

**Functional Analysis**  
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**Lecture No. 44**

**The Space  $L^1$  (contd)**

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**Prop.** Let  $\Omega \subset \mathbb{R}^N$  be open. Then  $L^1(\Omega)$  is not reflexive.

**Pf.** WLOG  $0 \in \Omega$ . Let  $n$  be suff. large s.t.  $B(0, \frac{1}{n}) \subset \Omega$ .  
 $= B_n$

$\alpha_n = |B_n|^{-1}$       $f_n(x) = \begin{cases} \alpha_n & \forall x \in B_n \\ 0 & \Omega \setminus B_n \end{cases}$



$f_n \in L^1(\Omega)$       $\int_{\Omega} |f_n| dx = \int_{B_n} \alpha_n dx = \alpha_n |B_n| = 1$ .

$\|f_n\|_1 = 1 \quad \forall n$ .

If  $L^1(\Omega)$  were reflexive  $\exists$  a w.-cpt. subseq.

If possible let  $f_{n_k} \rightarrow f$ .

i.e.  $\forall h \in L^{\infty}(\Omega)$       $\int_{\Omega} f_{n_k} h dx \rightarrow \int_{\Omega} f h dx$ .

We now have the following proposition.

**Proposition:** Let  $\Omega \subset \mathbb{R}^N$  be open. Then  $L^1(\Omega)$  is not reflexive. We saw that all the  $L^p$  spaces  $1 < p < \infty$  were reflexive and we said nothing about  $L^1$  and now, we show that it is not reflexive.

**Proof:** Without loss of generality, we can assume that  $0 \in \Omega$ . So, you can translate and by translation, you lose nothing. The Lebesgue measure is translation invariant and therefore we can always assume that  $0 \in \Omega$ . So, let  $n$  be sufficiently large such that, the ball  $B(0, \frac{1}{n}) \subset \Omega$ . So, all sufficiently large, this thing. Let us assume that  $\alpha_n$  is the Lebesgue measure of  $B_n$ , call this  $\alpha_n = |B_n|^{-1}$ . Then you set,  $f_n(x) = \alpha_n \quad \forall x \in B_n$  and

0 on  $\Omega \setminus B_n$ . So, then obviously  $f_n \in L^1(\Omega)$  and in fact, you have  $\int_{\Omega} |f_n| dx = \int_{B_n} \alpha_n dx = \alpha_n |B_n| = 1$ . So,  $\|f_n\|_1 = 1 \quad \forall n$ . So, if  $L^1(\Omega)$  were reflexive, there exists a weakly convergent subsequence. So, we have seen that any bounded sequence in a reflexive space has a weakly convergent subsequence. So, let us assume, if possible, let  $f_{n_k}$  weakly converge to  $f$ . So, this means,  $\forall h$  in the dual space which we have seen is  $L^\infty(\Omega)$ , you have  $\int_{\Omega} f_{n_k} h dx$  converges to integral  $\int_{\Omega} f h dx$ .

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$\forall h \in L^\infty(\Omega) \quad \int_{\Omega} f_{n_k} h dx \rightarrow \int_{\Omega} f h dx$   
 $h \equiv 1 \Rightarrow 1 = \int_{\Omega} f_{n_k} dx \rightarrow \int_{\Omega} f dx \Rightarrow \int_{\Omega} f dx = 1$   
 Now consider  $h \in C_c(\Omega \setminus \{0\})$ .  
 Then for  $k$  suff large  $\text{supp}(h) \cap B_{n_k} = \emptyset$ .  
 $\Rightarrow \int_{\Omega} f_{n_k} h = 0 \Rightarrow \int_{\Omega} f h = 0$ .  
 $\Rightarrow f = 0$  a.e. in  $\Omega \setminus \{0\}$  i.e.  $f = 0$  a.e. in  $\Omega$ .

So, if I take  $h \equiv 1 \Rightarrow 1 = \int_{\Omega} f_{n_k} dx \rightarrow \int_{\Omega} f dx \Rightarrow \int_{\Omega} f dx = 1$ .

