Functional Analysis
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Lecture No. 44

The Space L-1 (contd)

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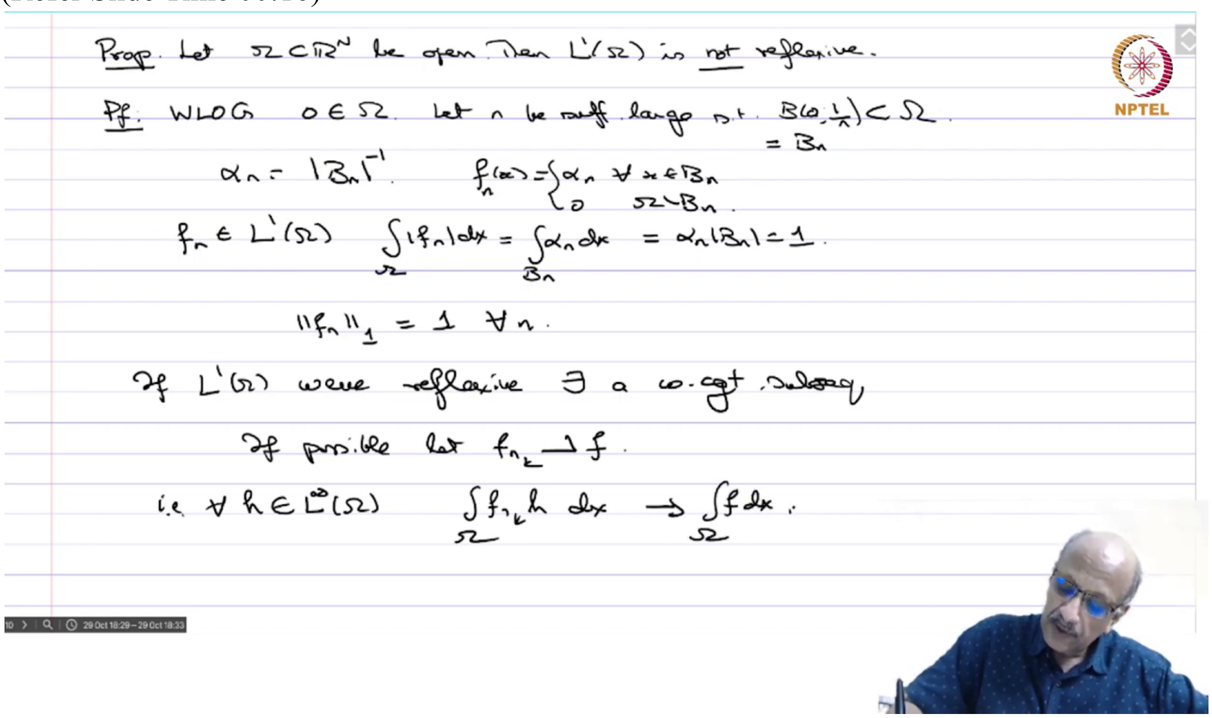

We now have the following proposition.

Proposition: Let $\Omega \subset \mathbb{R}^N$ be open. Then $L^1(\Omega)$is not reflexive. We saw that all the L^p spaces $1 < p < \infty$ were reflexive and we said nothing about L^1 and now, we show that it is not reflexive.

Proof: Without loss of generality, we can assume that $0 \in \Omega$. So, you can translate and by translation, you lose nothing. The Lebesgue measure is translation invariant and therefore we can always assume that $0 \in \Omega$. So, let n be sufficiently large such that, the ball $B(0, \frac{1}{n}) \subset \Omega$. So, all sufficiently large, this thing. Let us assume that α_n is the Lebesgue measure of B_n, call this $\alpha_n = |B_n|^{-1}$. Then you set, $f_n(x) = \alpha_n \ \forall x \in B_n$ and

