Functional Analysis

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Lecture No. 47

Exercises Part – 3

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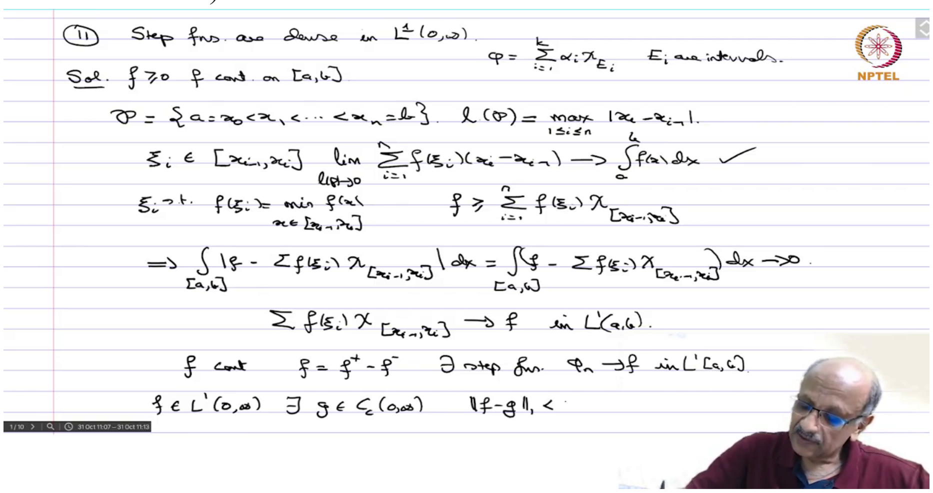

We continue the exercises.

Problem 11. Step functions are dense in $L^1(0, \infty)$. So, $(0, \infty)$ is the interval $x > 0$ in $\mathbb{R}$ and L^1 is the space with the Lebesgue measure and therefore, we want to show that step functions are dense. Now, what is a step function? A step function is a simple function. So, φ is a step function, this is $\sum_{i=1}^{k} \alpha_i \chi_{E_i}$, but now E_i are intervals. So this is a step function. Such functions are not just special simple functions, much smaller than simple functions as a set and we want to show that this is also dense in $L^1(0, \infty)$.

Solution. Let us take $f \geq 0$, f continuous on $[a, b]$. So, we are taking a fixed and finite interval. Then you take a partition $\wp = \{a = x_0 < x_1 < . . . . < x_n = b\}$ of this interval, and you set $\mu(\wp)$ or let us say $l(\wp)$ (if you like) is the maximum length, so $l(\wp) = \max_{1 \leq i \leq n} \left| x_i - x_{i-1} \right|$. So then you have, if $\xi_i \in [x_{i-1}, x_i]$ then $\lim_{l(\wp)\to 0} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \to \int_a^b f(x)\, dx$. —(1)

So, you take all possible partitions, the maximum should tend to 0, so the partitions become finer and finer, then the Riemann's sums we know converge to the Riemann integral, but we are dealing with continuous functions, therefore Riemann, Lebesgue integral are all one and the same. Then, if you choose, let us fix ξ_i, for instance, to be such that $f(\xi_i) = \min_{x\in[x_{i-1}, x_i]} f(x)$. You could choose anything you want to choose like this. So then, you have $f \geq \sum_{i=1}^{n} f(\xi_i)\, \chi_{[x_i - x_{i-1}]}$. So, this is a step function and this will imply that $\int_{[a,b]} |f - \sum f(\xi_i)\, \chi_{[x_i - x_{i-1}]}| dx$. I do not have to put the mod because I have taken the minimum, so

$\int_{[a,b]} |f - \sum f(\xi_i)\, \chi_{[x_i - x_{i-1}]}|\, dx = \int_{[a,b]} f - \sum f(\xi_i)\, \chi_{[x_i - x_{i-1}]} dx \to 0$ by condition (1),

because the $\int \chi$ is nothing but the measure which is the length of the interval and therefore you get that this tends to 0. So, you have $\sum f(\xi_i)\, \chi_{[x_i - x_{i-1}]} \to f$ in $L^1(a, b)$.

