Functional Analysis
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Lecture No. 49
Hilbert Spaces - Part 1

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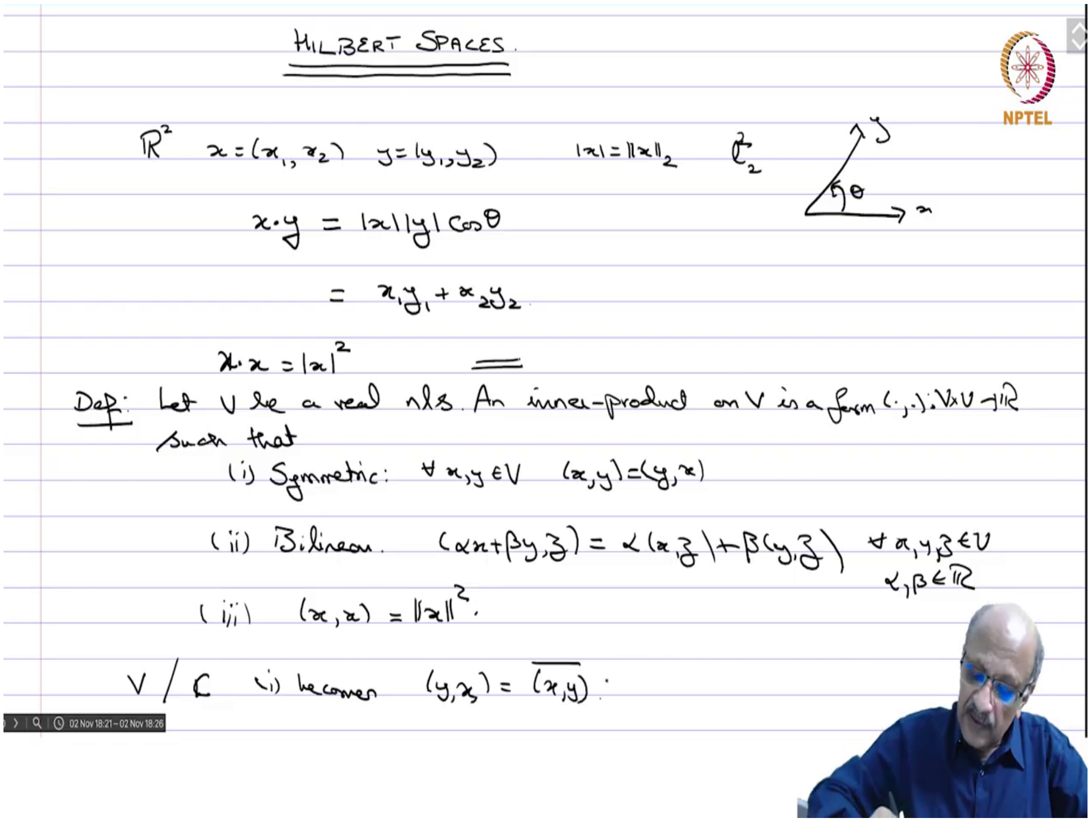

We will now start a very important chapter. This is called Hilbert Spaces. So, Banach spaces and Hilbert spaces dominate functional analysis. A Hilbert space is a Banach space with some geometry built in. In particular, we say when two vectors are orthogonal to each other or at right angles to each other, so we in fact, we introduce a notion of an angle.

How is it defined in $\mathbb{R}^2$? For instance if you take the plane $\mathbb{R}^2$ and you have two vectors, $x = (x_1, x_2)$, and $y = (y_1, y_2)$, then $|x| = ||x||_2$, that is the norm in l_2^2,. Then, what do you have? You have what is called the inner product $x\,.\,y = ||x||\,||y||\cos\theta$.

So, you say two vectors are orthogonal to each other, if θ is the angle between them is $\frac{\pi}{2}$. That is, $\cos(\frac{\pi}{2}) = 0$, so if $x \,.\, y = 0$, we say two vectors are orthogonal and $x \,.\, y$ can also be written as $x_1 y_1 + x_2 y_2$, and we have $x \,.\, x = |x|^2$, that is, the inner product generates the norm and it is linear in each of the variables. It is linear in x and y. So, we generalize all these things and define an inner product.

