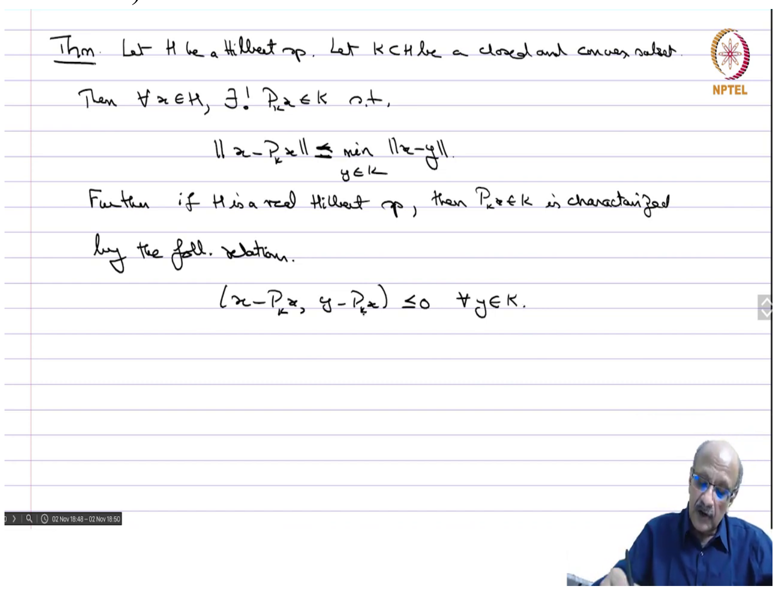
Functional Analysis
Professor. S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No. 50

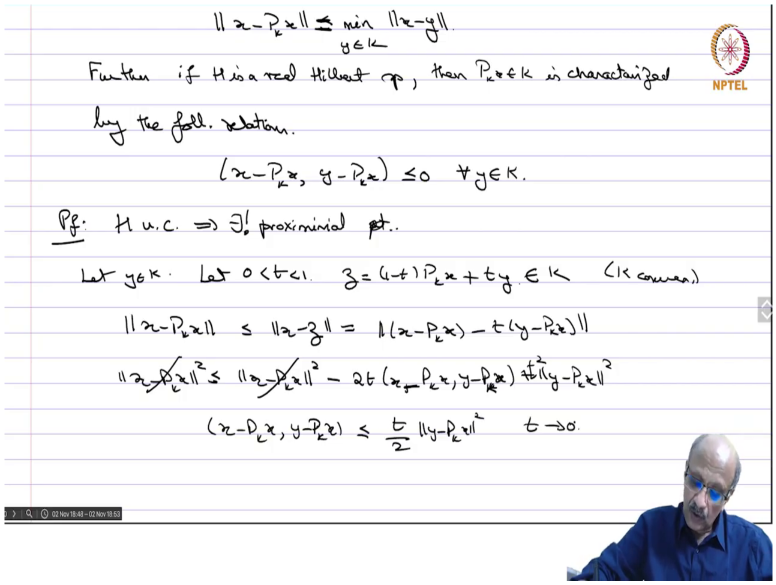
Hilbert Spaces - Part 2

(Refer Slide Time: 00:16)



Theorem: Let H be a Hilbert space. Let $K \subset H$ be a closed and convex set. Then, $\forall\, x \in H$, $\exists$a unique $P_k x \in K$ such that $||x - P_k x|| = \min_{y \in K} ||x - y||$. So, it is the closest point or proximinial point as I already mentioned on an earlier occasion and therefore you can find the unique thing. Further, if H is a real Hilbert space, then $P_k x \in K$ is characterized by the following relation. $(x - P_k x,\, y - P_k x) \leq 0$ $\forall\, y \in K$. So, whatever may be $y \in K$, this inner product should always be negative and there will be only one point which satisfies this.

(Refer Slide Time: 02:47)



Proof: So H is uniformly convex, implies there exists a proximinial point. That means there exists a unique proximinial point. We have already shown this. So, we only have to show the characterization.Let $y \in K$. Let $0 < t < 1$

$z = (1 - t) P_k x + t y \in K$ (as $P_k x \in K$ & $y \in K$ and therefore any convex combination will be in K, K is convex). So, we have $P_k x$ which is the closest point and we want to show that it satisfies this particular (given) inequality.