Now, if you take f to be an arbitrary continuous function, then you write as $f = f^+ - f^-$ and then, for each of them you can find a step function and therefore there exists a step function $\varphi_n \to f$ in $L^1(a, b)$. So now, if $f \in L^1(0, \infty)$, $\exists$ a $g \in C_c(0, \infty)$, continuous function with compact support, such that $||f - g||_1 < \epsilon$.

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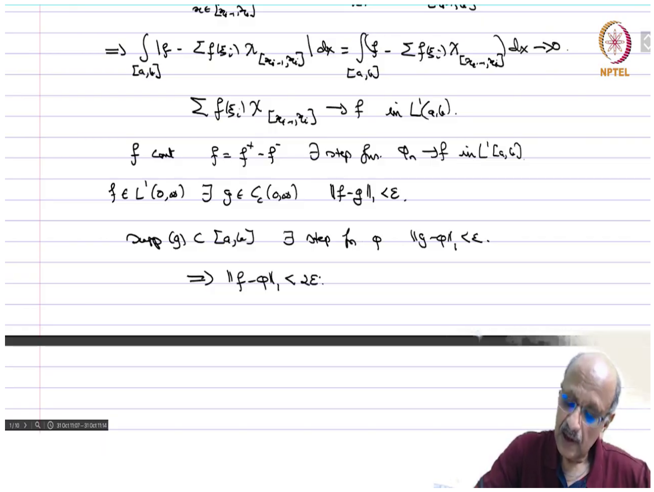

Then, let $supp(g) \subset [a, b]$, then $\exists$ a step function ϕ such that $|| g - \phi||_1 < \epsilon$.

Therefore, that implies that $||f - \phi||_1 < 2\epsilon$, and that proves that step functions are dense.

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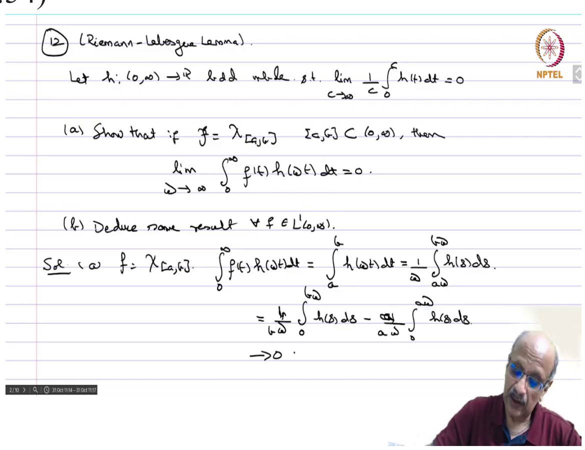

Problem 12. This is one of the versions of the Riemann-Lebesgue Lemma.

Let $h: (0, \infty) \rightarrow \mathbb{R}$ bounded and measurable, such that $\lim_{c \rightarrow \infty} \frac{1}{c}\int_0^c h(t)\, dt = 0$. h is bounded. So on any finite interval, you can integrate it, it is integrable. You are taking the average, so you are taking the integral over c dividing by the length of the interval, so that is nothing but the average. So, the average of this function, as the interval becomes larger and larger, tends to 0.

(a) Show that if $f = \chi_{[a,\, b]}$, $[a,\, b] \subset (0,\, \infty)$ then $\lim_{\omega \to \infty} \int_0^{\infty} f(t)\, h(\omega t)\, dt = 0$.

(b) Deduce the same result $\forall\, f \in L^1(0,\, \infty)$.

Solution. (a) $f = \chi_{[a,\, b]}$. So, what is $\int_0^{\infty} f(t)\, h(\omega t)\, dt$?

$$\int_0^{\infty} f(t)\, h(\omega t)\, dt = \int_a^{b} h(\omega t)\, dt \quad \text{because } f = \chi_{[a,\, b]}$$

$$= \frac{1}{\omega} \int_{a\omega}^{b\omega} h(s)\, ds \quad \text{this is just a change of variable}$$

$$= \frac{b}{b\,\omega} \int_0^{b\omega} h(s)\, ds - \frac{a}{a\,\omega} \int_0^{a\omega} h(s)\, ds$$

This gives the average and as $\omega \to \infty$, $a\omega,\, b\omega \to \infty$, so each term $\to 0$, by the given hypothesis and therefore you have $\int_0^{\infty} f(t)\, h(\omega t)\, dt \to 0$. So that proves the part (a).