Definition: Let V be a real non linear space and inner product on V is a function $(.\,,\,.)$: $V \times V \rightarrow \mathbb{R}$ such that

(i) it is symmetric, that is, for every $x, y \in V$, you have $(x, y) = (y, x)$.

(ii) it is bilinear, that means $(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$ for all $x, y, z \in V$ and for all $\alpha, \beta \in \mathbb{R}$. It is also linear in the second variable because you know, you can either change by symmetry, it also means that it is linear in the second variable.

(iii) $(x, x) = ||x||^2$.

Such a bilinear form is called an inner product. So, now what happens if you have complex, so if V is over $\mathbb{C}$, then (i) becomes $(x, y) = \overline{(y, x)}$.

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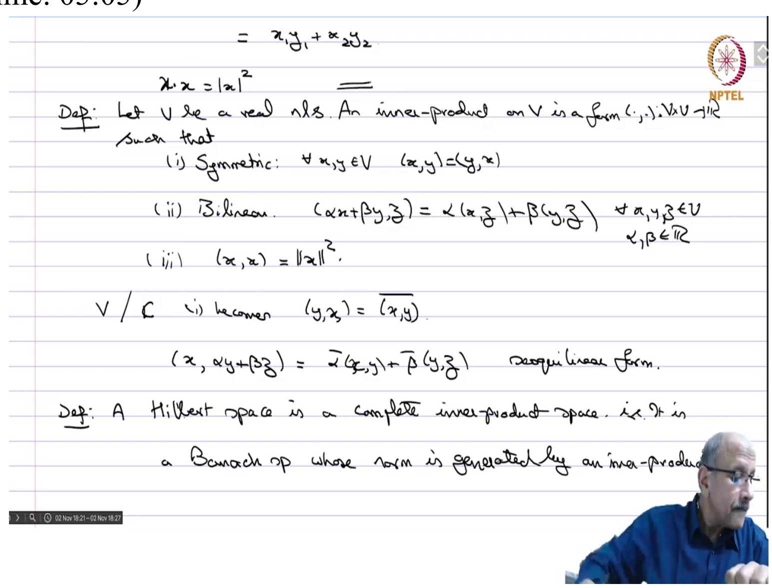

It is linear in the first variable and therefore this means we have $(x, \alpha x + \beta y) = \overline{\alpha}(x, z) + \overline{\beta}(y, z)$. So it is linear in the first variable. It is conjugate linear in the second variable and such a form is called a sesquilinear form. So that is, these are the changes you have to make when you are dealing with $\mathbb{C}$.

Definition. A Hilbert space is a complete inner product space, that is it is a Banach space whose norm is generated by an inner product.

