Functional Analysis
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Lecture No. 48

Exercises Part – 4

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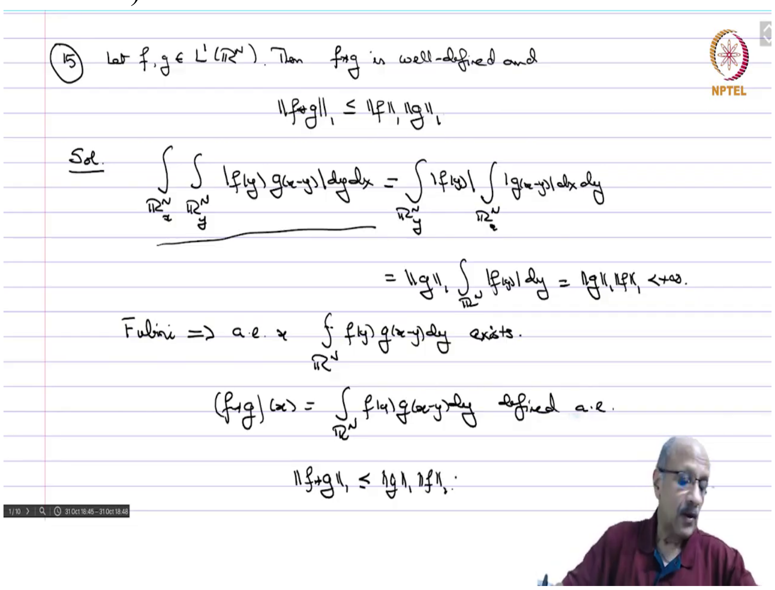

15. Let $f, g \in L^1(\mathbb{R}^N)$. Then, $f * g$ is well defined and $||f * g||_1 \leq ||f||_1 ||g||_1$. So, this is Young's Inequality when $p = 1$. We have already done when it is p, so this is easier, in fact.

Solution. We look at the integral, $\int_{\mathbb{R}^N_x} \int_{\mathbb{R}^N_y} |f(y)\, g(x - y)|\, dy\, dx$, everything is non-negative in the integrand, so you can interchange the order of integration.

$$\int_{\mathbb{R}^N_x} \int_{\mathbb{R}^N_y} |f(y)\, g(x - y)|\, dy\, dx = \int_{\mathbb{R}^N_y} |f(y)| \int_{\mathbb{R}^N_x} |g(x - y)|\, dx\, dy\,.$$

So, y is a constant as far as the inner integral is concerned and $g(x - y)$ is just a translated g and therefore by the translation invariance of Lebesgue measure, $\int_{\mathbb{R}^N_x} |g(x - y)|\, dx$ is in fact equal to $||g||_1$. It is a constant, it will come out. Therefore,

$$\int_{\mathbb{R}^N_y} |f(y)| \int_{\mathbb{R}^N_x} |g(x - y)|\, dx\, dy = ||g||_1 \int_{\mathbb{R}^N_y} |f(y)|\, dy = ||g||_1 ||f||_1 < \infty.$$

So, with the absolute value this is integrable. Therefore, by Fubini's theorem, for almost every x, $\int_{\mathbb{R}^N} f(y)\, g(x - y)\, dy$ exists and therefore for almost every x,

$(f * g)(x) = \int_{\mathbb{R}^N} f(y)\, g(x - y)\, dy$ is defined almost everywhere and of course

$|(f * g)(x)| \le \int_{\mathbb{R}^N_x} \int_{\mathbb{R}^N_y} |f(y)\, g(x - y)|\, dy\, dx$, therefore you have

$||f * g||_1 \le ||g||_1 ||f||_1$. So, that completes that exercise. So, this is much easier than the case for one of the functions is in L^p.