0 on $\Omega \setminus B_n$. So, then obviously $f_n \in L^1(\Omega)$ and in fact, you have $\int_{\Omega} |f_n|\, dx = \int_{B_n} \alpha_n\, dx = \alpha_n |B_n| = 1$. So, $||f_n||_1 = 1 \ \forall\, n$. So, if $L^1(\Omega)$ were reflexive, there exists a weakly convergent subsequence. So, we have seen that any bounded sequence in a reflexive space has a weakly convergent subsequence. So, let us assume, if possible, let f_{n_k} weakly converge to f. So, this means, $\forall\, h$ in the dual space which we have seen is $L^{\infty}(\Omega)$, you have $\int_{\Omega} f_{n_k} h\, dx$ converges to integral $\int_{\Omega} f\, h\, dx$.

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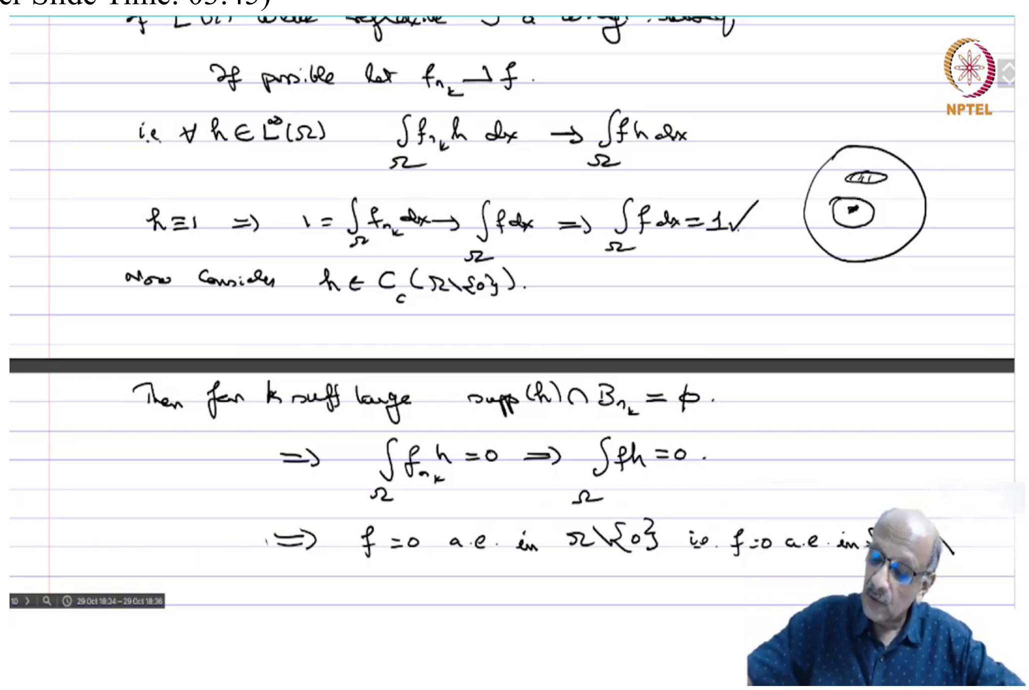

So, if I take $h \equiv 1 \Rightarrow 1 = \int_{\Omega} f_{n_k} dx \to \int_{\Omega} f\, dx \Rightarrow \int_{\Omega} f\, dx = 1$.

On the other hand, let us consider $h \in C_c(\Omega \setminus \{0\})$, this is also an open set and of course is contained in $C_c(\Omega)$, it is also L^{∞} function because it vanishes outside a compact set and then, so this is also in L^{∞}. So, you can put h there. Then, so what do you have? You have Ω, whereas the origin is there and then you have $\Omega \setminus 0$, so we have removed this point.

