Functional Analysis
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Lecture 5
Continuous Linear Maps - Part 2

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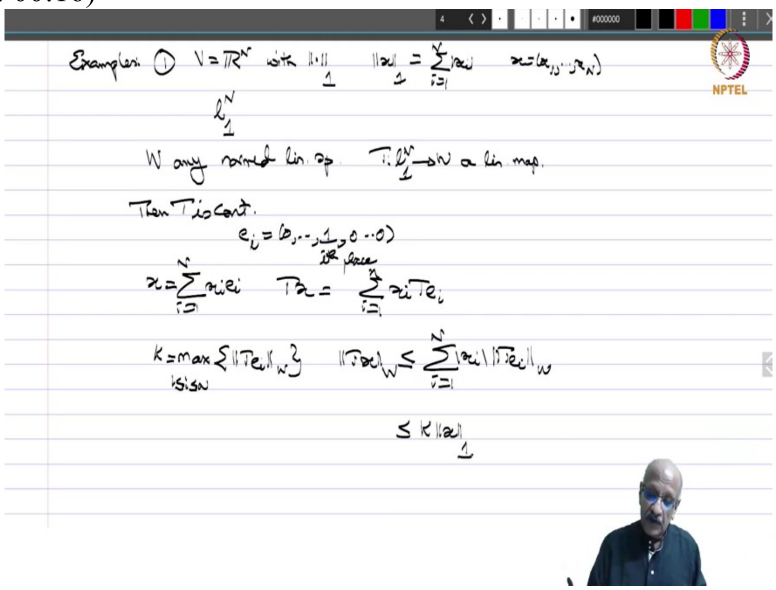

See some examples.

Example 1. Let us take $V=R^N$ with the $\|\cdot\|_1$. Recall that $\|x\|_1=\sum_{i=1,\dots,N}|x_i|$, where $x=(x_1,\dots,x_N)$. So, in short notation, R^N with $\|\cdot\|_1$ will be denoted as l_1^N.

Let W be any norm linear space, and $T: l_1^N \mapsto W$ be a linear map, then T is continuous. So, every linear map from l_1^N into any norm linear space is automatically continuous. So, how do we show this?

Let us take $e_i=(0,\dots,1(ith),..0)$ to be the standard basis vector. Then, every x can be written as $x=\sum_{i=1,..,N} x_i e_i$. Therefore, by linearity, $T(x)=\sum_{i=1,..,N} x_i T(e_i)$. Let us take $K=\max_{i=1,\dots,}\{\|T(e_i)\|\}$. Now, by the triangle inequality, we get that $\|T(x)\|_W \le \sum_{i=1,..,N} |x_i| \|T(e_i)\|_W \le K\|x\|_1$. Therefore, by the definition of continuity, T is a continuous map. So, this shows that every linear map from l_1^N to any norm linear space is automatically continuous.

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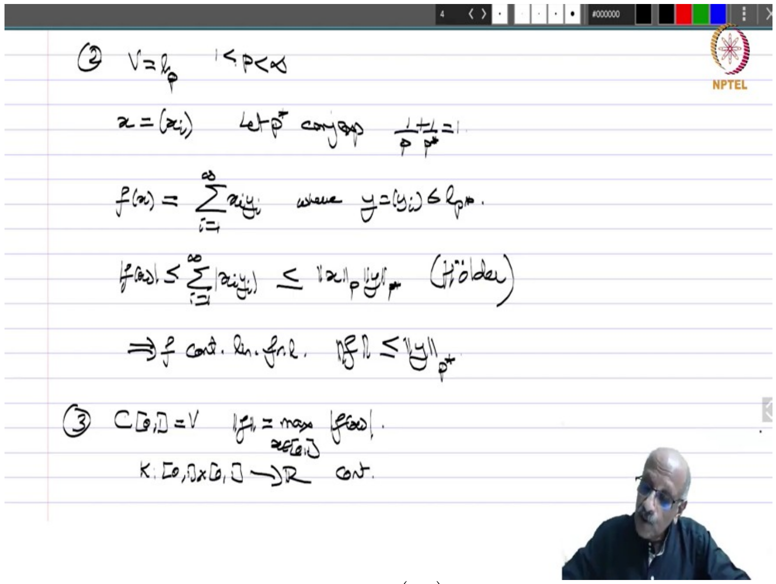

Example 2. Let us take $V = l_p, 1 < p < \infty$ and $x = (x_i)$ would be a sequence. And you know that $\sum_{i=1,\ldots,\infty} |x_i|^p < \infty$. Recall $p^{¿}$ is the conjugate exponent of p, that means $\frac{1}{p} + \frac{1}{p^{¿}} = 1$. Then define $f(x) = \sum_{i=1,\ldots,\infty} x_i y_i$ where $y = (y_i) \in l_{p^{¿}}$. Then, this is a linear functional. First of all, is it well defined? It is, because $|f(x)| \leq \sum_{i=1,\ldots,\infty} ¿ x_i y_i \vee ¿ \leq \|x\|_p \|y\|_{p^{¿}} ¿$(Holder's inequality).