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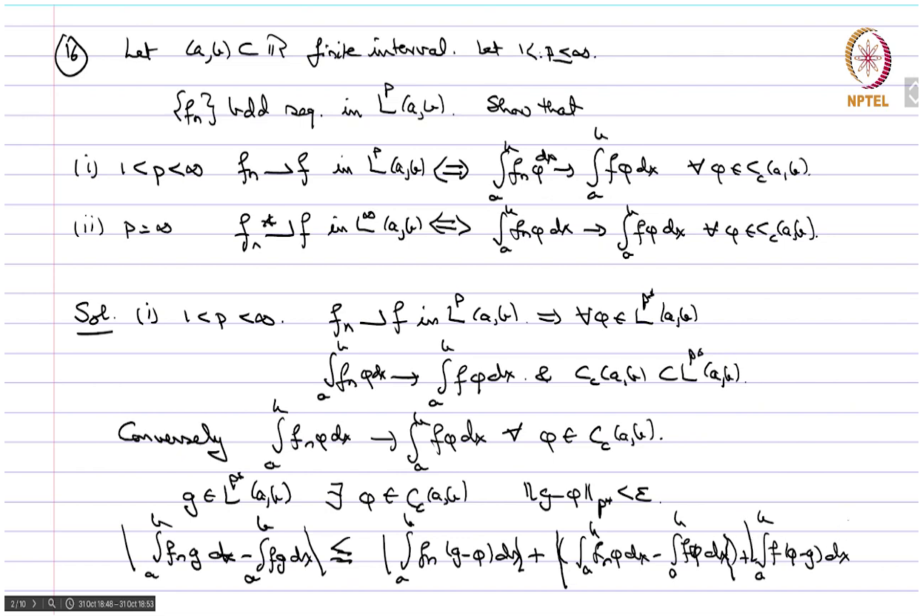

Problem 16. Let $(a, b) \subset \mathbb{R}$, finite interval. Let $1 < p \leq \infty$ and $\{f_n\}$ bounded sequence in $L^p(a, b)$. Then show that

(i) $1 < p < \infty$, f_n weakly converges to f in $L^p(a,b) \Leftrightarrow \int_a^b f_n \, \phi \, dx \to \int_a^b f \, \phi \, dx$, $\forall \phi \in C_c(a, b)$. (ii) $p = \infty$, so $f_n \to f$ weak star in $L^\infty(a, b) \Leftrightarrow \int_a^b f_n \, \phi \, dx \to \int_a^b f \, \phi \, dx$, $\forall \phi \in C_c(a, b)$

Solution. Let us look at $1 < p < \infty$. So, let us assume that f_n weakly converges to f in L^p. So, this implies for every $\phi \in L^{p^*}(a, b)$, you have $\int_a^b f_n \, \phi \, dx \to \int_a^b f \, \phi \, dx$

and, in particular, $C_c(a, b) \subset L^{p^*}$. It is in any L^p space and therefore in particular you have that for every ϕ, this happens. Conversely, let $\int_a^b f_n \, \phi \, dx \to \int_a^b f \, \phi \, dx$,

$\forall \phi \in C_c(a, b)$. Then, given any $g \in L^{p^*}(a, b)$, there exists $\phi \in C_c(a, b)$, such that $||g - \phi||_{p^*} < \epsilon$. So,

$$\left|\int_a^b f_n g\, dx - \int_a^b f g\, dx\right| \le \left|\int_a^b f_n (g - \phi)\, dx\right| + \left|\int_a^b f_n \phi\, dx - \int_a^b f \phi\, dx\right| + \left|\int_a^b f(\phi - g) dx\right|$$

$$\le ||f_n||_p \, ||g - \phi||_{p^*} + \left|\int_a^b f_n \phi\, dx - \int_a^b f \phi\, dx\right| + ||f||_p \, ||\phi - g||_{p^*}$$

(f_n is given to be a bounded sequence in L^p, i.e. $||f_n||_p \le M$ and by the Holder inequality)

$$\le M\epsilon + \left|\int_a^b f_n \phi\, dx - \int_a^b f \phi\, dx\right| + ||f||_p\, \epsilon.$$

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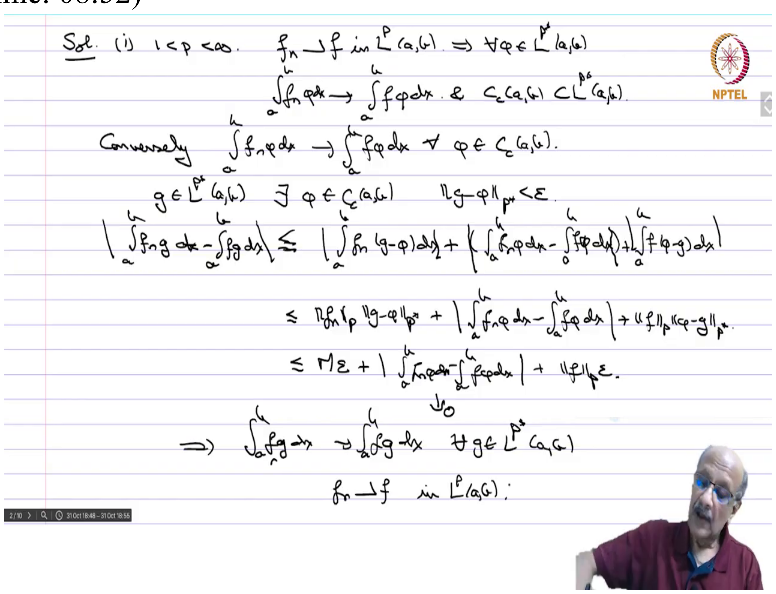

