Functional Analysis
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences

Lecture No. 45

Exercises Part – 1

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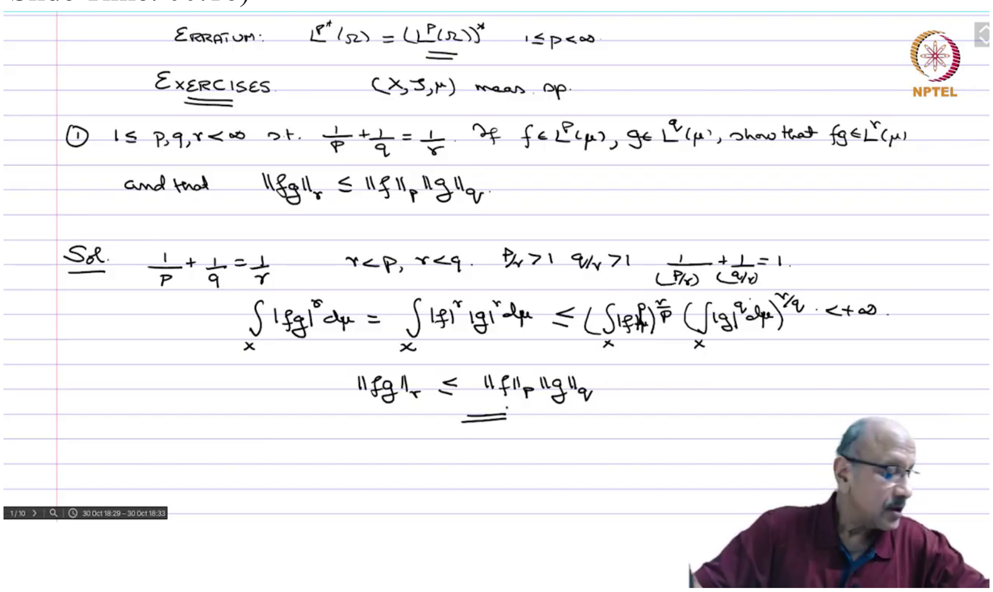

Before I start I just have one correction to make. Erratum. So, in the last lecture, just before proving Young's inequality I summarized the results of $L^p(\Omega)$ so I wrote $L^{p^*}(\Omega) = L^p(\Omega)$ by mistake. So, this should be actually $(L^p(\Omega))^*$, so $1 \le p < \infty$. So, this is the correction which you should do. So today, we will do some exercises.

Problem 1. $1 \le p,\ q,\ r < \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p(\mu)$, so throughout we will have $(X,\ S,\ \mu)$ is a measure space. I will not specify it in every exercise, this is the general hypothesis we always, so if $f \in L^p(\mu)$ and $g \in L^q(\mu)$, show that $fg \in L^r(\mu)$ and that norm $||fg||_r \le ||f||_p\, ||g||_q$.

Solution: So, you have that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ therefore you have that $r < p$ and $r < q$. So, $\frac{p}{r} > 1$, $\frac{q}{r} > 1$ and $\frac{1}{(p/r)} + \frac{1}{(q/r)} = 1$. So, now, if you take $\int_X |fg|^r\, d\mu$, so that is

$\int_X |fg|^r \, d\mu = \int_X |f|^r \, |g|^r \, d\mu$. Now, f is p integrable, so this is, f^r is $\frac{p}{r}$ integrable. And similarly, g^r is $\frac{q}{r}$ integrable so we can apply Holder's inequality.

$$\int_X |f|^r \, |g|^r \, d\mu \leq \left(\int_X |f|^p d\mu\right)^{1/(p/r)} \left(\int_X |g|^{r^{q/r}} d\mu\right)^{1/(q/r)} = \left(\int_X |f|^p \, d\mu\right)^{r/p} \left(\int_X |g|^q \, d\mu\right)^{r/q} < \infty,$$

now you take the r-th root on both sides ($|fg|^r$ is integrable) and $||fg||_r \leq ||f||_p \, ||g||_q$. So, that is the solution.