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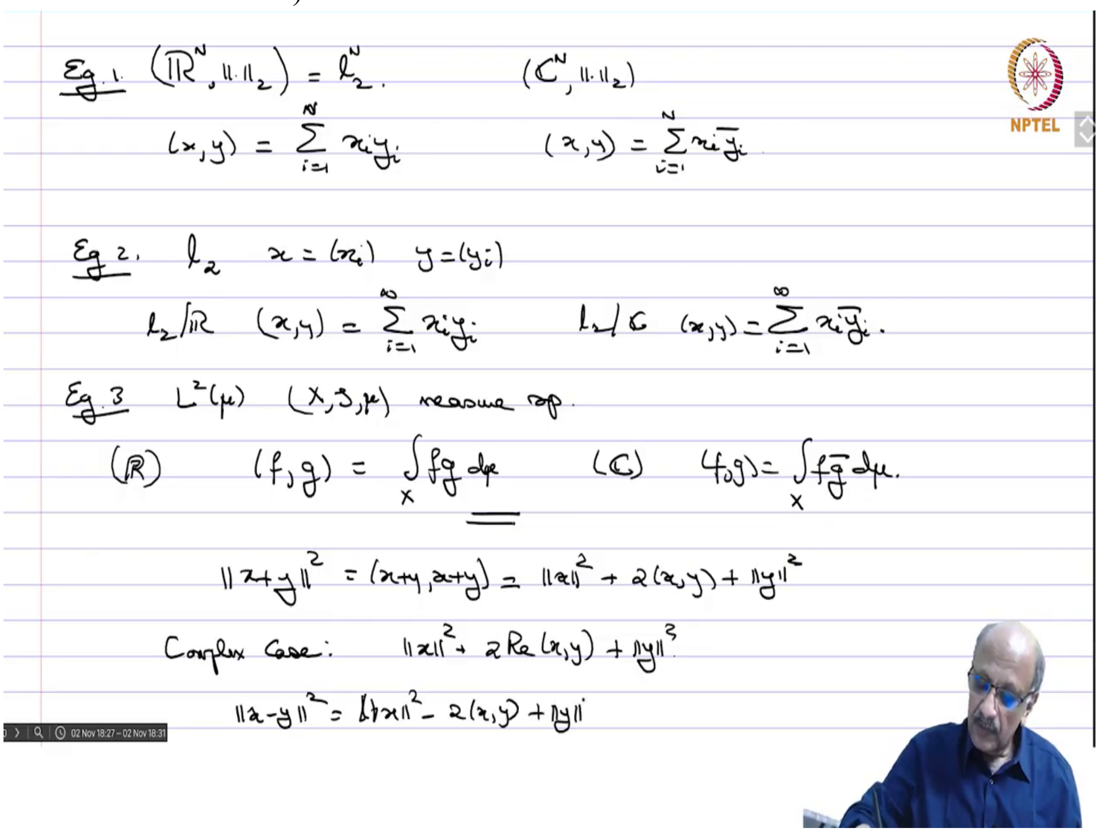

Example 1: We have $(\mathbb{R}^N, ||.||_2)$. This is the space l_2^N. So, you define the inner product $(x, y) = \sum_{i=1}^{n} x_i y_i$. If you take $(\mathbb{C}^N, ||.||_2)$, then you define $(x, y) = \sum_{i=1}^{n} x_i \overline{y_i}$. So, this is obviously the inner product because you have $(x, x) = ||x||^2$ and the other properties are immediate to verify.

Example 2: l_2 again, if you have $x = (x_i)$, $y = (y_i)$. Then if l_2 is over $\mathbb{R}$, then you define the inner product as $(x, y) = \sum_{i=1}^{\infty} x_i y_i$ and this is well defined, it is convergent because by Holder's inequality $(x, y) \leq ||x||\,||y||$ and therefore this is well defined and then if you have l_2 over $\mathbb{C}$, then you define $(x, y) = \sum_{i=1}^{\infty} x_i \overline{y_i}$.

Example 3: $L^2(\mu)$. So, if you have (X, S, μ) is a measure space, then $L^2(\mu)$ is Hilbert space, well it is complete and now the inner product, you can easily guess is nothing but $(f, g) = \int_X f\, g\, d\mu$, if the base field is $\mathbb{R}$. If the base field is $\mathbb{C}$, you have $(f, g) = \int_X f\, \overline{g}\, d\mu$. So this gives you Hilbert spaces. So now, I will mostly deal with reals, wherever complex needs to be mentioned, I will tell you what is the change to make.

So, $||x + y||^2 = (x + y, x + y) = ||x||^2 + 2(x, y) + ||y||^2$, you use linearity and develop this. In the complex case, you have

$||x + y||^2 = ||x||^2 + (x, y) + \overline{(x, y)} + ||y||^2 = ||x||^2 + 2\,\Re(x, y) + ||y||^2$.

So now, if I similarly write $||x - y||^2 = ||x||^2 - 2(x, y) + ||y||^2$ (real case) and

$||x - y||^2 = ||x||^2 - 2\Re(x, y) + ||y||^2$ (complex case).