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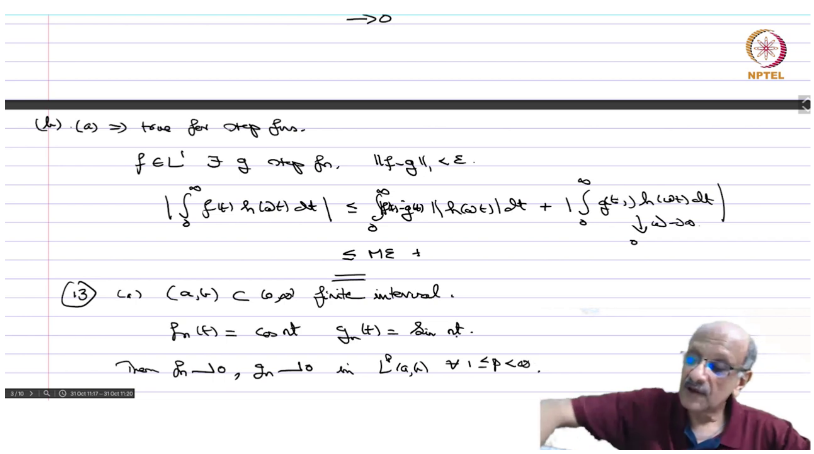


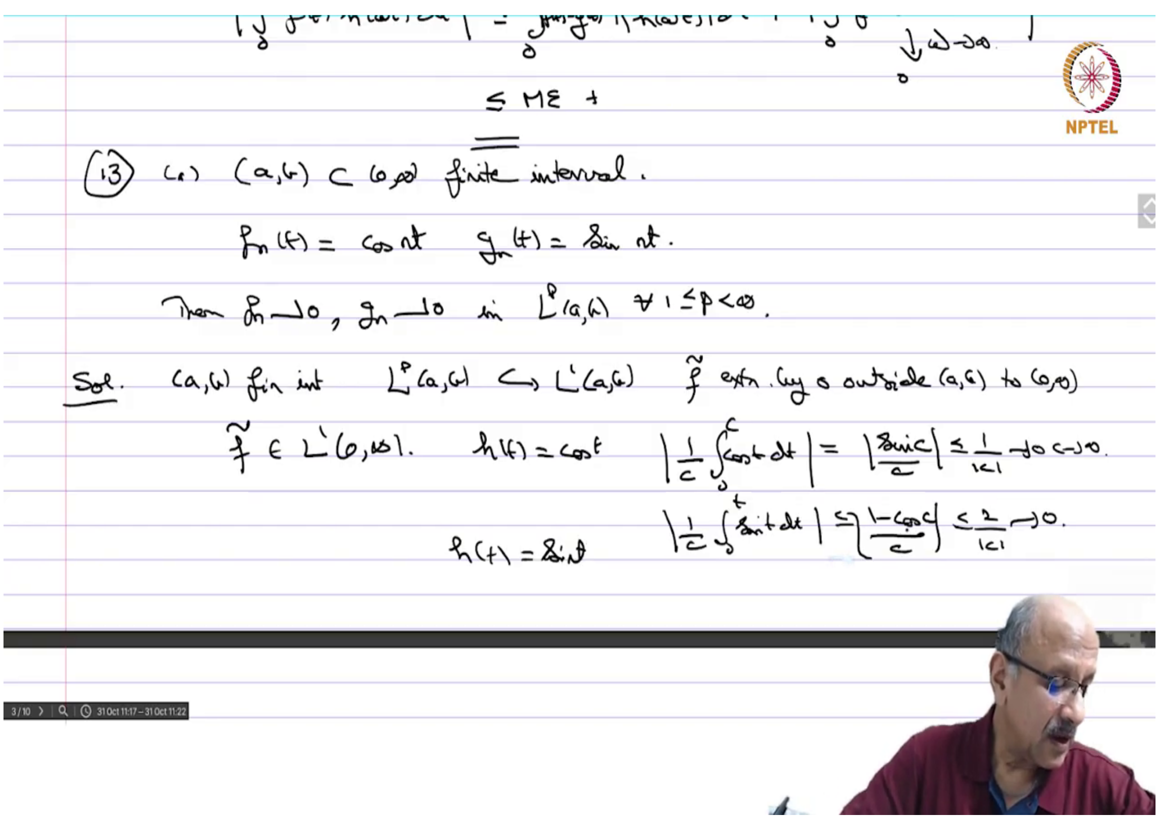
(b) (a) $\Rightarrow$ true for step functions. It is true for every characteristic function and therefore by linearity, it is true for any linear combination of them, so it is true for step functions.