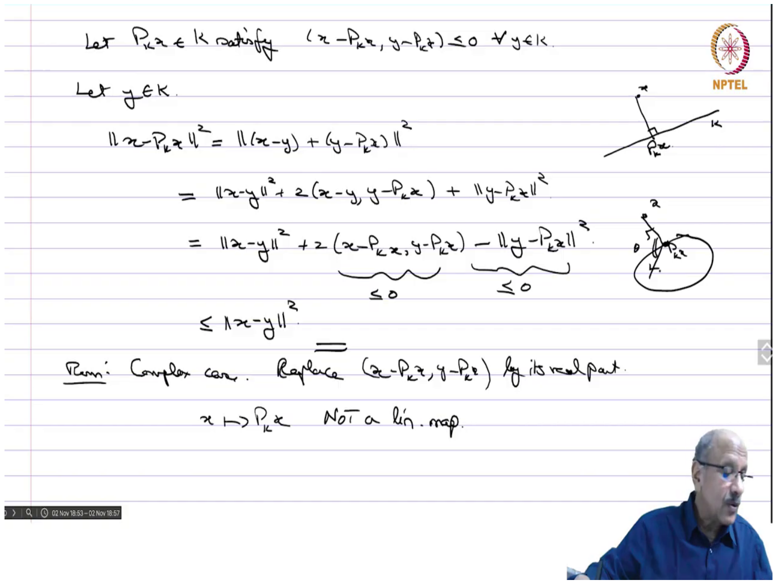
$$||x - P_k x|| \leq ||x - z|| = ||(x - P_k x) - t(y - P_k x)||$$

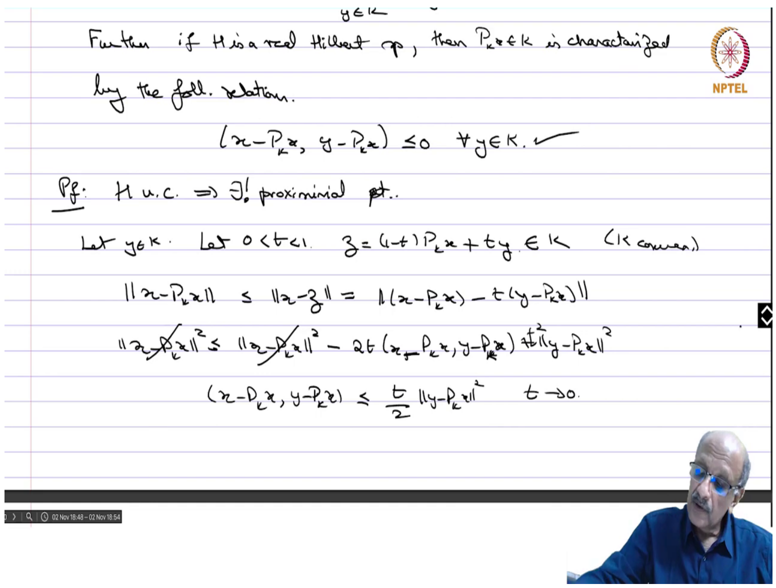
So now, let us square this and develop it. So, you have that

$$||x - P_k x||^2 \leq ||x - P_k x||^2 - 2t(x - P_k x, y - P_k x) + ||y - P_k x||^2 t^2$$

So, $||x - P_k x||^2$ get cancelled, and you get $(x - P_k x, y - P_k x) \leq \frac{t}{2}||y - P_k x||^2$. Now, you let $t \to 0$, you will get $(x - P_k x, y - P_k x) \leq 0$. So, that proves one part.

