Functional Analysis
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Lecture No. 46

Exercises Part – 2

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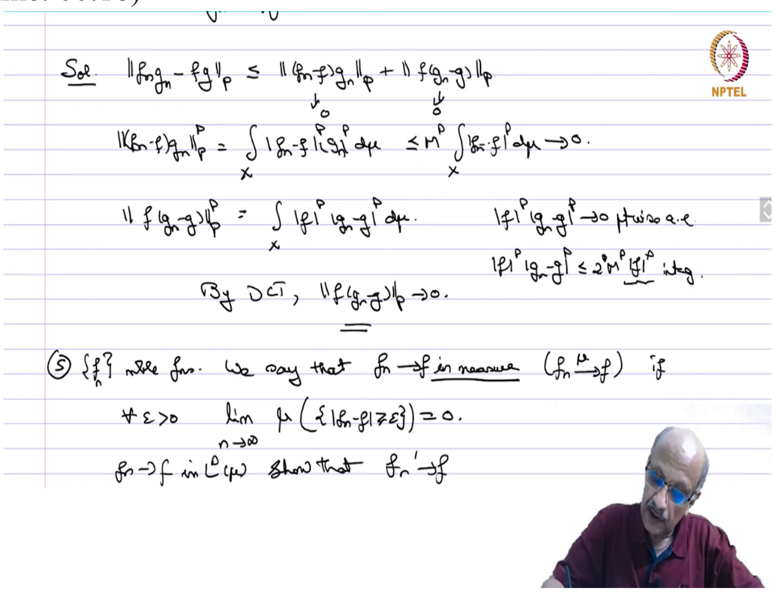

Problem 5. Let f_n measurable functions. We say that f_n converges to f in measure, this is one kind of convergence, $\forall \epsilon > 0$, $\lim_{n \to \infty} \mu(\{|f_n - f| \geq \epsilon\}) = 0$. This set is measurable so its measure has to go to 0. So, if $f_n \to f$ in $L_p(\mu)$ show that $f_n \to f$ in μ.

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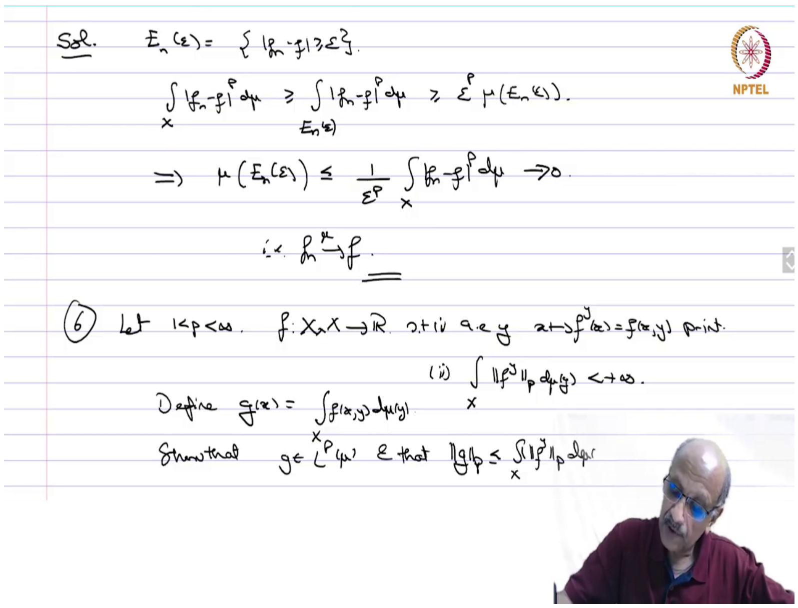

Solution. Let $E_n(\epsilon) = \{|f_n - f| \geq \epsilon\}$.

Now, $\int_X |f_n - f|^p d\mu \geq \int_{E_n(\epsilon)} |f_n - f|^p d\mu \geq \epsilon^p \mu(E_n(\epsilon))$ (as we are taking a smaller set and the integrand is nonnegative).