On the other hand, let us consider  $h \in C_c(\Omega \setminus \{0\})$ , this is also an open set and of course is contained in  $C_c(\Omega)$ , it is also  $L^\infty$  function because it vanishes outside a compact set and then, so this is also in  $L^\infty$ . So, you can put  $h$  there. Then, so what do you have? You have  $\Omega$ , whereas the origin is there and then you have  $\Omega \setminus \{0\}$ , so we have removed this point.

Then, for  $n, k$  sufficiently large,  $\text{supp}(h) \cap B_{n_k}$  will be empty. This implies that

$$\int_{\Omega} f_{n_k} h \, dx = 0 \text{ and this implies that } \int_{\Omega} f h \, dx = 0. f \in L^1(\Omega) \text{ and therefore, for every}$$

$h \in C_c(\Omega \setminus \{0\}), \int_{\Omega} f h \, dx = 0$ . So, by the theorem which we have proved, we have,  $f = 0$

almost everywhere in  $\Omega \setminus \{0\}$ . Because for every continuous function with compact support

$\int_{\Omega} f h \, dx = 0$  and this is compact support in  $\Omega \setminus \{0\}$ . But then if you add  $\{0\}$ , still the set is of

measure 0 so that is  $f = 0$  almost everywhere in  $\Omega$ . But that is a contradiction because

$\int_{\Omega} f \, dx = 1$  and  $f = 0$ , so this is a contradiction and therefore, you cannot have a

weakly convergent subsequence for this sequence and therefore  $L^1$  is not reflexive.

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Then for  $k$  suff large  $\text{supp}(h) \cap B_{n_k} = \emptyset$ .

$$\Rightarrow \int_{\Omega} f_{n_k} h = 0 \Rightarrow \int_{\Omega} f h = 0.$$

$$\Rightarrow f = 0 \text{ a.e. in } \Omega \setminus \{0\} \text{ i.e. } f = 0 \text{ a.e. in } \Omega. \quad \times$$

Cor.  $\Omega \subset \mathbb{R}^N$  open, Then  $L^\infty(\Omega)$  is not reflexive.

To sum up:

- $L^p(\Omega) = L^q(\Omega) \quad 1 \leq p < \infty$ .
- $L^p(\Omega)$  reflexive & separable  $1 < p < \infty$ .
- $L^1(\Omega)$  separable but not reflexive
- $L^\infty(\Omega)$  not sep. not ref.

**Corollary:**  $\Omega \subset \mathbb{R}^N$  open, then  $L^\infty(\Omega)$  is not reflexive.

So, this is immediate. Because if  $V$  is reflexive,  $V^*$  is reflexive and conversely. Therefore

$(L^1(\Omega))^*$  is  $L^\infty(\Omega)$ ,  $L^1$  is not reflexive therefore  $L^\infty$  cannot be reflexive either.

To sum up, we have  $(L^p(\Omega))^* = L^{p^*}(\Omega)$ ,  $1 \leq p < \infty$ .  $p^*$  is the conjugate exponent. This is not true for  $L^\infty$ . We have not proved it, we will see in the exercises actually that there is a continuous linear function which we will do. But then, you have that  $L^p(\Omega)$  is reflexive and separable if  $1 < p < \infty$ .  $L^1(\Omega)$  is separable but not reflexive and finally  $L^\infty(\Omega)$  is not separable, not reflexive. So, this is the summary of all that we have done so far.

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Thm. (Young's Ineq.) Let  $1 < p < \infty$ . Let  $f \in L^1(\mathbb{R}^N)$ ,  $g \in L^p(\mathbb{R}^N)$ .

Then the map  $x \mapsto \int_{\mathbb{R}^N} f(y)g(x-y)dy$  is well-def for a.e.  $x$ . The fn. thus defined is called the convolution of  $f$  and  $g$  and is denoted  $f * g$ . Further  $f * g \in L^p(\mathbb{R}^N)$

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

Pf: Let  $h \in L^{p^*}(\mathbb{R}^N)$ .  $(x,y) \mapsto f(y)g(x-y)h(x)$  is mble.  $\mathbb{R}^N \times \mathbb{R}^N$ .