Then, for n, k sufficiently large, $supp(h) \cap B_{n_k}$ will be empty. This implies that $\int_{\Omega} f_{n_k} h\, dx = 0$ and this implies that $\int_{\Omega} f\ h\, dx = 0$. $f \in L^1(\Omega)$ and therefore, for every $h \in C_c(\Omega \backslash 0)$, $\int f\, h\, dx = 0$. So, by the theorem which we have proved, we have, $f = 0$ almost everywhere in $\Omega \setminus 0$. Because for every continuous function with compact support $\int f\, h\, dx = 0$ and this is compact support in $\Omega \setminus 0$. But then if you add 0, still the set is of measure 0 so that is $f = 0$ almost everywhere in Ω. But that is a contradiction because $\int f\, dx = 1$ and $f = 0$, so this is a contradiction and therefore, you cannot have a weakly convergent subsequence for this sequence and therefore L^1 is not reflexive.

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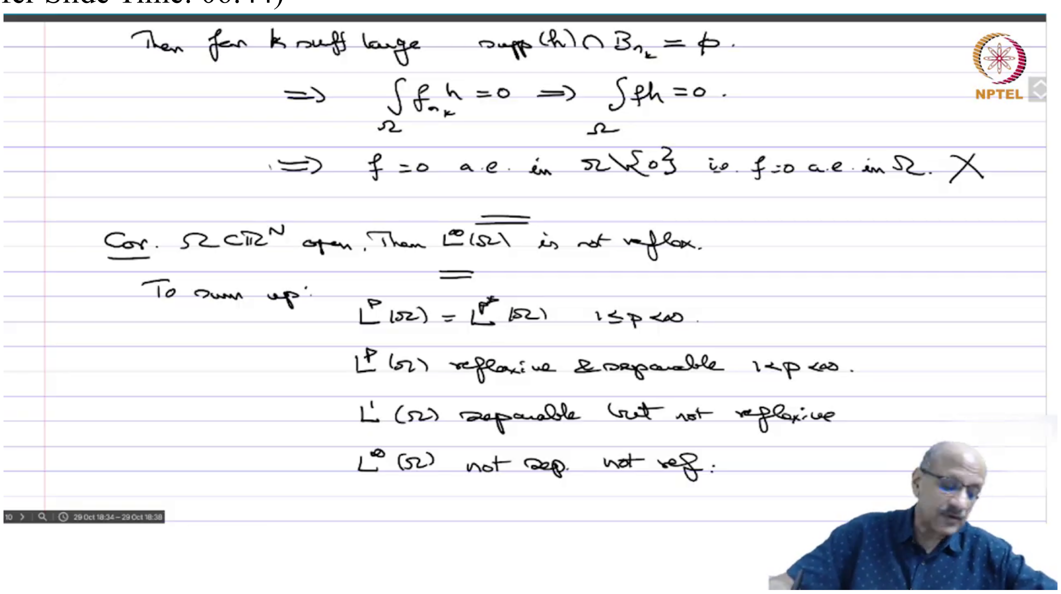

Corollary: $\Omega \subset \mathbb{R}^N$ open, then L^{∞} (Ω) is not reflexive.

So, this is immediate. Because if v is reflexive, v^* is reflexive and conversely. Therefore $(L^1(\Omega))^*$ is $L^{\infty}(\Omega)$, L^1 is not reflexive therefore L^{∞} cannot be reflexive either.

To sum up, we have $(L^p(\Omega))^* = L^{p^*}(\Omega)$, $1 \le p < \infty$. p^* is the conjugate exponent. This is not true for L^∞. We have not proved it, we will see in the exercises actually that there is a continuous linear function which we will do. But then, you have that $L^p(\Omega)$ is reflexive and separable if $1 < p < \infty$. $L^1(\Omega)$ is separable but not reflexive and finally $L^\infty(\Omega)$ is not separable, not reflexive. So, this is the summary of all that we have done so far.