If $f \in L^1$, then $\exists$ a step function g such that $||f - g||_1 < \epsilon$ by the previous exercise.

So, $|\int_0^{\infty} f(t)\, h(\omega t)\, dt\,| \ \leq\ \int_0^{\infty} |f(t)\ -\ g(t)|\ |h(\omega t)|\, dt\ +\ |\int_0^{\infty} g(t)\, h(\omega t)\, dt\,|$

In the first term, h is a bounded measurable function, so $|h|\ \leq M$. Now, you get $\int |f\ -\ g|$ is the L^1 norm and therefore $<\ \epsilon$. So that first term $< M\epsilon$ and the second term, of course g is the step function, therefore $\int_0^{\infty} g(t)\, h(\omega t)\, dt\ \rightarrow\ 0$ as $\omega\ \rightarrow\ \infty$. Therefore, you can make this arbitrarily small for $\omega\ \rightarrow\ \infty$ and consequently, you have the result.

Problem 13. **(a)** Let $(a,\ b)\subset(0,\ \infty)$ be a finite interval and $f_n(t)\ =\ \cos(nt)$ and $g_n(t)\ =\ \sin(nt)$. Then $f_n\rightarrow 0$ weakly and $g_n\rightarrow 0$ weakly in $L^p(a,\ b)$ $\forall\, 1\leq p\ <\ \infty$.



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Solution. (a, b) is a finite interval. So, $L^p(a, b) \subset L^1(a, b)$ because it is a finite measure space and therefore any L, bigger L^p is continuously embedded in a smaller L^p. So, now you take $\tilde{f}$ extension by 0 outside (a, b) to $(0, \infty)$. So you extend it, so then you have $\tilde{f} \in L^1(0, \infty)$. So, if you take $h(t) = \cos(t)$ then what is $\left|\frac{1}{c}\int_0^c \cos(t)\, dt\right|$?

$$\left|\frac{1}{c}\int_0^c \cos(t)\, dt\right| = \left|\frac{\sin c}{c}\right| \leq \frac{1}{|c|} \to 0 \text{ as } c \to \infty.$$

Similarly, $\left|\frac{1}{c}\int_0^t \sin(t)\, dt\right| \leq \left|\frac{1-\cos(c)}{c}\right| \leq \frac{2}{|c|} \to 0$ as $c \to \infty$.

So, $h(t) = \cos(t)$, $h(t) = \sin(t)$ satisfy the previous properties of the function h given in the previous exercises.

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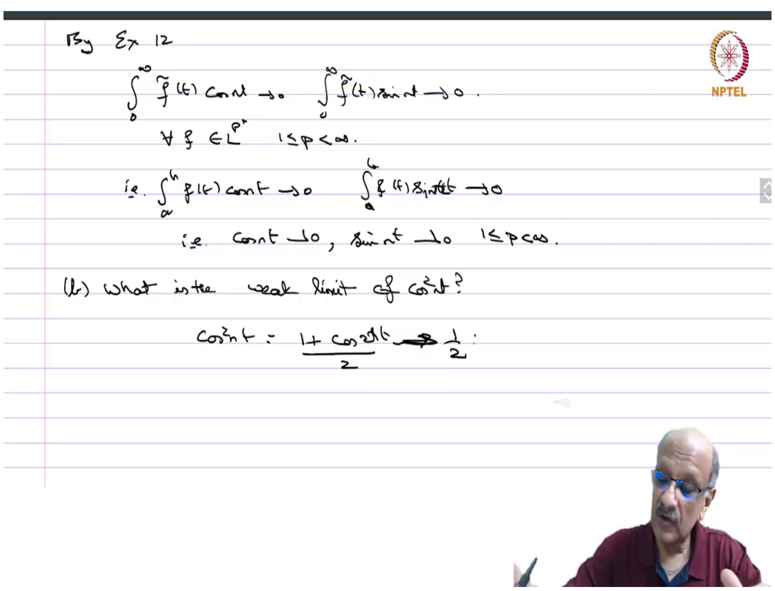