Given that $\left|\int_a^b f_n \phi\, dx - \int_a^b f \phi\, dx\right| \to 0$. Therefore everything can be made arbitrarily small, so this implies that $\int_a^b f_n g\, dx \to \int_a^b f g\, dx$, $\forall g \in L^{p^*}(a, b)$ and that is, f_n weakly converges to f in L^p. So, that proves (i).

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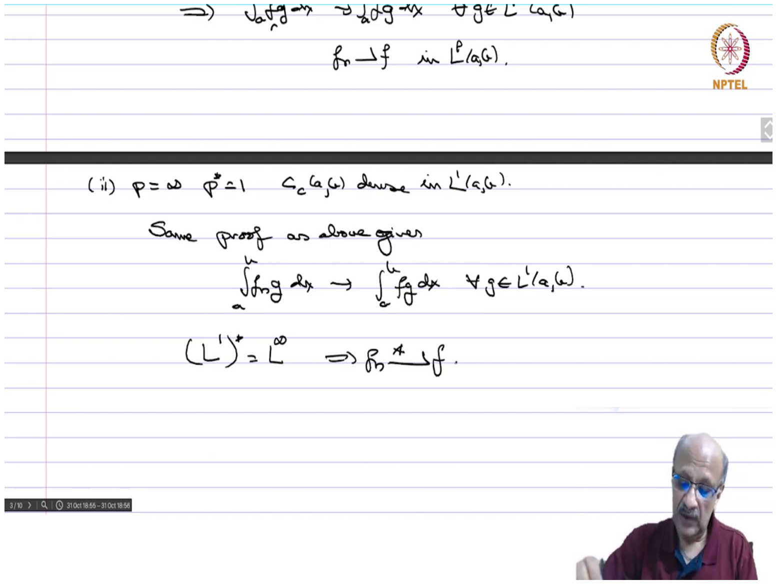

(ii) $p = \infty$ so $p^{*} = 1$ and you have $C_{c}(a, b)$ is dense in L^{1}. So, same proof as above and therefore gives $\int_{a}^{b} f_{n}\, g\, dx$ goes to $\int_{a}^{b} f\, g\, dx$ $\forall g \in L^{1}(a, b)$. We recall that $(L^{1})^{*} = L^{\infty}$ and therefore if you are doing this, $f_{n} \in L^{\infty}$ and if this happens in the pre dual space, then this implies that $f_{n} \rightarrow f$ in weak star. So, this is the characterization we had and therefore we have this.