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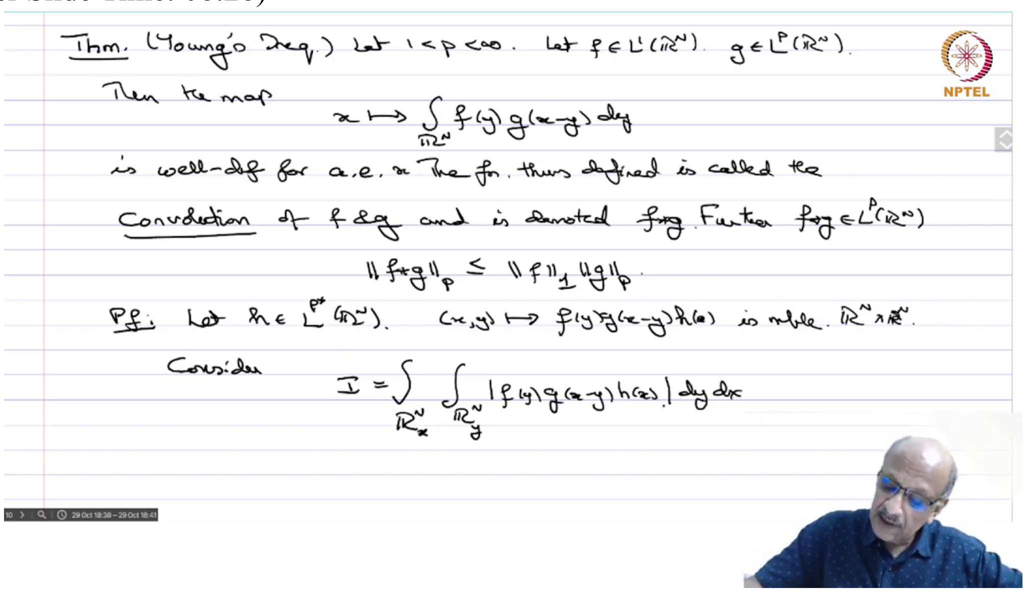

So now, we will conclude with an important inequality. So,

Theorem: (Young's Inequality) Let $1 < p < \infty$. Let $f \in L^1(\mathbb{R}^N)$. So, we are now in the entire space $L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$. Then, the map $x \mapsto \int_{\mathbb{R}^N} f(y) g(x-y)\,dy$ is well defined for almost every x. The function thus defined is called the convolution of f and g and is denoted by $f * g$. Further, $f * g \in L^p(\mathbb{R}^N)$ and you have the Jensen's inequality, $||f * g||_p \le ||f||_1 ||g||_p$. So, this is called Young's inequality.

Proof: Let $h \in L^{p^*}(\mathbb{R}^N)$, p^* is the conjugate exponent. Then, you consider the function $(x, y) \to f(y)\, g(x-y)\, h(x)$. So, this is measurable in the product space. Now, you consider the iterated integral. So, consider, $I = \int\limits_{\mathbb{R}^N_y} \int\limits_{\mathbb{R}^N_x} |f(y)\, g(x - y)h(x)|\, dy\, dx$. This is well defined because we are having a measurable function and modulus is non negative, therefore you can do.

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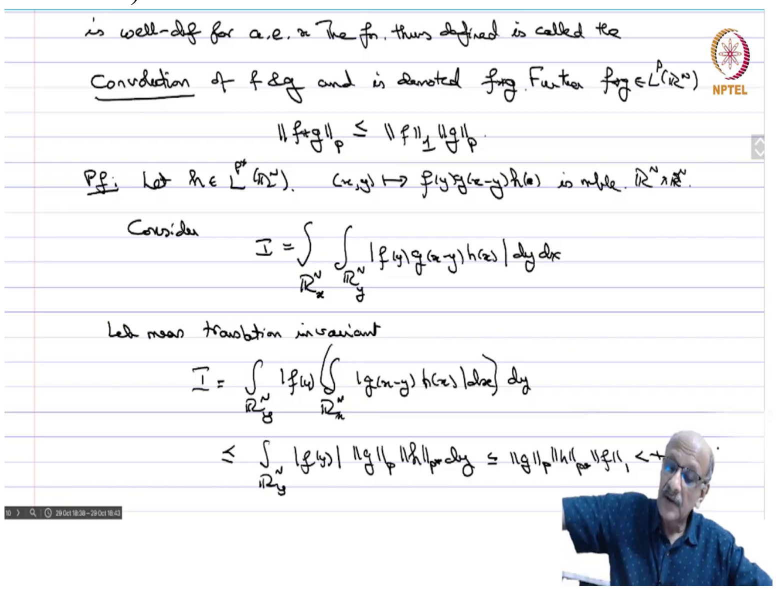