Therefore, $\mu(E_n(\epsilon)) \leq \frac{1}{\epsilon^p}\int_X |f_n - f|^p d\mu \to 0$, i.e. $f_n \to f$ in μ.

This kind of argument you have seen in measure theory or probability theory, something called **Chebyshev's inequality**, is almost proved in the same fashion. So, this is the same kind of proof that one uses and the very useful argument where you use just the fact that the integral, if you take a nonnegative integrand, then the integral over the smaller set is smaller.

Problem 6. Let $1 < p < \infty$ and $f: X \times X \to \mathbb{R}$ such that, (i) for almost every y, $x \mapsto f^y(x) = f(x, y)$ $p$integrable, (ii) $\int_X ||f^y||_p \, d\mu(y) < \infty$.

Define, $g(x) = \int_X f(x, y) \, d\mu(y)$ i.e., this is a combination of two variables. Show that,

$g \in L^p(\mu)$ and $||g||_p \le \int_X ||f^y||_p \, d\mu$.

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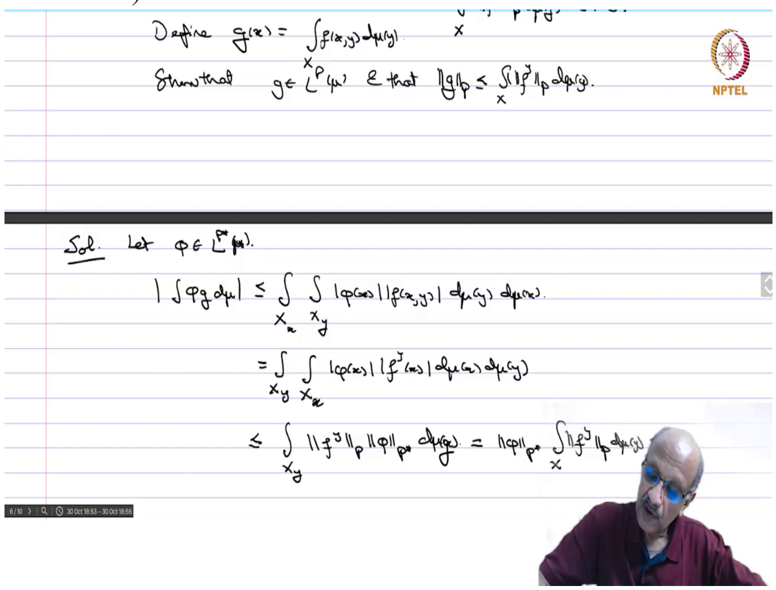

Solution. This is an argument very similar to our proof of Young's inequality.

Let $\phi \in L^{p^*}(\mu)$. So, this is the conjugate exponent. Then,

$|\int \phi g \, d\mu| \le \int_{X_x} \int_{X_y} |\phi(x)||f(x, y)| \, d\mu(y) \, d\mu(x)$[as everything is non-negative so you can interchange the order of integration]

$$|\int \phi g \, d\mu| \le \int_{X_x} \int_{X_y} |\phi(x)||f(x, y)| \, d\mu(y) \, d\mu(x) = \int_{X_y} \int_{X_x} |\phi(x)||f^y(x)| \, d\mu(x) \, d\mu(y)$$

Here, f^y is in L^p, ϕ is in L^{p^*}, by the Holder inequality,

$$\int_{X_y} \int_{X_x} |\phi(x)||f^y(x)| \, d\mu(x) \, d\mu(y) \le \int_{X_y} ||f^y||_p \, ||\phi||_{p^*} d\mu(y) = ||\phi||_{p^*} \int_{X_y} ||f^y||_p \, d\mu(y) < \infty.$$