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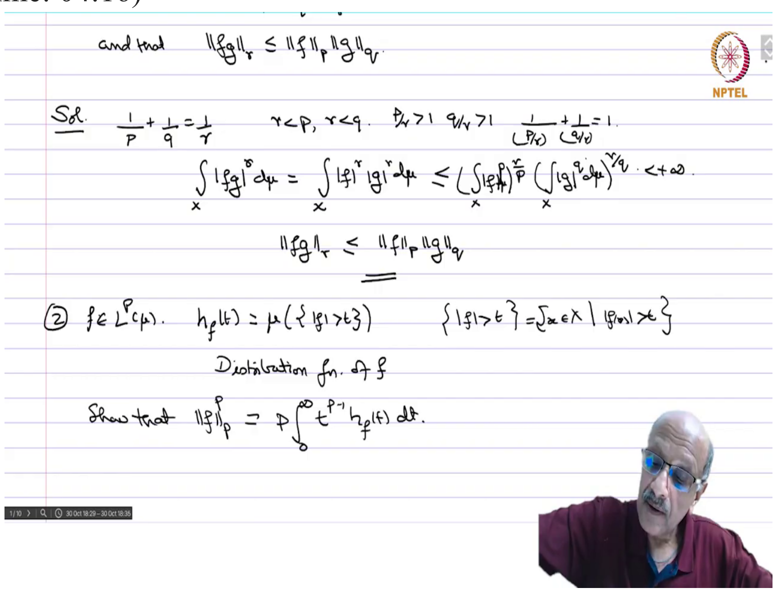

Problem 2. Let $f \in L^p(\mu)$. We will define $h_f(t) = \mu(\{|f| > t\})$, recall $\{|f| > t\} = \{x \in X \ / \ |f(x)| > t\}$. So, this is a measurable set and therefore its measure can be defined and so this is called the distribution function of f. So then, show that, $||f||_p^p = p \int_0^\infty t^{p-1} \, h_f(t) \, dt$.

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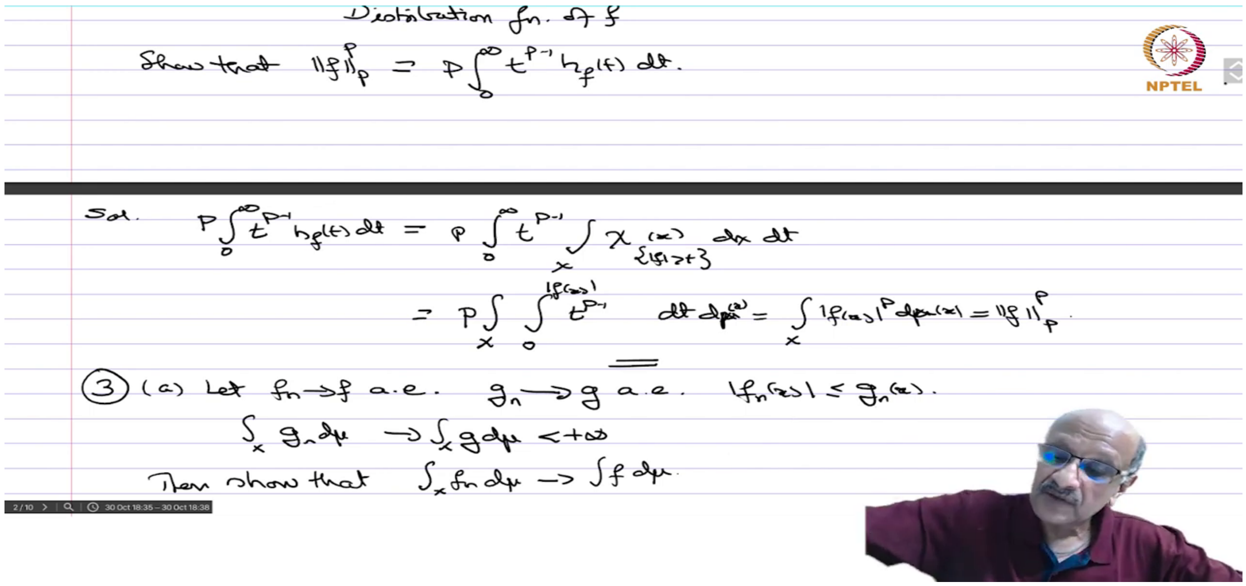

Solution. Let us take the right-hand side, i.e. $p\int_0^\infty t^{p-1}\, h_f(t)\, dt$. Now, what is $h_f(t)$? It is a measure of some sets so you can write $\int_X \chi_{\{|f|>t\}}(x)\, dx$. $p\int_0^\infty t^{p-1}\, h_f(t)\, dt =$ $p\int_0^\infty t^{p-1}\int_X \chi_{\{|f|>t\}}(x)\, dx\, dt$. So, you have, $\int_X \chi_{\{|f|>t\}}(x)\, dx$ is nothing but the measure, integral of characteristic function gives you the measure of the set so this is precisely, what you have as $h_f(t)$. Now, everything is nonnegative so we can interchange the order of integration without bothering about anything.

$$p\int_0^\infty t^{p-1}\int_X \chi_{\{|f|>t\}}(x)\, dx\, dt = \int_X \int_0^{|f(x)|} p\, t^{p-1} dt\, d\mu(x) = \int_X |f(x)|^p d\mu(x) = ||f||_p^p.$$

Problem 3 (a). Let $f_n \to f$ almost everywhere (all functions are measurable), $g_n \to g$ almost everywhere and $|f_n(x)| \le |g_n(x)|$. And also assume that $\int_X g_n\, d\mu$ converges to $\int_X g\, d\mu$ which is given to be finite. Then, show that $\int_X f_n\, d\mu \to \int_X f\, d\mu$. So, this is a generalized dominated convergence theorem. In dominated convergence theorem, you would have f_n's are all bounded by a single g and g is integrable. Now, you are relaxing. You are saying f_n is bounded by g_n, g_n converges to g and g is integrable.

