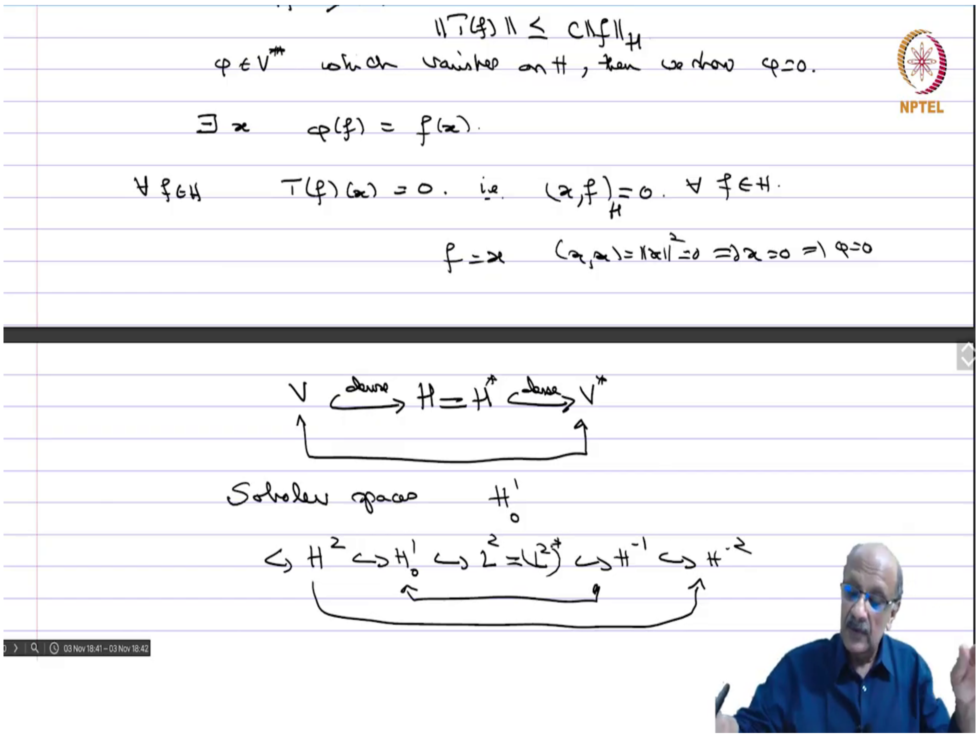

Why is that so? Because, well V is a Hilbert space so it is also reflexive. So, that implies $\exists\, x$ such that $\phi(f) = f(x),\ \ \forall f \in V^{*}$. So, that means $\forall f \in H$, we have $T(f)(x) = 0$ that is, $(x, f)_{H} = 0\ \forall f \in H$. But then $V \subset H$ and therefore I can put $f = x$. So, I get $(x, x) = ||x||^{2} = 0 \Rightarrow x = 0$, that means $\phi = 0$.

Conclusion. So, if V is continuously embedded in H and this is a dense inclusion and if I identify H with H^{*}, then $H = H^{*}$ is also embedded in V^{*} and this inclusion is also dense. Now, if I also identify V with V^{*}, that will be absurd because if $V = V^{*}$ and you have H in between V and V^{*}; then all of them are equal and you know, they are not. So, it is not correct to identify.

Therefore, when you have a family of Hilbert Spaces which are connected to each other like this, then choose one space which we call the pivot space where we identify the space with dual but all other spaces, you will have, that the duals are the same.When you study, especially in the case of Sobolev Spaces, this will be the case. So, we will have a Sobolev space called H_0^1. There will be spaces like this. H_0^2 contained in H_0^1 which is contained in L^2 and that we will assume is its dual and that will be H^{-1}, that is the dual of this space contained in H^{-2} which will be the dual of this space and so on.

And therefore, we will have a whole chain of Sobolev spaces and their duals. We will not confuse H^1, there is a Riesz map, when you want to only study these two. But when you are studying it together with L^2, you must have only one space which is identified and the other spaces should not. So, we will continue with this.