(Refer Slide Time: 05:59)





Conversely, let $P_K x \in K$ satisfy $(x - P_k x, y - P_k x) \leq 0 \, \forall \, y \in K$. So it is the condition you have here. So, we are going to assume that condition and show that $x - P_k x$ is in fact the minimum. So let $y \in K$. So you have

$$||x - P_k x||^2 = ||(x - y) + (y - P_k x)||^2$$

$$= ||x - y||^2 + 2(x - y, y - P_k x) + ||y - P_k x||^2$$

$= ||x - y||^2 + 2(x - P_k x, y - P_k x) - 2||y - P_k x||^2 + ||y - P_k x||^2$

$= ||x - y||^2 + 2(x - P_k x, y - P_k x) - ||y - P_k x||^2$

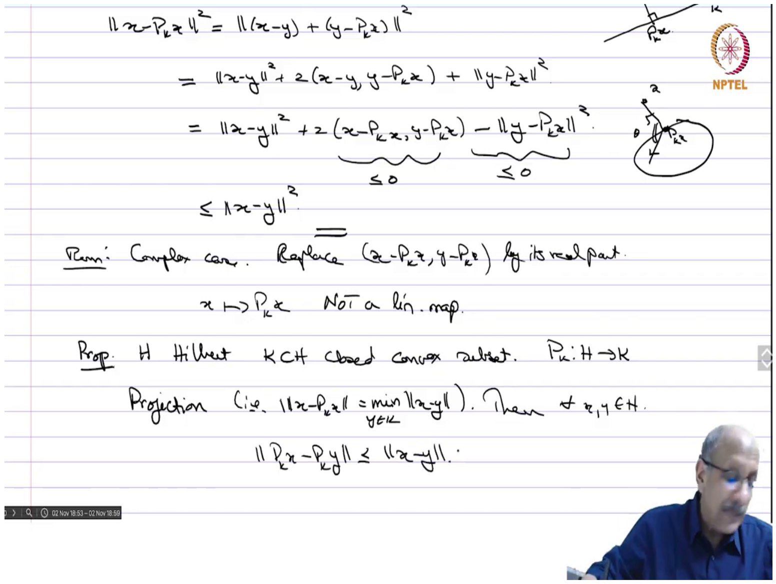
So, $(x - P_k x, y - P_k x)$ is given to be less than equal to 0, $-||y - P_k x||^2$ is anyway less than equal to 0 and therefore $||x - P_k x||^2 \leq ||x - y||^2$. This is true for all $y \in K$ and therefore $P_k x$ is the closest point.

Remark: Complex case: Replace $(x - P_k x, y - P_k x)$ by its real part and you can try to prove it yourself.

In general $x \mapsto P_K x$ is not a linear map, it is a projection. So what are you doing? For instance if you had a line in the plane then if you have this the closed convex set. It is nothing but the foot of the perpendicular from x, so this will be the point $P_k x$. So, this is what we have. Now what does this condition tell you? $(x - P_k x, y - P_k x)$. So, if you have a convex set here, so, you have $P_k x$, let us say it is the closest point to x. So, it tells you that this angle between $x - P_k x$, $y - P_k x$ should always be obtuse.

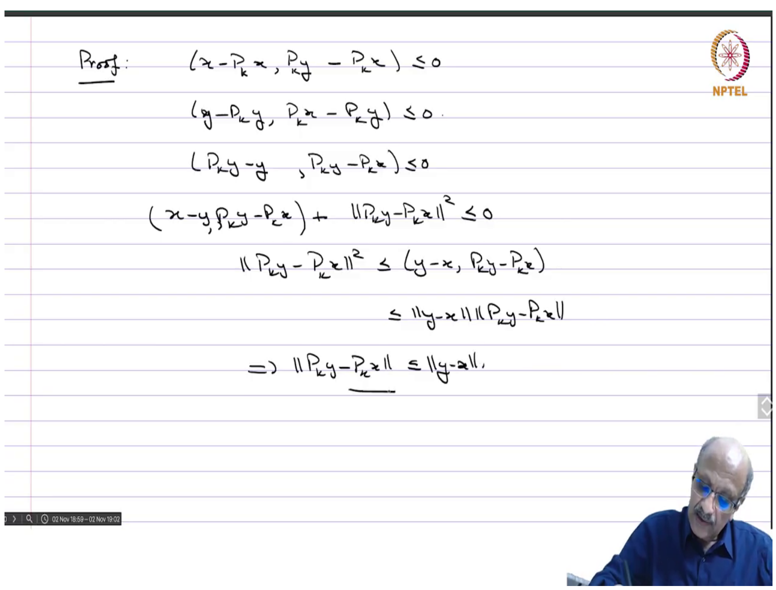
So, the geometric condition is that this angle in the plane, if you had in the plane, it is saying the closest point from the closed convex set to x in the in the 2 norm would be this unique point such that the angle between two vectors joining $P_k x$ to x and $P_k x$ to any element of K should be an obtuse angle or a right angle, but then $(x - P_k x, y - P_k x)$ will always be less than or equals to 0. So, that is the thing.