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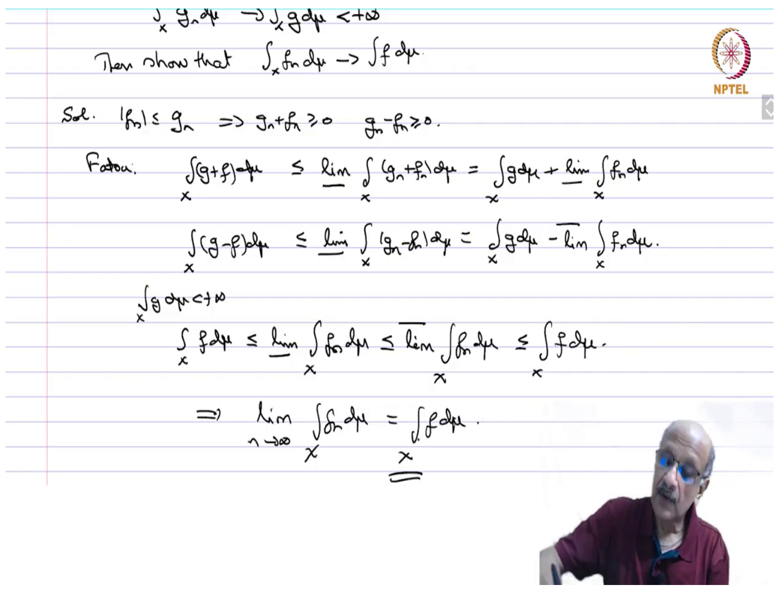

Solution. So, $|f_n| \le |g_n|$. So, this implies $g_n + f_n \ge 0$ and $g_n - f_n \ge 0$. So, we can apply Fatou's Lemma. So, what does Fatou's Lemma say? So, $g_n + f_n \to g + f$. So, $\int_X g + f\, d\mu \le \lim\ \inf \int_X (g_n + f_n) d\mu = \int_X g\, d\mu + \lim\ \inf \int_X f_n\, d\mu$ as g_n converges. Similarly, $\int_X g - f\, d\mu \le \lim\ \inf \int_X (g_n - f_n) d\mu = \int_X g\, d\mu - \lim\ \sup \int_X f_n\, d\mu$.

Now, integral $\int_X g\, d\mu$ is finite and therefore we are allowed to cancel it in both these relations. So, we get $\int f\, d\mu \le \lim\ \inf \int f_n\, d\mu \le \lim\ \sup \int f_n\, d\mu \le \int f\, d\mu$. So, we have a sandwich in which either end is the same so this implies, $\lim\inf \int f_n\, d\mu = \lim\sup \int f_n\, d\mu = \int f\, d\mu$. Therefore, $\lim_{n \to \infty} \int_X f_n\, d\mu = \int_X f\, d\mu$.

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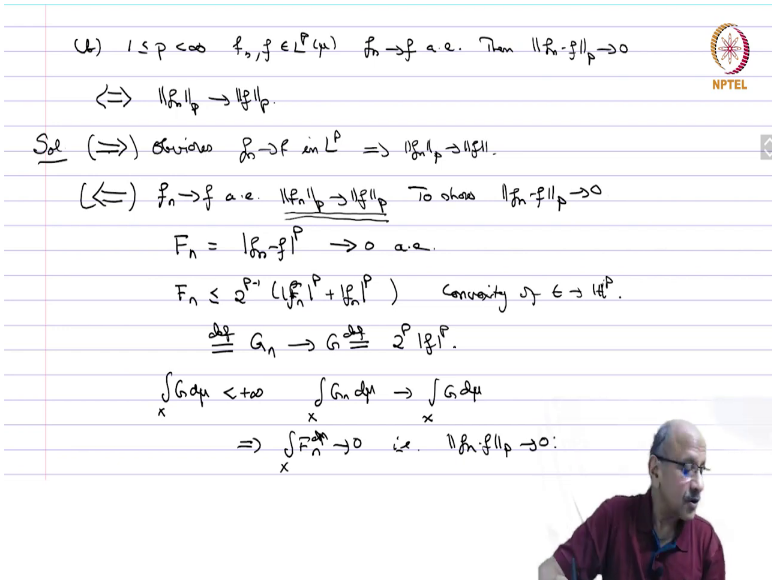

(b) we are going to use this. Let $1 \leq p < \infty$ and $f_n, f \in L^p(\mu)$, and $f_n \to f$ almost everywhere, then $||f_n - f||_p \to 0$, that is $f_n \to f$ in L^p if and only $||f_n||_p \to ||f||_p$.