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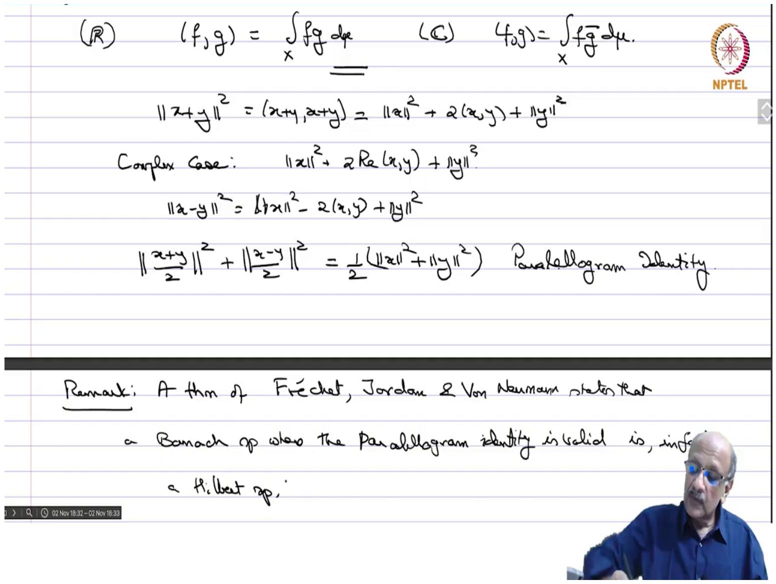

If you add these two, you get $\left|\left|\frac{x+y}{2}\right|\right|^2 + \left|\left|\frac{x-y}{2}\right|\right|^2 = \frac{1}{2}\left(||x||^2 + ||y||^2\right)$, this is what is called the parallelogram identity, and this in 2 dimensions is called the Apollonius theorem and we have seen this.

Remark. A theorem of Frechet Jordan and Von Neumann states that a Banach space where the parallelogram law identity is valid, is in fact a Hilbert Space. That means you can write down an inner product which generates a norm. Once you have the parallelogram identity, you can use it to do it.

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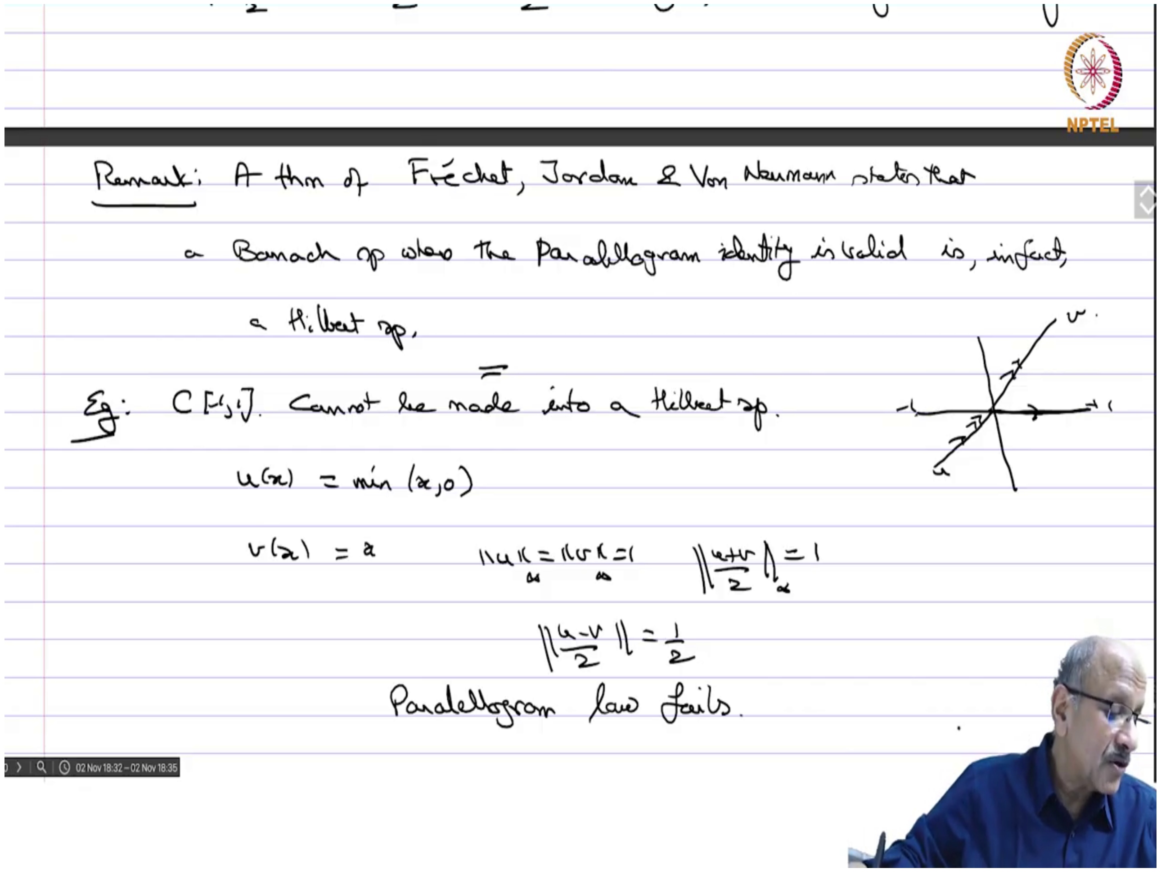