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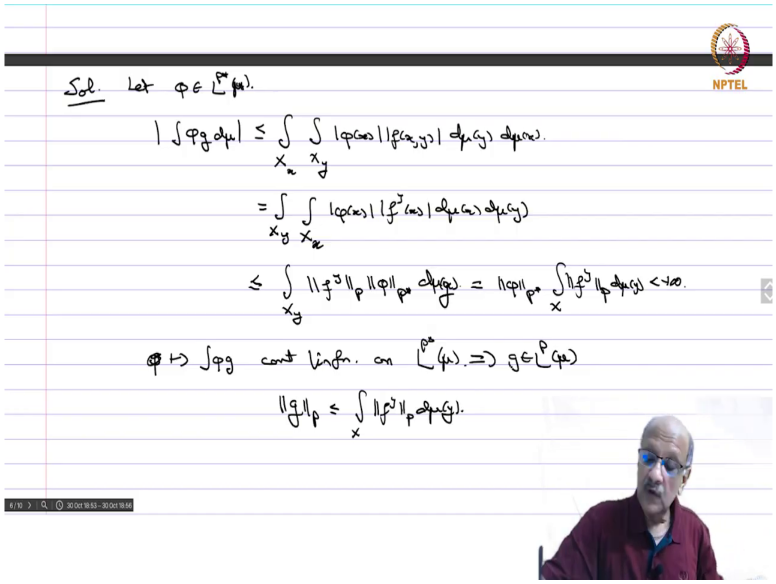

They have given that $1 < p < \infty$ so $\phi \in L^{p^*}(\mu)$ therefore $\phi \to \int \phi\, g$ is a continuous linear functional on $L^{p^*}(\mu)$ and therefore this implies that $g \in L^{p}(\mu)$ and therefore $||g||_{p} \leq \int\limits_{X} ||f^{y}||_{p}\, d\mu(y)$. So, it is just exactly like we did for Young's inequality. So, we have used the duality argument and then we do this problem.

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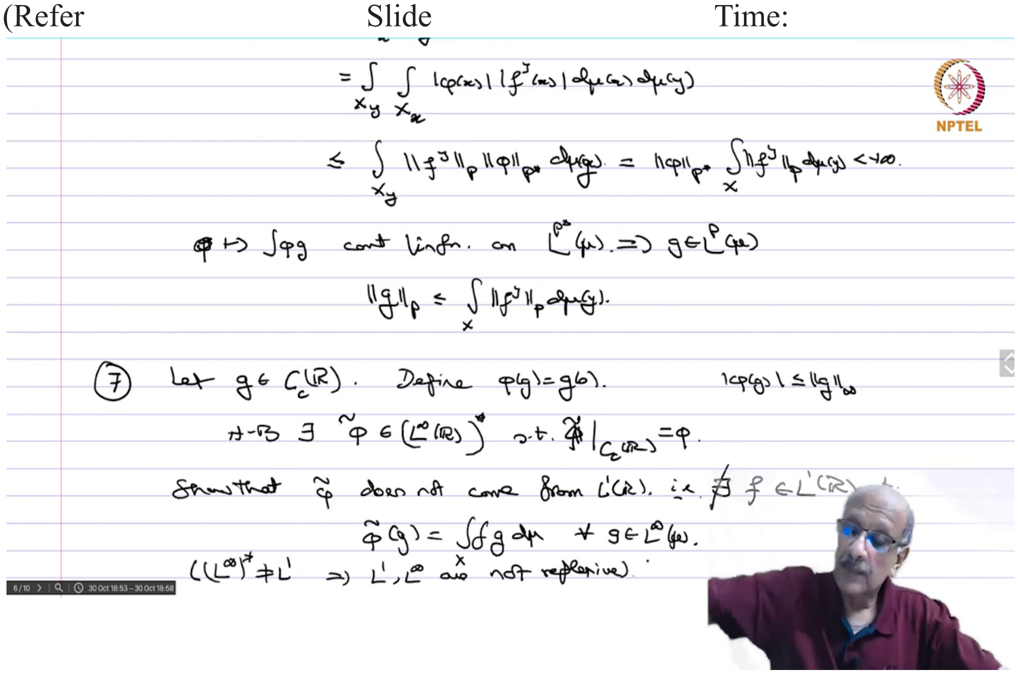