Therefore, this is not only well defined, it tells you that this is a continues linear function. So this implies that f is a continuous linear functional and $\|f\| \leq \|y\|_{p^{¿}}$.

One of the theorems which we will prove in this course is that every continuous linear functional on $l_p$will occur in this way. This is the only way, these are the only functionals. And in fact, you have equality $\|f\| = \|y\|_{p^{¿}}$ and we can show that the dual space $l_p^{¿}$ is nothing but $l_{p^{¿}}$. That is why we have given the conjugate exponent's notation as $p^{¿}$. This is the theorem which we will prove later.

The above example gives you an example of a continuous linear functional. We have given examples in finite dimensions and example in sequence spaces. So, now let us look at function spaces.

Example 3. Let us take $V = C[0,1]$ to be a base space with $\|f\| = \max_{x \in [0,1]} ¿ f(x) \vee ¿¿$.

Now, let $K : [0,1] \times [0,1] \mapsto R$ be a continuous function i.e., we take a continuous function in two variables.

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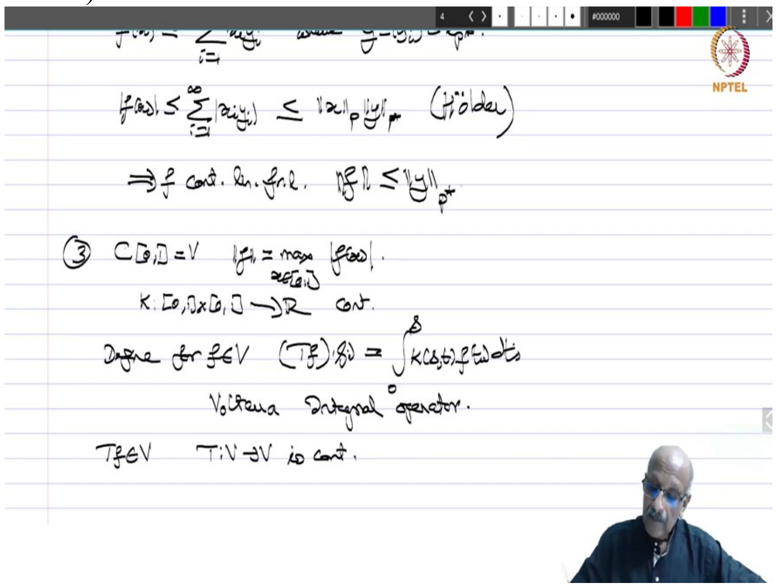

Then we define: for $f \in V$, I am going to define the following function.

$$(Tf)(s) = \int_{[0,s]} K(s,t) f(t) dt$$

This is called Volterra integral operator. We want to show two things. One is $Tf \in V$ and second is $T : V \mapsto V$ is continuous. Clearly, it is linear, so we want to show these two things.

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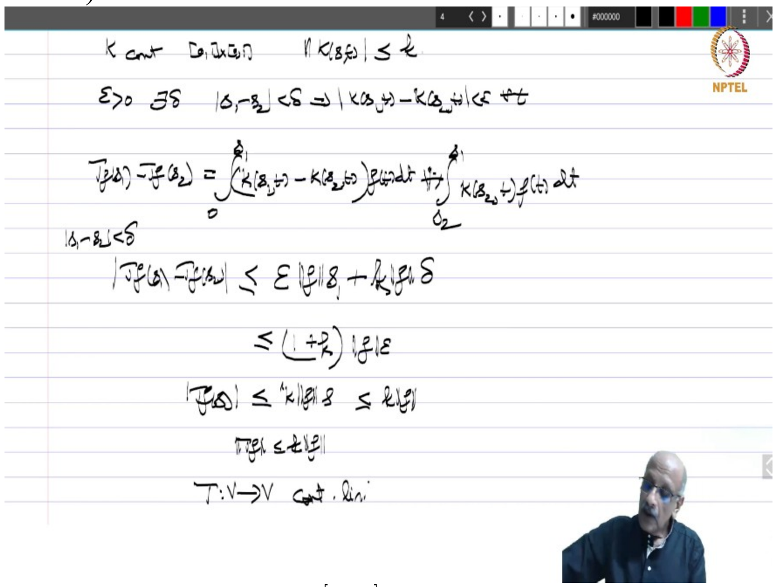