Consider 
$$I = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(y)g(x-y)h(x)| dy dx$$

So now, we will conclude with an important inequality. So,

**Theorem:** (Young's Inequality) Let  $1 < p < \infty$ . Let  $f \in L^1(\mathbb{R}^N)$ . So, we are now in the entire space  $L^1(\mathbb{R}^N)$  and  $g \in L^p(\mathbb{R}^N)$ . Then, the map  $x \mapsto \int_{\mathbb{R}^N} f(y)g(x - y)dy$  is well defined for almost every  $x$ . The function thus defined is called the convolution of  $f$  and  $g$  and is denoted by  $f * g$ . Further,  $f * g \in L^p(\mathbb{R}^N)$  and you have the Jensen's inequality,  $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ . So, this is called Young's inequality.

**Proof:** Let  $h \in L^{p^*}(\mathbb{R}^N)$ ,  $p^*$  is the conjugate exponent. Then, you consider the function  $(x, y) \rightarrow f(y)g(x - y)h(x)$ . So, this is measurable in the product space. Now, you

consider the iterated integral. So, consider,  $I = \int_{\mathbb{R}_y^N} \int_{\mathbb{R}_x^N} |f(y) g(x - y)h(x)| dy dx$ . This is

well defined because we are having a measurable function and modulus is non negative, therefore you can do.

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The slide contains handwritten mathematical notes. At the top, it says 'is well-def for a.e. x. The fn. thus defined is called the convolution of f and g and is denoted f \* g. Function f, g ∈ L^p(ℝ^N)'. Below this, it states the inequality  $\|f * g\|_p \leq \|f\|_1 \|g\|_p$ . A 'P.P.' (Proposition) follows: 'Let h ∈ L^p(ℝ^N). (x, y) ↦ f(y)g(x-y)h(x) is mble. ℝ^N × ℝ^N'. The word 'Consider' is written above the equation  $I = \int_{\mathbb{R}_x^N} \int_{\mathbb{R}_y^N} |f(y)g(x-y)h(x)| dy dx$ . Below this, it says 'Let meas translation invariant' followed by the equation  $I = \int_{\mathbb{R}_y^N} |f(y)| \left( \int_{\mathbb{R}_x^N} |g(x-y)h(x)| dx \right) dy$ . The final inequality is  $\leq \int_{\mathbb{R}_y^N} |f(y)| \|g\|_p \|h\|_p dy \leq \|g\|_p \|h\|_p \|f\|_1 < \infty$ . In the bottom right corner of the slide, there is a small video inset of a man in a blue shirt pointing at the notes.

So, Lebesgue measure is translation invariant and we can also interchange the order of integration, because you have a non negative function, Fubini's Theorem says that I can

interchange, so  $I = \int_{\mathbb{R}_y^N} |f(y)| \left( \int_{\mathbb{R}_x^N} |g(x - y)h(x)| dx \right) dy$ . Now, here you have  $g \in L^p$

and  $h \in L^{p^*}$ . So, by Holders inequality,

$$I \leq \int_{\mathbb{R}_y^N} |f(y)| \|g\|_p \|h\|_{p^*} dy \leq \|g\|_p \|h\|_{p^*} \|f\|_1 < \infty, \text{ since } y \text{ is not a fixed constant,}$$

so you are just translating the origin for this function and therefore as the Lebesgue measure is translation invariant so this will give you  $\|g\|_p \|h\|_{p^*}$ . But these are constants which come out and therefore the integral is less than equal to  $\|g\|_p \|h\|_{p^*}$ , then integral  $|f(y)|$  is just  $\|f\|_1$ . And that is of course finite.