Example: $C[-1, 1]$ is a Banach space with a sup norm and cannot be made into a Hilbert space. You can do it with $C[0, 1]$ but it is easier to write it with $C[-1, 1]$. That means the parallelogram law will fail. So, you define $u(x) = min\,(x, 0)$. So that means, how does a graph of $u(x)$ look like? When x is negative, it takes the value x and then when x becomes positive, then it is 0. Then $v(x) = x$. So, these are the two functions u and v. Then $||u||_{\infty} = ||v||_{\infty} = 1$, you can easily see. So $\left|\left|\frac{u+v}{2}\right|\right|_{\infty} = 1$ and $\left|\left|\frac{u-v}{2}\right|\right| = \frac{1}{2}$

and you see that the parallelogram law fails for these pairs of functions.

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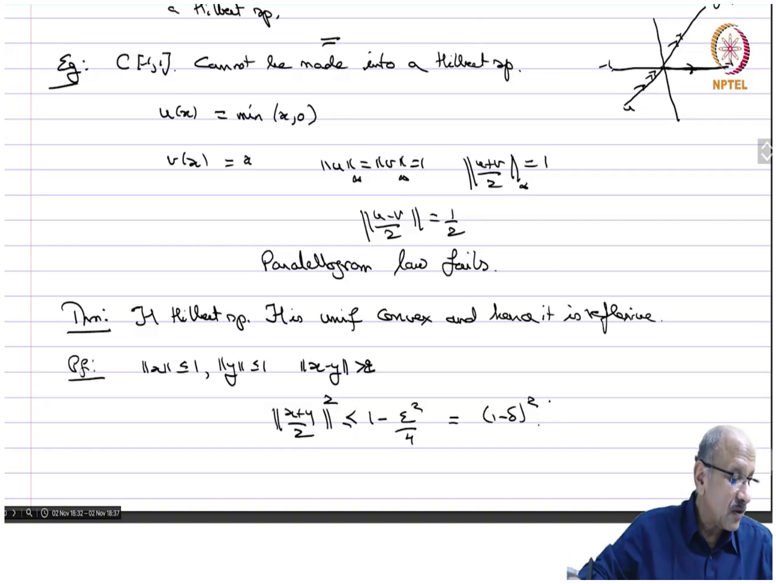

Theorem. H Hilbert. H is uniformly convex and hence, it is reflexive.