So, Lebesgue measure is translation invariant and we can also interchange the order of integration, because you have a non negative function, Fubini's Theorem says that I can interchange, so $I = \int\limits_{\mathbb{R}^N_y} |f(y)|\, (\int\limits_{\mathbb{R}^N_x} |g(x - y)h(x)|\, dx)\, dy$. Now, here you have $g \in L^p$ and $h \in L^{p^*}$. So, by Holders inequality,

$I \leq \int\limits_{\mathbb{R}^N_y} |f(y)|\; ||g||_p\, ||h|\, dy \leq ||g||_p\, ||h||_p\, ||f||_1 < \infty$, since y is not a fixed constant, so you are just translating the origin for this function and therefore as the Lebesgue measure is translation invariant so this will give you $||g||_p\, ||h||_{p^*}$ But these are constants which come out and therefore the integral is less than equal to $||g||_p\, ||h|$, then integral $|f(y)|$ is just $||f||_1$. And that is of course finite.

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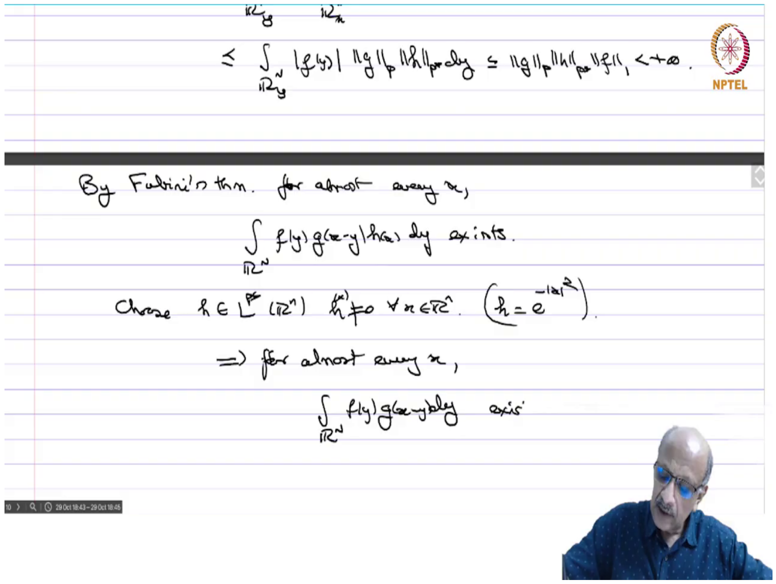

By Fubini's theorem, if the iterated integral with the modulus is finite then for almost every x, we have $\int_{\mathbb{R}^N} f(y)\, g(x-y)\, h(x)\, dy$ exists. That is part of Fubini's theorem. When you have the iterated integral with the modulus is finite then for almost every value of one variable, the function in the other variable, the integral will exist. Here, you notice that $h(x)$ does not depend on y. So, if we come out of the integral, so $h(x)$ times this integral exists. Now, you choose $h(x) \in L^{p^*}(\mathbb{R}^N)$ such that $h(x) \neq 0\ \forall x \in R^N$. For instance, you can take $h = e^{-|x|^2}$, this is a function which will never vanish in all the L^p spaces and therefore you can choose this, so, then you can divide by $h(x)$ and therefore this implies for almost every x, we have $\int_{\mathbb{R}^N} f(y)\, g(x-y)\, dy$ exists. So, mapping is well defined and therefore the convolution is also well defined.