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$\int_{\mathbb{R}^y} |f(y)| \|g\|_p \|h\|_p dy \leq \|g\|_p \|h\|_p \|f\|_1 < +\infty$

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By Fubini's thm. for almost every  $x$ ,  
 $\int_{\mathbb{R}^N} f(y) g(x-y) h(x) dy$  exists.  
 Choose  $h \in L^p(\mathbb{R}^N)$   $h \neq 0 \forall x \in \mathbb{R}^N$ . ( $h = e^{-|x|^2}$ )  
 $\Rightarrow$  for almost every  $x$ ,  
 $\int_{\mathbb{R}^N} f(y) g(x-y) dy$  exists.

By Fubini's theorem, if the iterated integral with the modulus is finite then for almost every  $x$ , we have  $\int_{\mathbb{R}^N} f(y) g(x - y) h(x) dy$  exists. That is part of Fubini's theorem.

When you have the iterated integral with the modulus is finite then for almost every value of one variable, the function in the other variable, the integral will exist. Here, you notice that  $h(x)$  does not depend on  $y$ . So, if we come out of the integral, so  $h(x)$  times this integral exists. Now, you choose  $h(x) \in L^{p^*}(\mathbb{R}^N)$  such that  $h(x) \neq 0 \forall x \in \mathbb{R}^N$ . For instance, you can take  $h = e^{-|x|^2}$ , this is a function which will never vanish in all the  $L^p$  spaces and therefore you can choose this, so, then you can divide by  $h(x)$  and therefore this implies for almost every  $x$ , we have  $\int_{\mathbb{R}^N} f(y) g(x - y) dy$  exists. So, mapping is well defined and therefore the convolution is also well defined.

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$\mathbb{R}^N$   
 Choose  $h \in L^{p^*}(\mathbb{R}^N)$   $h \neq 0 \forall x \in \mathbb{R}^N$ . ( $h = e^{-|x|^2}$ )  
 $\Rightarrow$  for almost every  $x$ ,  
 $\int_{\mathbb{R}^N} f(y)g(x-y)dy$  exists  
 $h \mapsto \int_{\mathbb{R}^N} h(x)(f * g)(x) dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h(x)f(y)g(x-y) dy dx$   
 is a cont. lin. fun. on  $L^{p^*}(\mathbb{R}^N)$   
 $|\int_{\mathbb{R}^N} h(x)(f * g)(x) dx| \leq \|h\|_{p^*} \|f\|_1 \|g\|_p$   
 $\Rightarrow (f * g)(x) \in L^p(\mathbb{R}^N)$   
 $\|f * g\|_p \leq \|f\|_1 \|g\|_p$

Now, we consider  $h \rightarrow \int_{\mathbb{R}^N} h(x)(f * g)(x) dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h(x) f(y) g(x - y) dy dx$ . And

therefore this is continuous, linear functional on  $L^{p^*}(\mathbb{R}^N)$  because  $h$  goes to this continuous linear functional and  $|\int_{\mathbb{R}^N} h(x)(f * g)(x) dx| \leq \|h\|_{p^*} \|f\|_1 \|g\|_p$ , we just calculated it earlier. This is a continuous linear functional and hence by the Riesz Representation Theorem,  $1 < p < \infty$ , so  $f * g \in L^p(\mathbb{R}^N)$  because that is a dual of  $L^{p^*}$  and you have  $\|f * g\|_p \leq \|f\|_1 \|g\|_p$  and that is exactly Young's inequality which we wanted to show.


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$\Rightarrow (f * g)(x) \in L^p(\mathbb{R}^n)$   
 $\|f * g\|_p \leq \|f\|_1 \|g\|_p$

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Remark. By a simple change of variable  
 $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y) dy$ .

Remark. Young's ineq. also true for  $p=1$ .  
 $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ .



**Remark.** By a simple change of variable, we also have  $f * g = \int_{\mathbb{R}^N} f(x-y)g(y)dy$ .

**Remark.** Young's inequality is also true for  $p = 1$ . So, if  $f$  and  $g$  are both in  $L^1$ ,  $f * g$  is well defined and you have  $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ . This is true and we will see it in the exercises.