By exercise 12, we have $\int_0^\infty \tilde{f}\ \cos(nt)\, dt$ and $\int_0^\infty \tilde{f}\ \sin(nt)\, dt \to 0 \quad \forall\, f \in L^{p^*}$, $1 \le p \le \infty$ that means, $\int_a^b f(t)\ \cos(nt)\ \to 0$ and $\int_a^b f(t)\ \sin(nt)\ \to 0\ \forall$ such f and i.e. $\cos(nt)\ \to 0$ and $\sin(nt)\ \to 0$ weakly $\ \forall\ 1 \le p \le \infty$.

(b) What is the weak limit of $\cos^2(nt)$?

$\cos^2(nt) = \frac{1+\cos(2nt)}{2}$, and that of course $\cos(nt)\ \to 0$ weakly,

so $\cos^2(nt)$ goes weakly to $\frac{1}{2}$. So, here is an example about weak convergence. If f converges weakly, f_n and g_n converges weakly, $f_n g_n$ need not converge weakly to the same limit. It may not even converge, weakly. So when you are doing nonlinear operations, you have to be very careful. Here, $\cos(nt)$ goes to 0 but $\cos^2(nt)$ does not go weakly to 0, it goes weakly to $\frac{1}{2}$. So, this is an important point to note and this is an example of how you should be careful when dealing with nonlinear functions of weakly convergent sequences.

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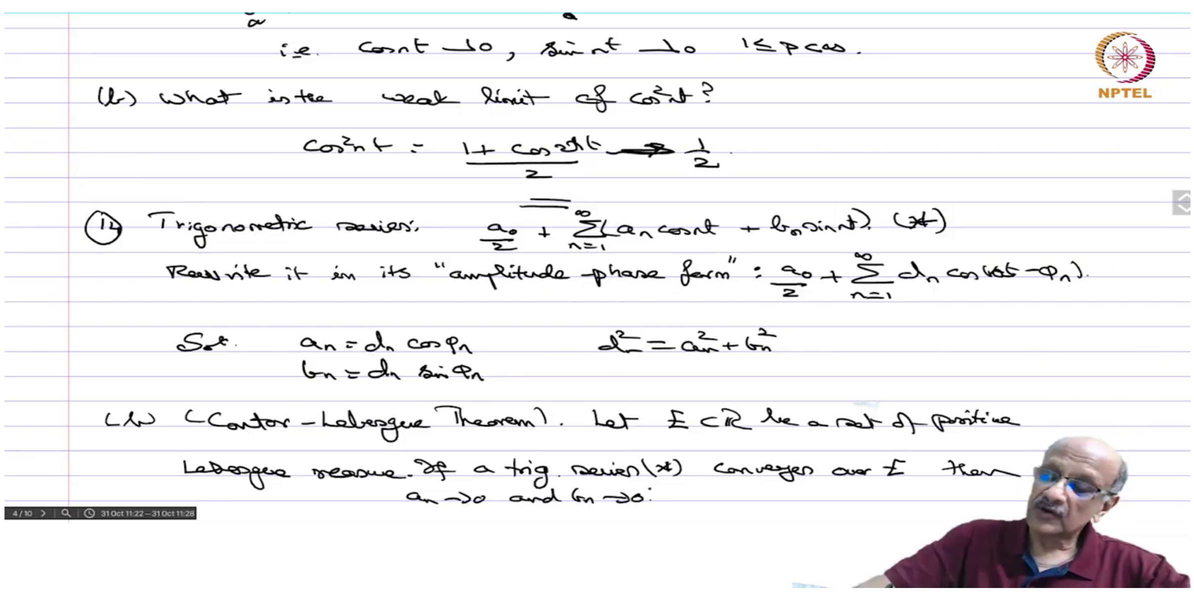

Problem 14. A trigonometric series : $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$ — (*)

Typically the Fourier series are trigonometric series. So, any such series written like this are called trigonometric series. So, rewrite it in it's amplitude phase form. Namely,

$\frac{a_0}{2} + \sum_{n=1}^{\infty} d_n \cos(nt - \phi_n)$ ϕ_n is the phase, d_n is called the amplitude.