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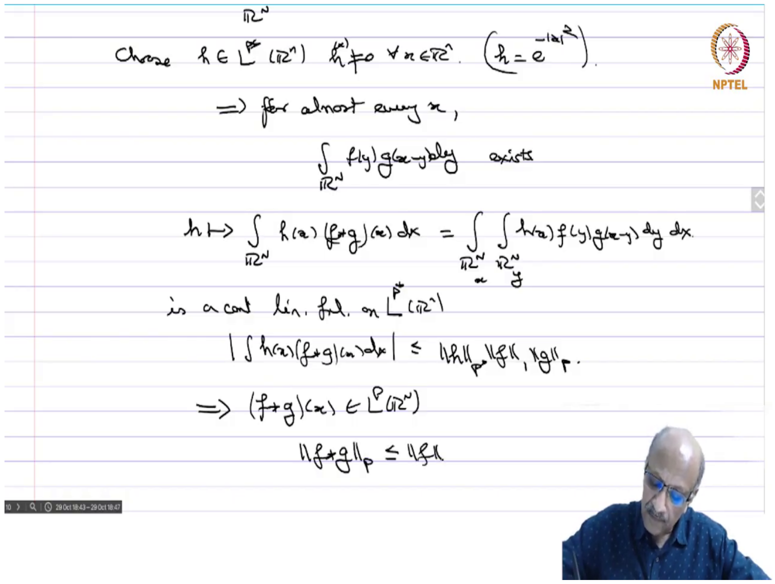

Now, we consider $h \to \int_{\mathbb{R}^N} h(x)(f * g)(x)dx = \int_{\mathbb{R}^N_x} \int_{\mathbb{R}^N_y} h(x)\, f(y)\, g(x - y)\, dy\, dx$. And therefore this is continuous, linear functional on $L^{p^*}(\mathbb{R}^N)$ because h goes to this continuous linear functional and $|\int h(x)\,(f * g)(x)dx| \leq ||h||_{p^*}||f||_1||g||_p$, we just calculated it earlier. This is a continuous linear functional and hence by the Riesz Representation Theorem, $1 < p < \infty$, so $f * g \in L^p(\mathbb{R}^N)$ because that is a dual of L^{p^*} and you have $||f * g||_p \leq ||f||_1||g||_p$ and that is exactly Young's inequality which we wanted to show.

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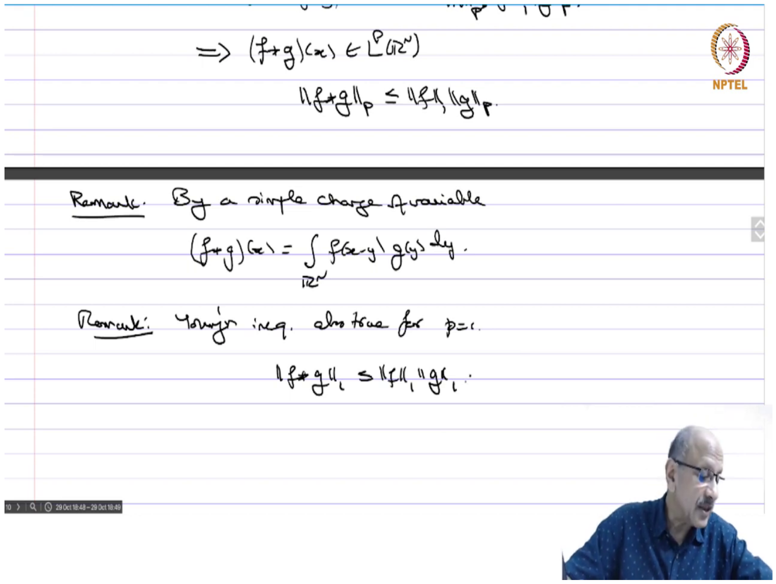

Remark. By a simple change of variable, we also have $f * g = \int_{\mathbb{R}^N} f(x-y)g(y)dy$.

Remark. Young's inequality is also true for $p = 1$. So, if f and g are both in L^1, $f * g$ is well defined and you have $||f * g||_1 \leq ||f||_1 ||g||_1$. This is true and we will see it in the exercises.