Since K is continuous on the compact set $[0,1]\times[0,1]$, it is uniformly continuous and bounded. Let us say we have the $|K(s,t)| \le k$ for some constant k. Also, given any $\epsilon > 0$, since it is uniformly continuous, therefore there exists $\delta > 0$ such that $¿ s_1 - s_2 \vee ¿ \delta$ implies $|K(s_1,t) - K(s_2,t)| < \epsilon$ for all t. In fact, much more is true. Here, I am using very little of the uniform continuity, you can also vary t within some similar δ, but I am going to take it only for a fixed t.

So, now we take

$$(Tf)(s_1) - (Tf)(s_2) = \int_{[0,s_1]} K(s_1,t) - K(s_2,t) f(t)\, dt + \int_{[s_2,s_1]} K(s_2,t) f(t)\, dt$$

So I have just added and subtracted things (if you write out the two formulae, you will get this).

Thus, if $|s_1 - s_2| < \delta$

$$|(Tf)(s_1) - (Tf)(s_2)| \le \epsilon \|f\| s_1 + k\|f\|\delta \le (1+k)\|f\|\epsilon.$$

This shows that Tf is a continuous function.

Also, we have that $\|(Tf)(s)\| \le k\|f\| s \le k \vee |f| \vee ¿$ t. If we take the maximum over s, we get that $\|Tf\| \le k\|f\|$. Therefore, T is continuous and linear.

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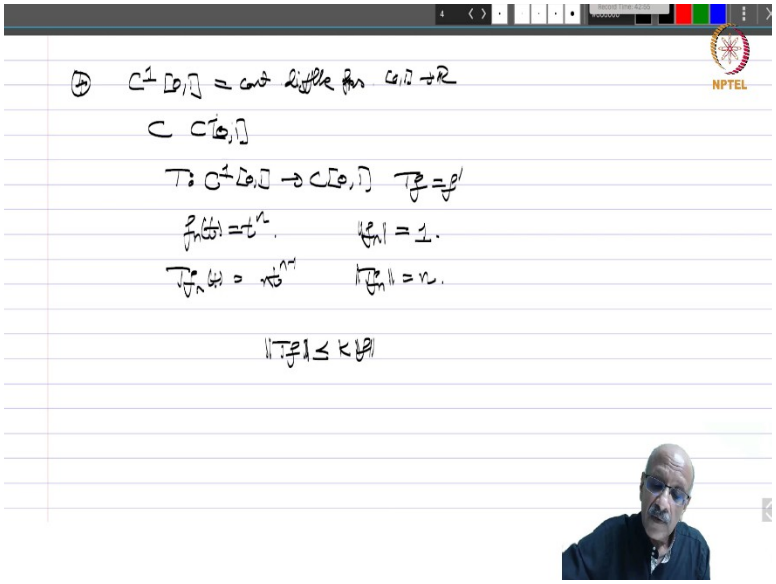

So we have seen some examples of mappings which are continuous linear mappings. Let me give you finally an example of a mapping which is not continuous but linear. Let me take $C^1[0,1]$, the space of continuously differentiable maps on [0,1] to R. Since this is a sub space of$C[0,1]$, we can put the same norm as in $C[0,1]$. Now we define $T: C^1[0,1] \mapsto C[0,1]$ (both with the spaces are equipped with the same sup norm). Define $T(f)=f'$, the first derivative of f. Since every function in $C^1[0,1]$ is differentiable and the derivative is continuous, so this makes sense.

I want to show that this is a linear map but this is not continuous. Let us take $f_n(t)=t^n$. Then what is $(f_n)=f'_n$? $T(f_n)=n t^{n-1}$. Now, $\|f_n\|=1$ and ¿

So, we we can never have an inequality like $\|T(f)\| \leq k\|f\|$. Because if we put $f=f_n$, then $n \leq k$ for all n. That is impossible because $n \to \infty$, but k is a fixed number. So, this is an example of a mapping which is not continuous mapping but which is nevertheless, linear. Now we will look at some other properties like isomorphism between norm linear spaces.