Solution: All you need to do is put $a_n = d_n \cos(\phi_n)$ and $b_n = d_n \sin(\phi_n)$

From this you get $d_n^2 = a_n^2 + b_n^2$. So, you can compute d_n, the positive square root and once you know d_n, from this you can, whichever relation you like, you can take $\tan(\phi_n) = \frac{b_n}{a_n}$ or $\cos(\phi_n)$ or $\sin(\phi_n)$ from whichever, from these three equations you can easily compute what is ϕ_n.

(b) **Cantor-Lebesgue Theorem**. Let $E \subset \mathbb{R}$ be a set of positive Lebesgue measure. If a trigonometric series (*) converges over E, that means at every point in E it converges, then $a_n \to 0$ and $b_n \to 0$. So, the coefficients of the trigonometric series have to go to 0, that is a necessary condition for the convergence on a set of positive measure.

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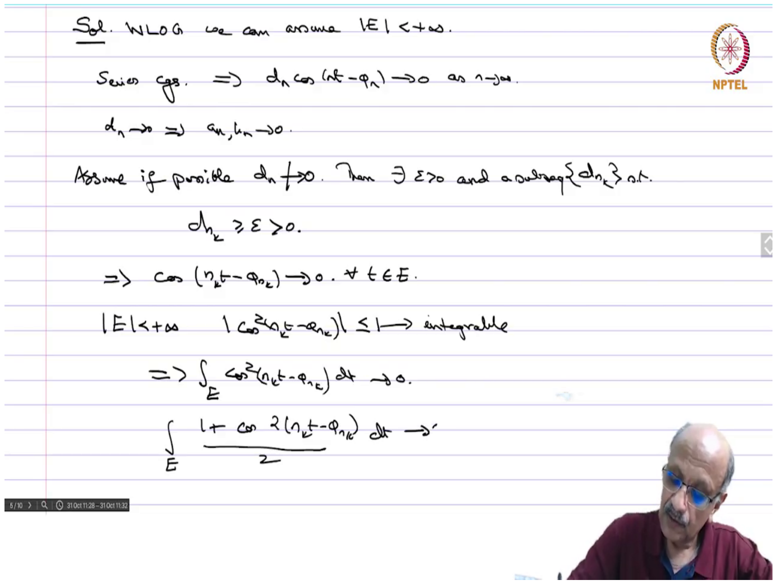

Solution. Without loss of generality, we can assume that the Lebesgue measure of E is strictly finite, because it converges over a set E, so if necessary you take a subset of positive, finite measure, there also it is going to converge. So, there is no problem in assuming that the measure is finite. Series convergence means the general term has to go to 0, so this implies that $d_n \cos(nt - \phi_n) \to 0$ as $n \to \infty$. That is the general term of any convergent series has to go to 0. So that is why, we wrote it as (*) because we had two terms, we combined it and wrote an amplitude phase form so there is only 1 term and so it is $\sum \alpha_n$, α_n has to go to 0. So, $d_n \cos(nt - \phi_n)$ has to go to 0.

So, this can happen in 2 ways. (i) $d_n \to 0$. $d_n^2 = a_n^2 + b_n^2$, so if $d_n \to 0$, then a_n and b_n automatically go to 0. So, $d_n \to 0 \Rightarrow a_n, b_n \to 0$. So, this is what we need. So assume if possible, d_n does not go to 0. Then, $\exists$ an $\epsilon > 0$ and a subsequence $\{d_{n_k}\}$ such that $d_{n_k} \geq \epsilon > 0$. So, it stays away from 0. So now, if d_{n_k} does not go to 0, the general term anyway goes to 0, so this implies that $\cos(n_k t - \phi_{n_k})$ has to go to 0 $\forall t \in E$.