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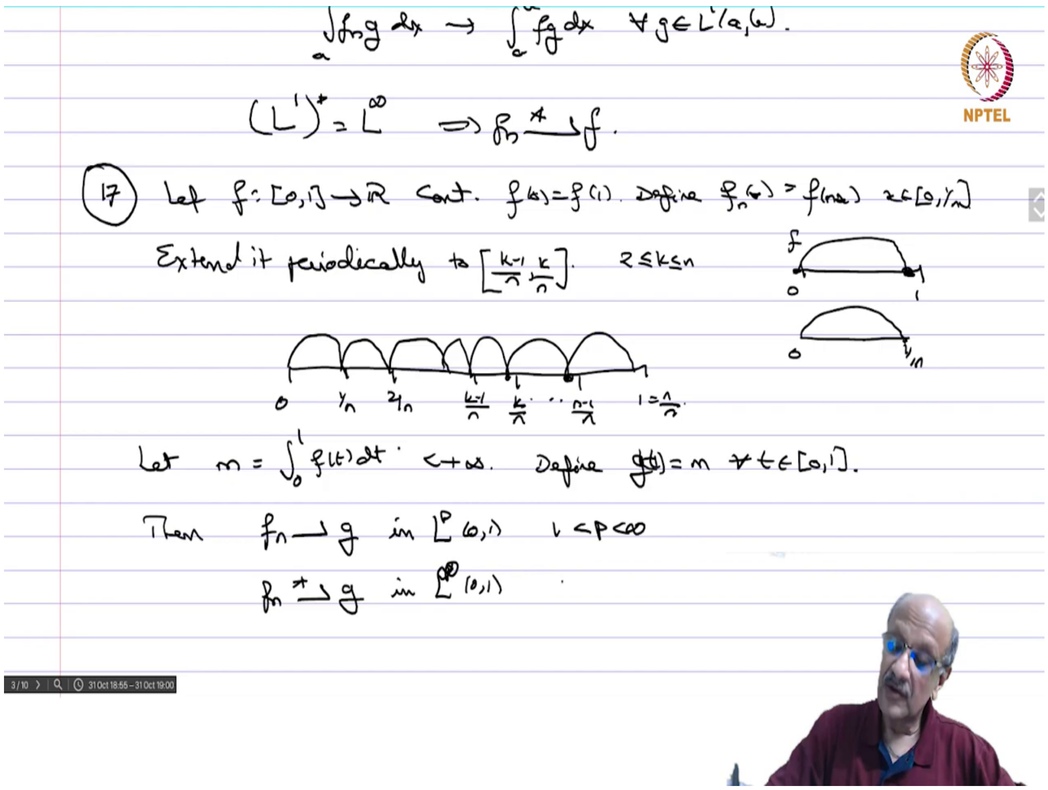
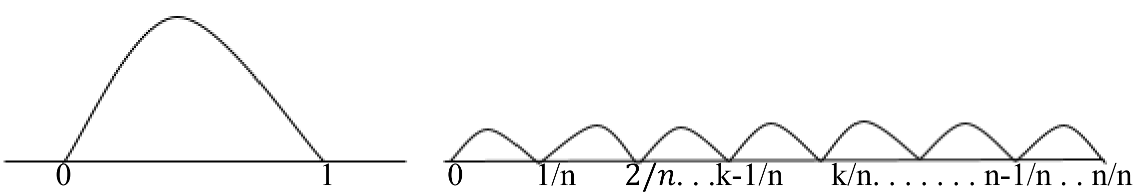


Problem 17. Let f: $[0, 1] \rightarrow \mathbb{R}$ continuous and $f(0) = f(1)$. So it is a periodic function. Define $f_n(x) = f(nx)$, $x \in [0, 1/n]$. So we are rescaling the same function. So I have a function f which is say, something like this. The end values need not be 0.



So now, we are taking $[0, 1]$ and dividing it into n equal parts, as shown here and then reproduce the same function in each interval $[\frac{k-1}{n}, \frac{k}{n}]$. So, after scaling you simply move that function. So, let $m = \int_0^1 f(t)\, dt$ which of course will be finite because f is a continuous function. Then, define $g(t) = m$, $\forall t \in [0, 1]$. Then, f_n is of course in all the L^p spaces because it is a continuous function on a compact set and that is why we took periodic, because when we take the same value at the end points and then you repeat, then it becomes a continuous function. So, f is a continuous function on a compact set, therefore it is in all the L^p spaces and therefore you have fn weakly converges to g in $L^p(0, 1)$, $1 < p < \infty$ and f_n weak star converges to g in $L^\infty(0, 1)$. So this function which we have scaled and reproduced, converges in L^p either weakly or in weak star to the average value because $\int f(t)\, dt$ is nothing but the integral of f divided by the length of the interval, so this is nothing but the mean value or the average value of the function in this. So, this periodic scaling and periodic repetition, if you do, then that sequence converges to the average value of f. So that is, in the weak or weak star topology depending on the space.