Problem 7. Let $g \in C_c(\mathbb{R})$ (continuous functions with compact support in $\mathbb{R}$). Define $\phi(g) = g(0)$, Then of course, $|\phi(g)| \leq ||g||_\infty$ and therefore continuous with respect to the L^∞ norm. Therefore, by Hahn Banach, $\exists \tilde{\phi} \in (L^\infty(\mathbb{R}))^*$ such that, $\tilde{\phi}|_{C_c(\mathbb{R})} = \phi$. Of course, the norm will be preserved, and so on. Show that $\tilde{\phi}$ does not come from $L^1(\mathbb{R})$, that is there does not exist $f \in L^1(\mathbb{R})$, such that $\tilde{\phi}(g) = \int_X fg\, dx \quad \forall g \in L^\infty(\mathbb{R})$. This is another proof that $(L^\infty)^* \neq L^1 \Rightarrow L^1, L^\infty$ are not reflexive. We give another type of proof for that. So here, we are directly producing a linear functional which does not come from it.

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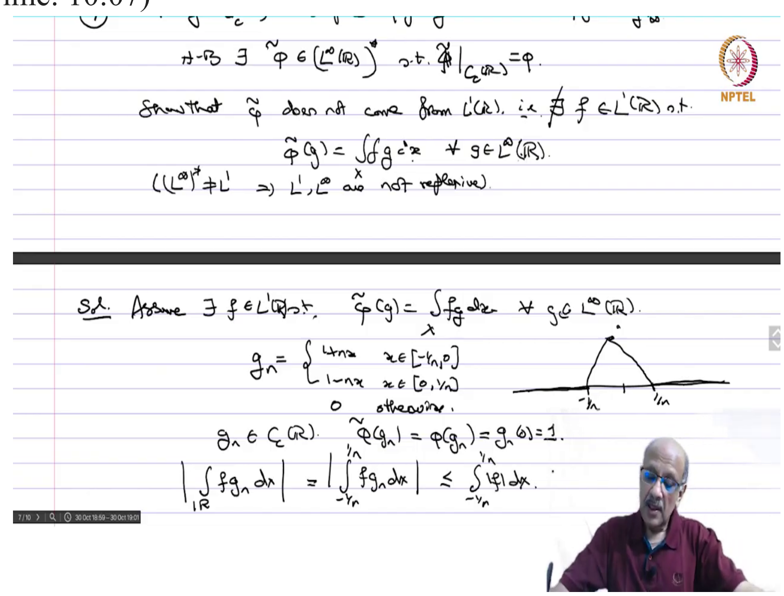

Solution. Assume $\exists f \in L^1(\mathbb{R})$ such that $\tilde{\phi}(g) = \int_X fg\, dx \quad \forall g \in L^\infty(\mathbb{R})$.

So now, we take $g_n = 1 + nx, \quad x \in [-1/n, 0]$

$$= 1 - nx, \quad x \in [0, 1/n]$$

$$= 0, \quad \text{otherwise.}$$

The function is a roof type function, so it is 0 for $x \leq -1/n$ and $x \geq 1/n$, at (0, 1/n) it has value 1. Inside $[-\frac{1}{n}, \frac{1}{n}]$, it takes $1 \pm nx$ kind of value. So, then $g_n \in C_c(\mathbb{R})$. And you have $\tilde{\phi}(g_n) = \phi(g_n) = g_n(0) = 1$. On the other hand, you have,

$$|\int_{\mathbb{R}} f\, g_n dx| = |\int_{-1/n}^{1/n} f\, g_n dx| \leq \int_{-1/n}^{1/n} |f|\, dx.$$