So, $|E|$ is finite. So constant functions are integrable. So, and you have $|\cos^2(n_k t - \phi_{n_k})| \leq 1 \rightarrow$ integrable, and

$$\cos(n_k t - \phi_{n_k}) \rightarrow 0 \Rightarrow \int_E \cos^2(n_k t - \phi_{n_k})dt \rightarrow 0 \Rightarrow \int_E \frac{1+\cos 2(n_k t - \phi_{n_k})}{2}dt \rightarrow 0$$

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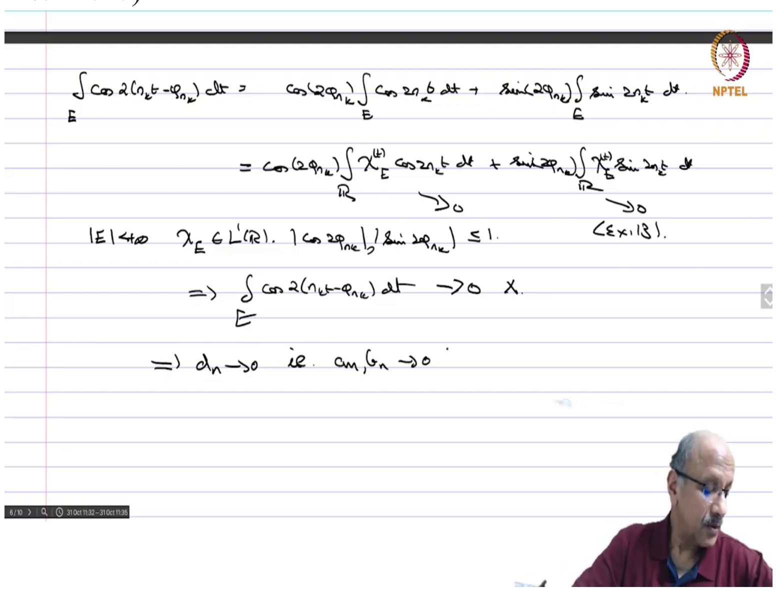

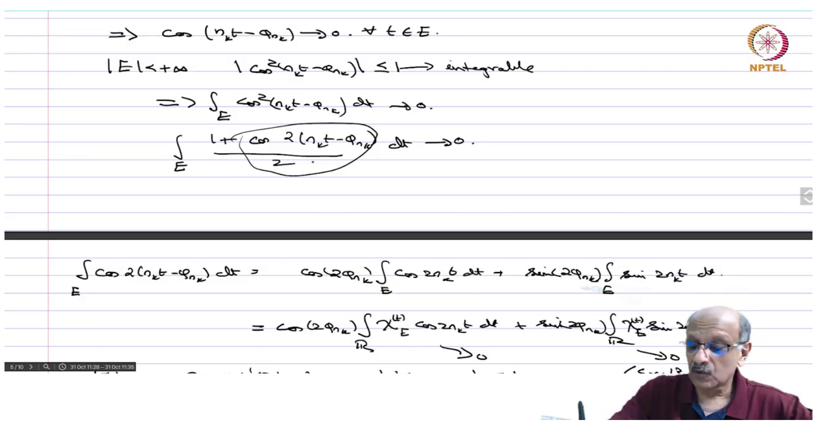

Let us look at $\cos\ (2n_k t)$.

So, $\int_E \cos\ 2(n_k t - \varphi_{n_k})dt\ = \cos(2\varphi_{n_k}) \int_E \cos\ (2n_k t)dt\ + \sin(2\varphi_{n_k}) \int_E \sin(2n_k t)dt$

$$= \cos(2\varphi_{n_k}) \int_R \chi_E^{(t)} \cos\,(2n_k t)dt + \sin(2\varphi_{n_k}) \int_R \chi_E^{(t)} \sin\,(2n_k t)dt$$

Now, $|E|$ is of finite measure. Therefore $\chi_E \in L^1(R)$ and $|\cos(2\varphi_{n_k})| \leq 1$, $|\sin(2\varphi_{n_k})| \leq 1$. Therefore, each of these terms has to go to 0 by exercise 13 and this implies that $\int_E \cos\, 2(n_k t - \varphi_{n_k})dt \to 0$ and that is a contradiction because $\int_E \frac{1+\cos 2(n_k t - \varphi_{n_k})}{2} dt \to 0$. Now, $\int_E \cos\, 2(n_k t - \varphi_{n_k})dt \to 0$ means $\int_E \frac{1+\cos 2(n_k t - \varphi_{n_k})}{2} dt$ has to go to 1/2, which is not, we have shown that it goes to 0 and therefore it is not possible and therefore it is a contradiction, so this implies $d_n \to 0 \Rightarrow a_n, b_n \to 0$