Proof. $||x|| \leq 1$ and $||y|| \leq 1$, $||x - y|| > \epsilon$, then from the parallelogram law, you will get $\left|\left|\frac{x+y}{2}\right|\right|^2 \leq 1 - \frac{\epsilon^2}{4} = (1 - \delta)^2$ for suitable ϵ and therefore you have uniform convexity and we know that every uniformly convex space is a reflex. So, every Hilbert space is reflexive.

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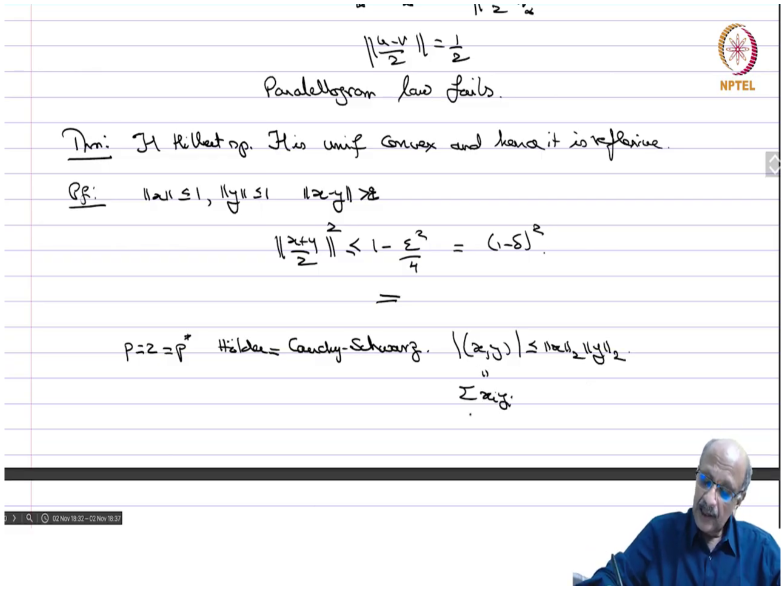

Now, when $p = 2 = p^*$, Holderz Cauchy-Schwartz and what does it mean? In the notation which I wrote earlier, $|(x, y)| \leq ||x||_2 \, ||y||_2$. So, this is the Holder inequality applied to $p = 2 = p^*$ and $(x, y) = \sum x_i y_i$. $i = 1$ to n or $i = 1$ to ∞, whatever space you are working with. So, this is in fact a fundamental property of Hilbert spaces. So, we have the following theorem.

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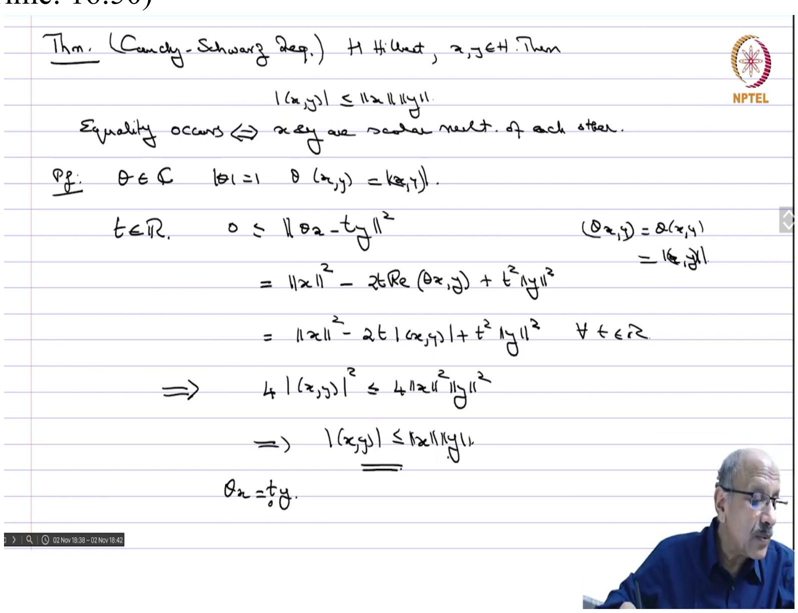

Theorem: (Cauchy-Schwarz inequality) H is a Hilbert space and $x, y \in H$. Then, $|(x, y)| \leq ||x||\,||y||$. This is a very fundamental inequality which will be used again and again and again. Equality occurs if and only if x and y are scalar multiples of each other.