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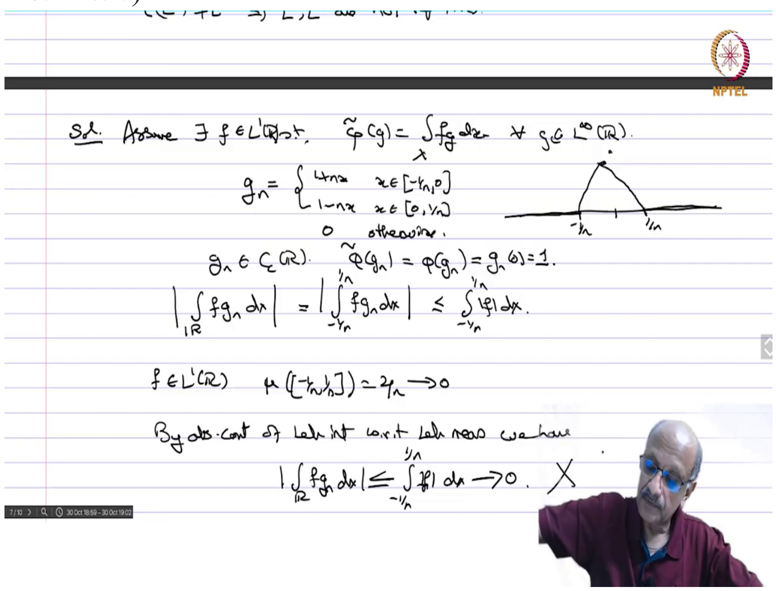

Now, $f \in L^1(\mathbb{R})$ and $\mu([-1/n, 1/n]) = 2/n \rightarrow 0$. The Lebesgue measure is just the length of the interval which goes to 0, therefore by absolute continuity of the Lebesgue integral with respect to the Lebesgue measure, we have $|\int_{\mathbb{R}} f\, g_n dx| \leq \int_{-1/n}^{1/n} |f|\, dx \rightarrow 0$. On the other hand, you have $\tilde{\phi}(g_n) = 1$, so these two cannot be equal. So, you have a contradiction. And that proved the result.

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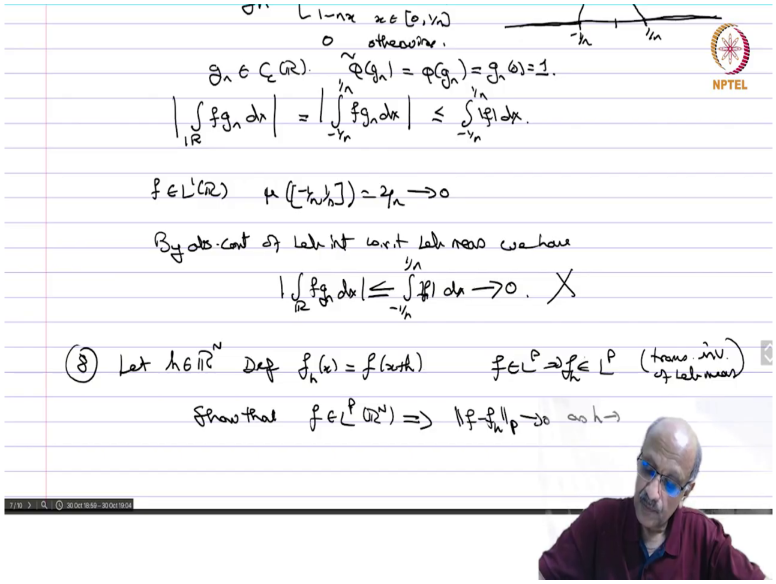

Problem 8. Let $h \in \mathbb{R}^N$ be a vector. Define $f_h(x) = f(x + h)$ i.e. just the translation. The lebesgue measure is translation invariant, so $f \in L^p \Rightarrow f_h \in L^p$. Show that if $f \in L^p(\mathbb{R}^N) \Rightarrow ||f - f_h||_p \to 0 \; as \; h \to 0, \;\; 1 \le p < \infty$.

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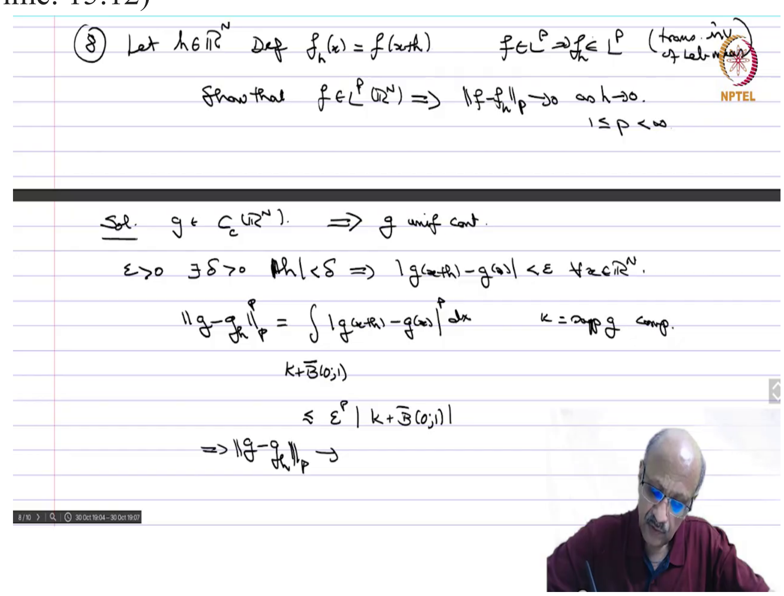