(Refer Slide Time: 10:42)



Proposition. This P_K still has some good properties. So, H Hilbert, $K \subset H$ closed convex and $P_k : H \to K$ is the projection that is $||x - P_k x|| = \min_{y \in K} ||x - y||$. Then, $\forall\, x,\, y\, \in H$ you have $||P_k x - P_k y|| \le ||x - y||$. So, it is a non expansive mapping. So this is a non linear map in general and it is a non expansive map.

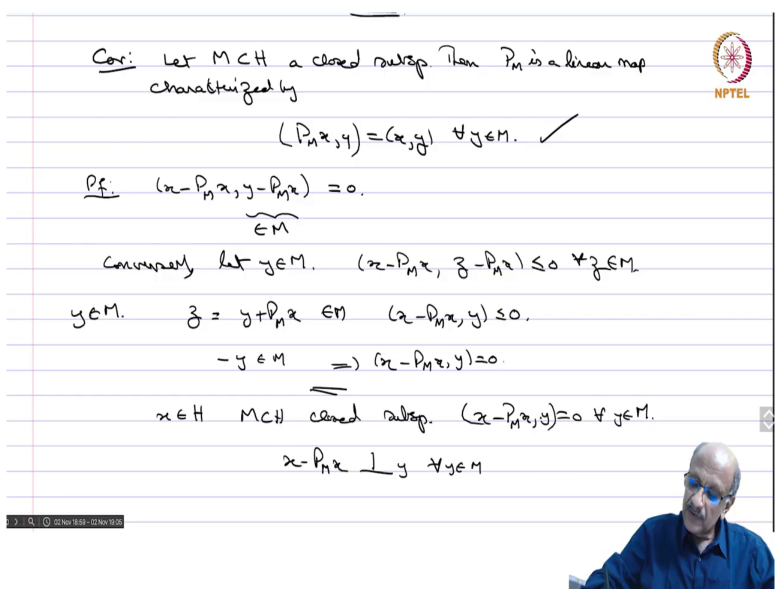
(Refer Slide Time: 11:50)



Proof. You know, $(x - P_K x,\ y - P_K x) \le 0$ for any $y \in K$. So, I am going to take here an element $P_K y \in K$. Similarly, you have $(y - P_K y,\ P_K x - P_K y) \le 0$. So, let us rewrite the second one as $(P_K y - y,\ P_K y - P_K x) \le 0$, so I have multiplied both the terms by -1, so that it will not change. So now, let us add these two. So, you have $(x - y,\ P_K y - P_K x) + ||P_K y - P_K x||^2 \le 0$.

So, $||P_K y - P_K x||^2 \le (y - x,\ P_K y - P_K x) \le ||y - x||\,||P_K y - P_K x||$ (by the Cauchy-Schwarz). So, I can cancel $||P_K y - P_K x||$, so that will give you $||P_K y - P_K x|| < ||y - x||$ and that proves this proposition.

(Refer Slide Time: 14:19)



Corollary. Let $M \subset H$ be a closed subspace (automatically convex). Then, P_M is a linear map characterized by