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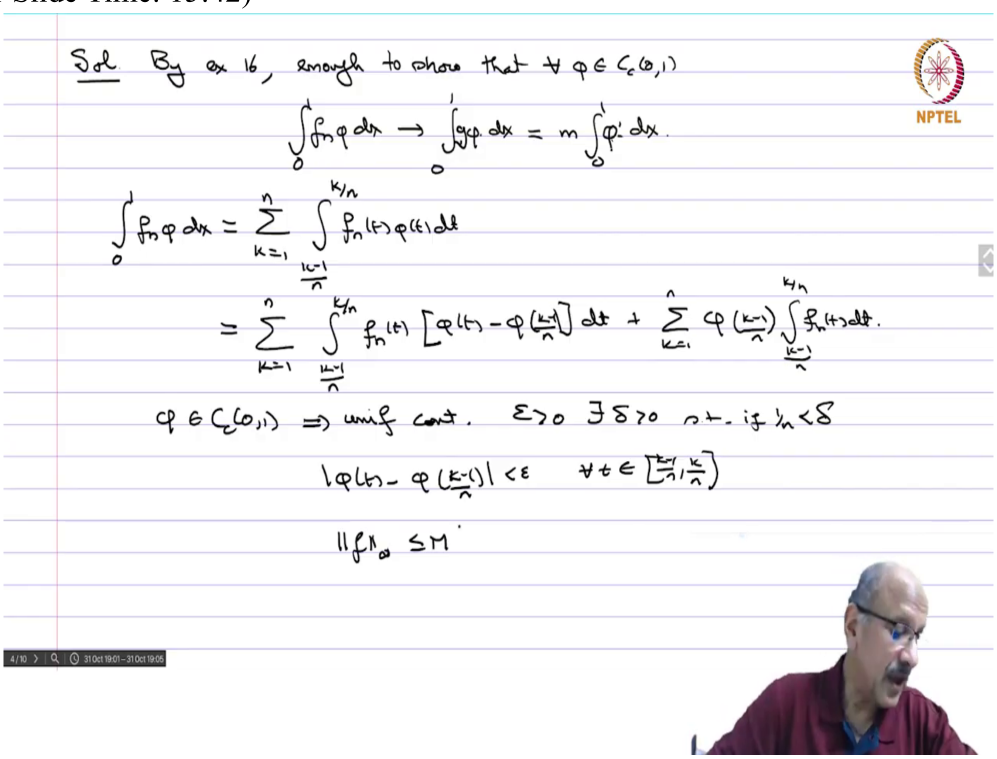

Solution. By exercise 16, enough to show that $\forall\ \phi \in C_c(0, 1)$, you have

$$\int_0^1 f_n \phi\, dx \to \int_0^1 g\, \phi\, dx = m \int_0^1 \phi\, dx$$

So, let us compute $\int_0^1 f_n \phi \, dx$. I am going to split into the various intervals.

$$\int_0^1 f_n \phi \, dx = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f_n(t)\phi(t) \, dt = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f_n(t)\left[\phi(t) - \phi(\tfrac{k-1}{n})\right] dt + \sum_{k=1}^{n} \phi(\tfrac{k-1}{n}) \int_{\frac{k-1}{n}}^{\frac{k}{n}} f_n(t) \, dt$$

Now, ϕ is continuous with compact support. Therefore, it is uniformly continuous. Therefore, given $\epsilon > 0$ $\exists\, \delta > 0$ such that, if $\frac{1}{n} < \delta$, we have $\left|\phi(t) - \phi(\frac{k-1}{n})\right| < \epsilon \quad \forall\, t \in [\frac{k-1}{n}, \frac{k}{n}]$. Because if $t \in [\frac{k-1}{n}, \frac{k}{n}]$, $t - \frac{k-1}{n} < \frac{1}{n} < \delta$, $\frac{1}{n}$ is the length of this interval. Therefore, $\left|\phi(t) - \phi(\frac{k-1}{n})\right| < \epsilon$ and also you have that $||f||_\infty \leq M$, say. So, let us look at the first term.

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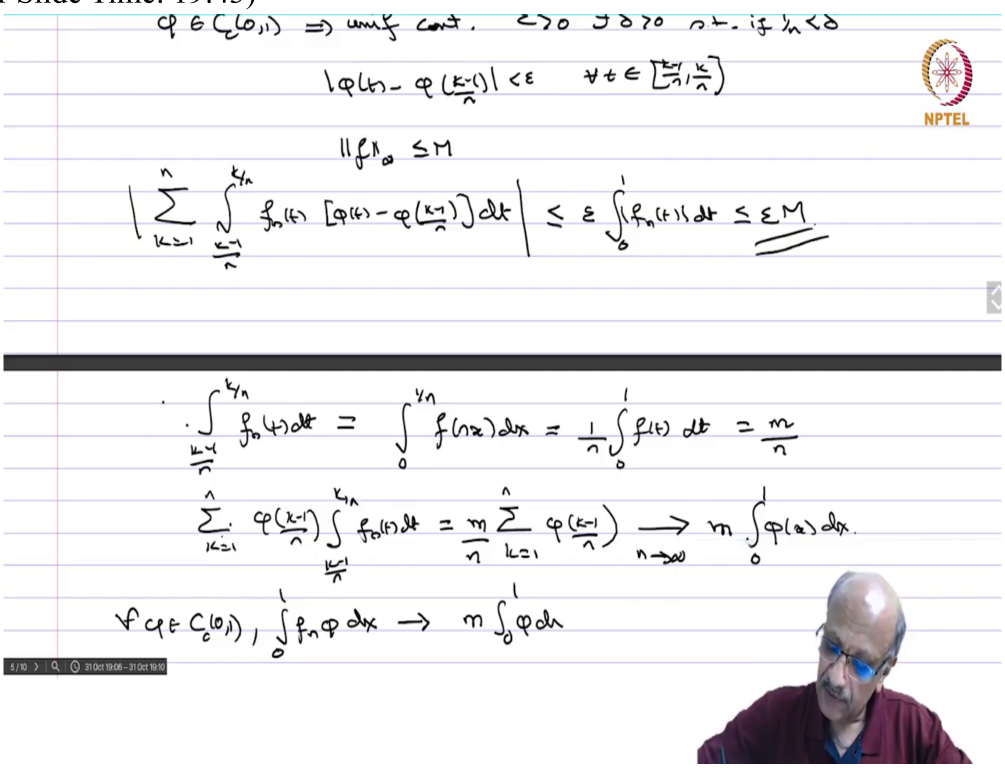