Solution. Let $g \in C_c(\mathbb{R}^N)$. This implies it is a continuous function with compact support g is uniformly continuous. So, given $\epsilon > 0$, $\exists\, \delta > 0$, such that,

$|h| < \delta \Rightarrow |g(x + h) - g(x)| < \epsilon\ \forall\, x \in \mathbb{R}^N$(because of the uniform continuity and note that $||h|| = |h|\ in\ \mathbb{R}^N$). So, $||g - g_h||_p^{\ p} = \int |g(x + h) - g(x)|^p\, dx$

I should integrate this over all of $\mathbb{R}^N$. But g has compact support and g_h support is just translated by h and $h < \delta$ which can be assumed without loss of generality to be less than 1 and therefore the support of these functions will be contained in $K + \bar{B}(0; 1)$. So, $K = supp\ g$ compact. So, $|g(x + h) - g(x)|^p$ will be non-zero indefinitely, it will be 0 outside $K + \bar{B}(0; 1)$. And so,

$||g - g_h||_p^{\ p} = \int |g(x + h) - g(x)|^p\, dx \le \epsilon^p |K + \bar{B}(0; 1)|$, and therefore this implies that $||g - g_h||_p \to 0$. So, this is true for any h.

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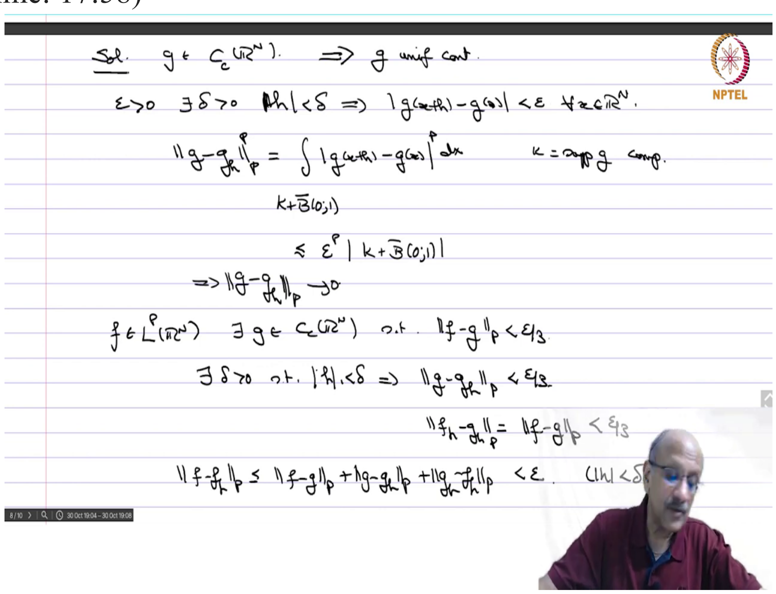

So now, let $f \in L^p(\mathbb{R}^N)$, we have, $\exists\, g \in C_c(\mathbb{R}^N)$ such that $||f - g||_p < \epsilon/3$. Then, $\exists\, \delta > 0$ such that $|h| < \delta \Rightarrow ||g - g_h||^p < \epsilon/3$. And then what about $||f_h - g_h||_p$? You are translating both f & g, by the translation invariance of Lebesgue measure, $||f_h - g_h||_p = ||f - g||_p < \epsilon/3$. Therefore,

$$||f - f_h||_p \leq ||f - g||_p + ||g - g_h||_p + ||g_h - f_h||_p < \epsilon$$

(as each is less than $\epsilon/3$) for all $|h| < \delta$ and therefore, that proves the statement.