$$(P_M x,\ y) = (x,\ y)\ \ \forall\, y \in M$$

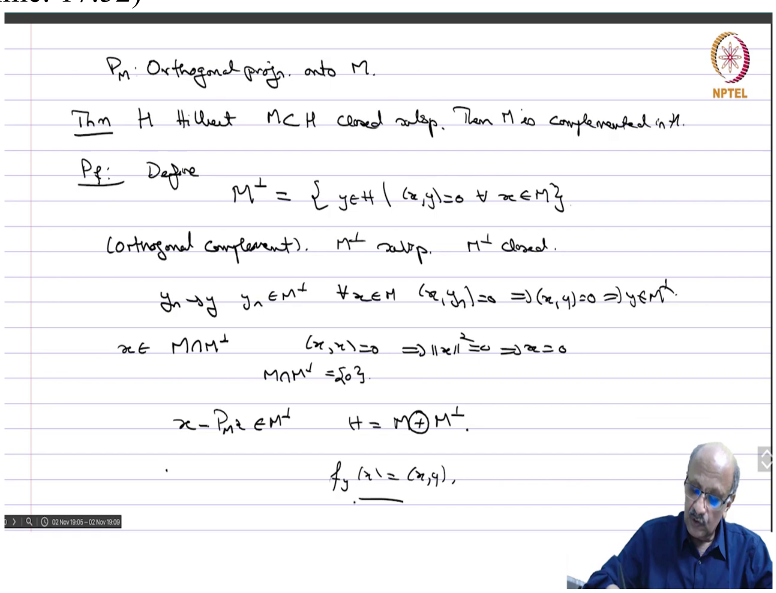
Proof. So if $(P_M x,\ y) = (x,\ y)$, then obviously

$(x - P_M x,\ y - P_M x) = 0\ ,\ as\ \ y - P_M x \in M$. So that way, you have this.

On the other hand, if you have, conversely, let $y \in M$. Then, we have to show that P_M satisfies this particular thing. So let me, what do we have? We have $(x - P_M x,\ z - P_M x) \leq 0\ \forall\, z \in M$. This is given to us. If you take any $y \in M$, you take $z = y + P_M x \in M$. So, I get $(x - P_M x,\ y) \leq 0.\ -y \in M$. So you have

$(x - P_M x,\ -y) \leq 0$, so this implies that $(x - P_M x,\ y) = 0$ and that is exactly this particular condition and that tells you immediately, once you have P_M is defined in this way, it is linear. So, you can easily check from this that P_M is a linear map. If $x \in H$ and M is a closed subspace, then what do you know about $x - P_M x$? $(x - P_M x,\ y) = 0\ \ \forall\, y \in M$. That means $x - P_M x \perp y,\ \forall\, y \in M$. That means the inner product is 0. That is what we mean by saying x and y are orthogonal.

(Refer Slide Time: 17:52)



P_M is called orthogonal projection onto M.

Theorem. H Hilbert space, $M \subset H$ is a closed subspace. Then, M is complemented in H. Recall that not every closed subspace can be complemented, that is what we saw and here we are proving that **in a Hilbert space, every closed subset is complemented.**

Proof. Define $M^{\perp} = \{y \in H \,/\, (x,\ y) = 0,\ \forall\, x\ \in M\}$. This is called an orthogonal complement. So, $M^{\perp}$ is clearly a subspace. Because if you have, it is linear in y and therefore automatically. So $M^{\perp}$ is a subspace. It is also closed. So, if $y_n \rightarrow y$, and $y_n \in M^{\perp}$, then $\forall\, x\ \in M$, you have $(x,\ y_n) = 0 \Rightarrow (x,\ y) = 0 \Rightarrow y \in M^{\perp}$. So, $M^{\perp}$ is a closed subspace. What about $M \cap M^{\perp}$? Let $x \in M \cap M^{\perp}$. Then, $(x,\ x) = 0\ \Rightarrow ||x||^2 = 0 \Rightarrow x = 0$. So, $M \cap M^{\perp} = \{0\}$. Then, we saw that $x - P_M x\ \in M^{\perp}$, we know all this and therefore $H = M \oplus M^{\perp}$. So, every closed subspace is complemented by it's orthogonal complement and therefore this proof is complete.

So, I have already defined a $\perp$ which is an annihilator. We will see actually that every continuous linear functional in H occurs as an inner product only and therefore this $M^{\perp}$, the orthogonal complement which I defined, is in fact nothing but the annihilator of M. So the two notations are not in conflict with each other. So, our next task is to look at the dual of a Hilbert space and show that every continuous linear functional in fact occurs like $f_y(x) = (x,\ y)$. So, this is what we will have to show in the next.