Proof. We will do it for the complex case, the real case will be a particular case of that. So, let us assume, $\theta \in \mathbb{C}$ such that $|\theta| = 1$ and $\theta(x, y) = |(x, y)|$. We have done this before, in various examples. Given a complex number can be written as $r\, e^{i\theta}$. So any complex number can be written as a number with modulus 1.

If $t \in \mathbb{R}$, we have $0 \leq ||\theta\, x - t\, y||^2$. So, let us expand this. This will give you $||x||^2 - 2t\, \Re(\theta\, x, y) + t^2 ||y||^2$ as $|\theta| = 1$. Now, what is $\Re(\theta\, x, y)$? So, $(\theta\, x, y) = \theta(x, y) = |(x, y)|$, it is a real number anyway. So,

$||\theta\, x - t\, y||^2 = ||x||^2 - 2t\, |(x, y)| + t^2 ||y||^2 \geq 0$. So, you have a quadratic polynomial which is non negative, which is of constant sign. That means, you are having a parabola which never changes sign. That means it does not have any real roots, that means its roots are imaginary. So, this is true for all $t \in \mathbb{R}$ and therefore, you have that the roots of this quadratic should be imaginary. So, $b^2 < 4ac$, that means $4|(x, y)|^2 \leq 4\, ||x||^2\, ||y||^2$ and that implies $|(x, y)|^2 \leq ||x||^2\, ||y||^2$. So, that proves the Cauchy-Schwarz inequality which is a fundamental inequality as I said.

So, if you want equality in this, you should have that the roots should be equal. That means $\theta\, x$ should be equal to $t_0\, y$, where t_0 is a root of this quadratic polynomial and that proves whatever we wanted to prove. Because you want equality means, $||x||^2 - 2t\, |(x, y)| + t^2 ||y||^2 = 0$ only then, so that means t must be a root of this thing and it should be a perfect square and therefore you have $\theta\, x = t_0 y$.

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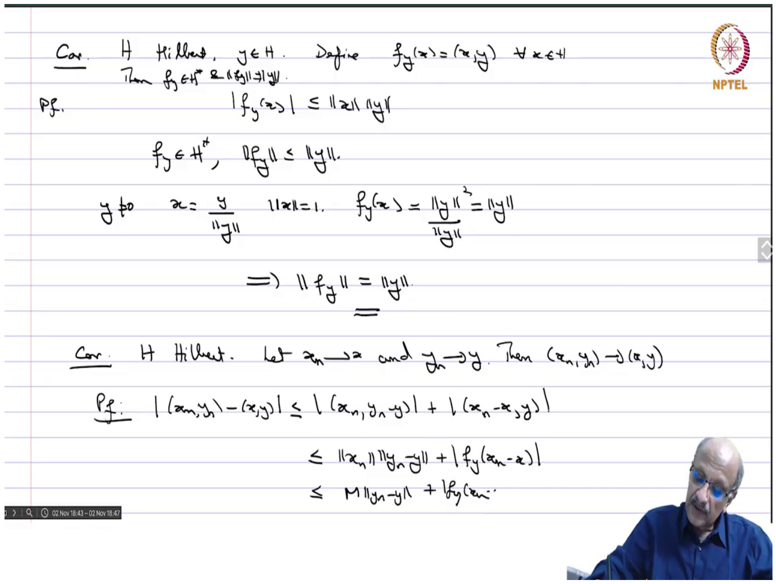

Corollary. H Hilbert and $y \in H$. So, define $f_y(x) = (x, y)$ $\forall\, x \in H$. Now, I am writing in this fashion with y in the second coordinate, so that it is linear in the first coordinate and therefore even in the complex case or real case, it does not matter which, this is always a linear functional. Therefore, by the Cauchy-Schwarz $|f_y(x)| \le ||x||\,||y||$