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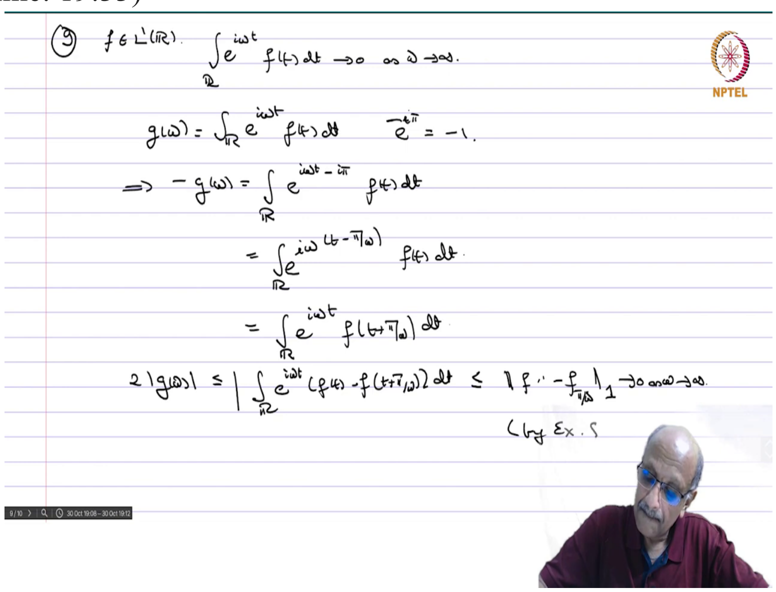

Problem 9. As an application of this result, we take $f \in L^1(\mathbb{R})$, then $\int_{\mathbb{R}} e^{i\omega t} f(t)dt \to 0 \text{ as } \omega \to \infty$. So, we are taking a complex function, of course, complex function is the same Lebesgue integrable. If you take the modulus power p, should be integrable, as $f \in L^1$, $e^{i\omega t}$ has modulus is 1 and therefore this is still integrable and therefore this is well defined. Let us define $g(\omega) = \int_{\mathbb{R}} e^{i\omega t} f(t)dt$. Now, you have $e^{i\pi} = -1$ and $e^{-i\pi} = -1$. So, I am going to multiply by -1. So, $-g(\omega) =$

$\int_{\mathbb{R}} e^{i\omega t - i\pi} f(t)dt = \int_{\mathbb{R}} e^{i\omega(t - \pi/\omega)} f(t)dt = \int_{\mathbb{R}} e^{i\omega t} f(t + \pi/\omega)\, dt$. [I make a change of variable]. Therefore,

$2|g(\omega)| \le |\int_{\mathbb{R}} e^{i\omega t}(f(t) - f(t + \pi/\omega))\, dt| \le ||f - f_{\pi/\omega}||_1 \to 0\ as\ \omega \to \infty$ [as $|e^{i\omega t}| = 1$ and $\pi/\omega \to 0$ as $\omega \to \infty$] by exercise 8.

the h translation, this is by exercise 8.

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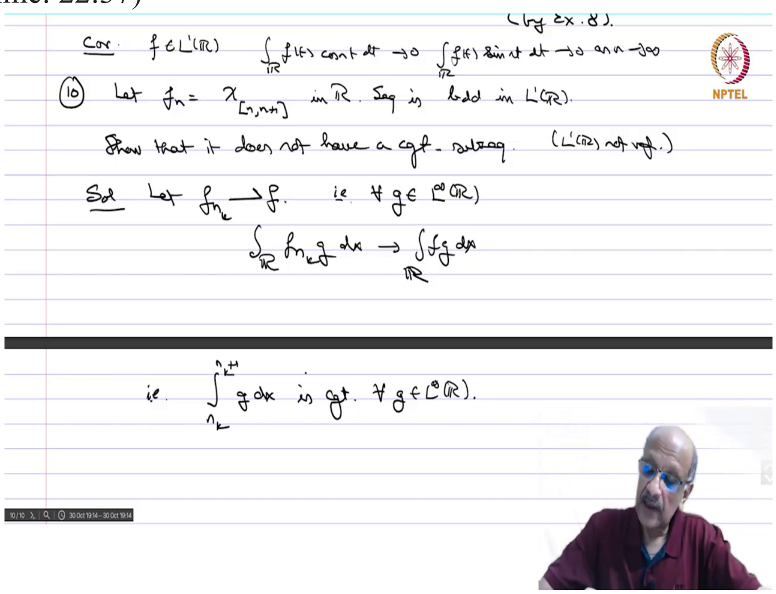