So, this is a useful result because you have, if you know that $f_n, f \in L^p(\mu)$, if $f_n \to f$ pointwise, and the norm converges, this is usually easier to show, then you have that it actually converges in the L^p norm. So, this is a very useful result.

Solution. As the norm is a continuous function so if $f_n \to f$ in L^p, this implies that $||f_n||_p \to ||f||_p$ so that is true in any nonlinear space, so you don't have to do much. Now we are going to do the converse which is the interesting part, so $f_n \to f$ almost everywhere and we are given that $||f_n||_p \to ||f||_p$. And you have to show, $||f_n - f||_p \to 0$. So, let me define $F_n = |f_n - f|^p \to 0$ almost everywhere. And then by convexity, $|F_n| \leq 2^{p-1}(|f_n|^p + |f|^p)$ the convexity of $t \to |t|^p$.

Define $2^{p-1}(|f_n|^p + |f|^p) \equiv G_n \to G \equiv 2^p|f|^p$. And also, you have that $\int_X G\, d\mu < \infty$ because $\int_X G\, d\mu$ is $\int_X 2^P |f|^p$, $f \in L^p$. And also, $\int_X G_n\, d\mu \to \int_X G\, d\mu$ and that is precisely why we are using $||f_n||_p \to ||f||_p \Rightarrow f_n \to f$ part of the hypothesis. So, therefore, you have that all the conditions of the previous part are true and therefore $\int_X F_n d\mu \to 0$ that is $||f_n - f||_p \to 0$. So, this proves the converse.

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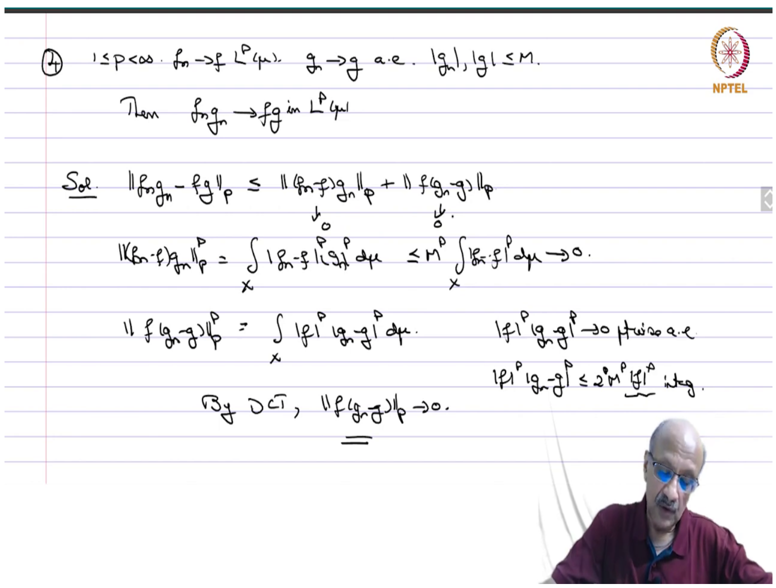

Problem 4. $1 \le p < \infty$, $f_n \to f$ in $L^p(\mu)$ and $g_n \to g$ almost everywhere and $|g_n|$, $|g|$ are all bounded by a fixed end constant. Then, $f_n g_n \to fg$ in $L^p(\mu)$. These kinds of theorems are very useful because we want to know the convergence of integral, which is the key thing in all this.

Solution. $||f_n g_n - fg||_p \le ||(f_n - f)g_n||_p + ||f(g_n - g)||_p$ by the Triangle inequality. So, let us take the first one. $||(f_n - f)g_n||_p$.

$$||(f_n - f)g_n||_p^p = \int_X |f_n - f|^p |g_n|^p d\mu \le M^p \int_X |f_n - f|^p d\mu \to 0.$$

Now, $||f(g_n - g)||_p^p = \int_X |f|^p |g_n - g|^p d\mu$, $|f|^p |g_n - g|^p \to 0$ pointwise, that is given to us. Also, $|f|^p |g_n - g|^p \leq 2^p M^p |f|^p$ and this is integrable. By dominated convergence theorem, we have $||f(g_n - g)||_p \to 0$. So, $f_n g_n \to fg$ in L^p.