So, $\left|\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} f_n(t)\left[\phi(t) - \phi(\frac{k-1}{n})\right] dt\right| \leq \epsilon \int_0^1 |f_n(t)|\, dt \leq \epsilon M$. So, that is the first term. Now, let us look at the second term.

$\int_{\frac{k-1}{n}}^{\frac{k}{n}} f_n(t)\, dt = \int_0^{\frac{1}{n}} f(nx)\, dx = \frac{1}{n}\int_0^1 f(t)\, dt = \frac{m}{n}$. Because we have just repeated the function again and again, so integral $\int_{\frac{k-1}{n}}^{\frac{k}{n}} f_n(t)\, dt$ on any such interval $[\frac{k-1}{n}, \frac{k}{n}]$ is nothing but the integral $\int_0^{\frac{1}{n}} f(nx)\, dx$.

So, this is what you get from this, for every k. So, this is for every k and therefore you get that second term $\sum_{k=1}^{n} \varphi(\frac{k-1}{n}) \int_{\frac{k-1}{n}}^{\frac{k}{n}} f_n(t)dt = \frac{m}{n}\sum_{k=1}^{n} \varphi(\frac{k-1}{n}) \rightarrow m\int_0^1 \phi(x)\, dx \quad n \rightarrow \infty$.

Now, as $n \rightarrow \infty$, $\frac{1}{n}\sum_{k=1}^{n} \varphi(\frac{k-1}{n})$ is nothing but Riemann's sum. $\frac{1}{n}$ is the length of each of these intervals, where I am taking the value of the lowest point and ϕ is a continuous function. So, for any point it should converge to $m\int_0^1 \phi(x)\, dx$. So, that is exactly what we want. We have shown that for every $\phi \in C_c(0, 1)$, $\int_0^1 f_n \phi\, dx$ converges. So, the first term goes to 0 because it is arbitrarily small, it is less than ϵ_m. The second term converges to $m\int_0^1 \phi\, dx$ and that is precisely what we want to show.

$$\forall \phi \in C_c(0, 1), \int_0^1 f_n \phi\, dx \rightarrow n\int_0^1 \phi\, dx$$

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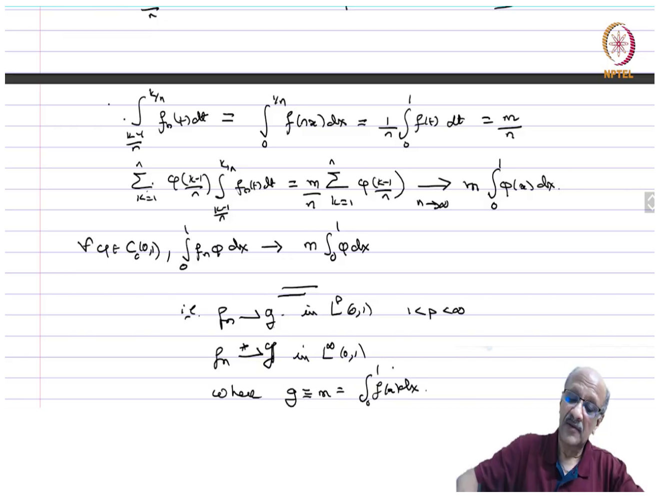

And that is $f_n \rightarrow f$ weakly in $L^p(0,\ 1),\quad 1 < p < \infty$.

and it goes to weak star, g in $L^\infty(0,\ 1)$ where $g \equiv m = \int_0^1 f(x)\,dx$.

So that is, so we will wind up with this and start a next, new chapter next time.