Corollary: If $f \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} f(t)\ \cos(nt)dt \to 0$ and $\int_{\mathbb{R}} f(t)\ \sin(nt)\, dt \to 0$as $n \to \infty$.

That is the corollary of the previous exercise if you take real and imaginary parts of that integral.

Problem 10. Let $f_n = \chi_{[n, n+1]}$ in $\mathbb{R}$ is a bounded sequence in $L^1(\mathbb{R})$. Show that it does not have a convergent subsequence. So again, you show that $L^1(\mathbb{R})$ is not reflexive. That is another example to show that.

Solution. Let $f_{n_k} \to f$ weakly, i.e. $\forall g \in L^{\infty}(\mathbb{R})$ we have $\int_{\mathbb{R}} f_{n_k} g\, dx \to \int_{\mathbb{R}} f\, g\, dx$. But what is f_{n_k}? i.e., $\int_{n_k}^{n_k+1} g\, dx$ is convergent $\forall g \in L^{\infty}(\mathbb{R})$.

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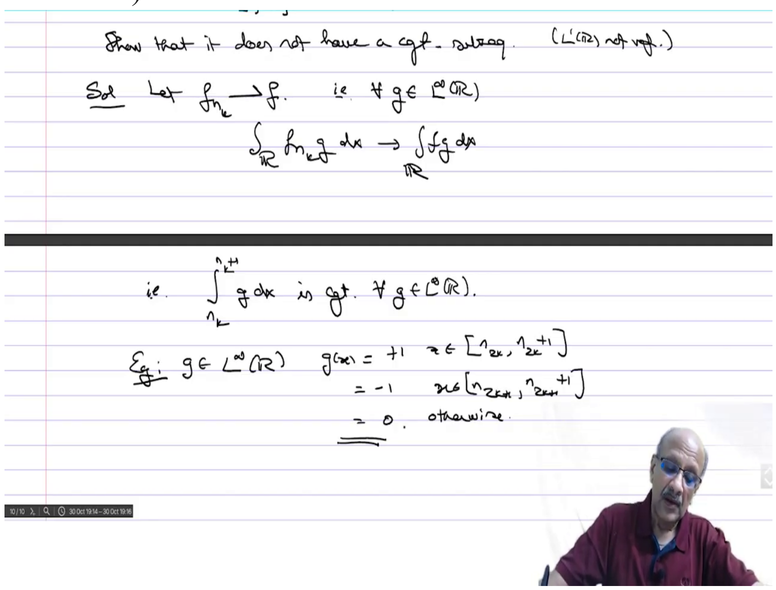

And that is not possible. You can take arbitrary; L^{∞} functions can be arbitrarily defined. Similar argument we showed for l^{∞}, we showed the sequence does not have a convergent subsequence. We have seen such an argument before, so it is the same argument.

Example: $g \in L^{\infty}(\mathbb{R})$,.

You take $g(x) = 1$, $x \in [n_{2k}, n_{2k} + 1]$

$$= -1, \ x \in [n_{2k+1}, n_{2k+1} + 1]$$

$$= 0, \quad \text{otherwise.}$$

So, such a thing will alternately give you +1, -1, and so on, and therefore that integral cannot converge. So, if you compute this, you will alternately get +1, -1, +1, -1, for the subsequence and that is not a convergent subsequence. So, you cannot have a convergent subsequence. We will continue with more exercises.