Therefore, $f_y \in H^*$ and $||f_y|| \le ||y||$. So, assume that $y \neq 0$ and you take $x = \frac{y}{||y||}$. So, $||x|| = 1$ and $f_y(x) = (\frac{y}{||y||}, y) = \frac{||y||^2}{||y||} = ||y||$. So, the maximum is reached on the unit sphere and this implies $||f_y|| = ||y||$. So, this is the proof. Let me state the theorem. $f_y \in H^*$ and $||f_y|| = ||y||$. So, that was the corollary and I have already proved it.

So, we will see in the next chapter that every continuous linear functional, in fact arises in this fashion. So, H is isomorphic to its own dual and so very often, especially in the real case we can identify the two.

Corollary. H Hilbert. Let x_n weakly converge to x and y_n converge to y. Then, (x_n, y_n) converges to (x, y). So, when you have two, an inner product and one term converges weakly and one term converges strongly, then the limit is still natural, whatever you expect. So, if both of them are weak, then you cannot expect it.Like, for instance in L_p

spaces also we saw an example. cos nt went weakly to 0, but $\cos^2 nt$ went weakly to $\frac{1}{2}$. So, you cannot expect normally, when you are dealing with the product of weak convergence to behave properly, you do not know what will happen but if one is weak and one is strong, then you are on safe ground.

Proof. $|(x_n, y_n) - (x, y)| \leq |(x_n, y_n - y)| + |(x_n - x, y)|$.

Now, by Cauchy-Schwarz inequality, $|(x_n, y_n - y)| \leq ||x_n|| \, ||y_n - y||$.

$|(x_n, y_n) - (x, y)| \leq ||x_n|| \, ||y_n - y|| + |f_y(x_n - x)| \leq M||y_n - y|| + |f_y(x_n - x)|$

as every weakly convergence sequence is bounded.

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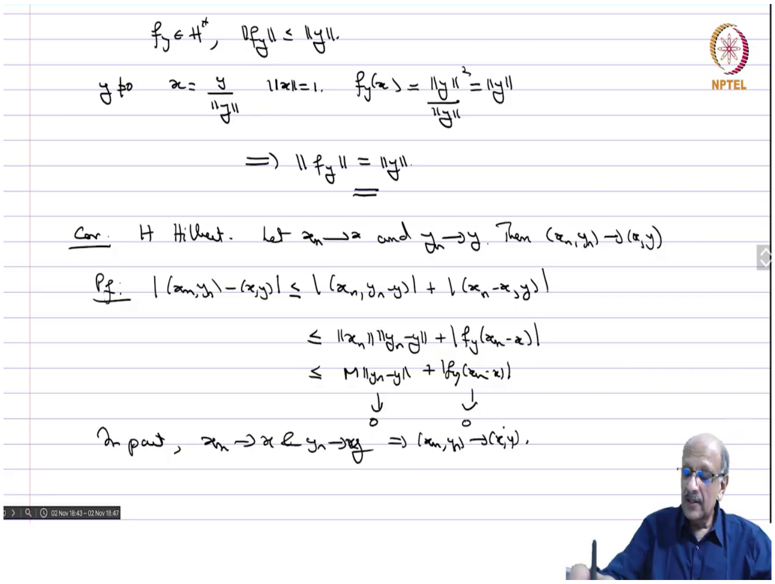

The first one goes to 0 because y_n converges to y strongly and the second term goes to 0 because x_n converges to x weakly and therefore (x_n, y_n) converges to (x, y). In particular, if $x_n \to x$ and $y_n \to y$, both of them are strong, then $(x_n, y_n) \to (x, y)$. That is now obvious that you could have done it in many ways directly. But now you know if it converges in norm, then it converges weakly. So, $x_n \to x$ weakly as well and $y_n \to y$ in norm and therefore you have this.