Functional Analysis
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Lecture No. 42
L-p Spaces in Euclidean spaces - Part 2

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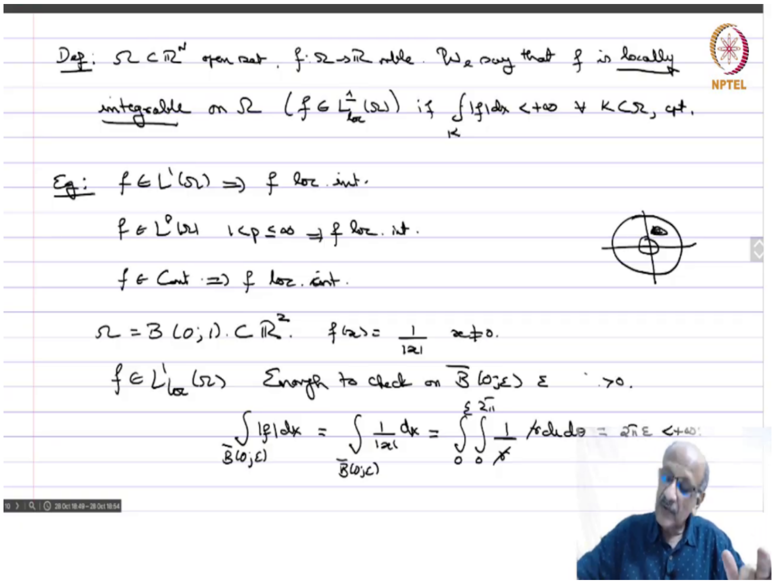

Do any, give an important definition.

Definition: So, Ω in R^N open set, f: $\Omega \rightarrow R$ measurable. We say that f is locally integrable on Ω and notation for that is $f \in L^1_{loc}(\Omega)$. If $\int_K |f(x)|dx$ is finite, for every K contained in Ω compact. So, if it is integrable on every compact subset, then you say it is locally integrable.

So, examples.

Example: So, $f \in L^1(\Omega)$ automatically implies that f is locally integrable. $f \in L^p(\Omega)$, $1 \leq p \leq \infty$, then also f is also locally integrable. Because, if it is L^∞, it is a bounded function and bounded function is always integrable on any compact set because compact sets are finite measure. And if it is in L^p, then it is L^p in any compact set also, and if it is L^p in a compact set, compact sets are finite measure and we have seen that it is L^1 in a compact measure. Then if f is continuous, f

continuous implies that f is locally integrable. Because here also it is bounded and therefore any continuous function on a compact set is bounded and therefore it is locally integrable.

Let me give one more example of locally integrable function which is an unbounded function and which does not come really in any of these categories.

Example: So, let us take $\Omega = B(0; 1)$ the ball center origin, radius 1. And contained in R^2. And then you take $f(x) = \frac{1}{|x|}$, if $x \neq 0$. So, this function is defined only outside the origin, it is defined almost everywhere. It is not defined at one point which has measure zero and it is unbounded. So, because as you go near 0, this becomes bigger and bigger and therefore the function blows up near the origin. We want to show that f is locally integrable. So, $f \in L^1_{loc}(\Omega)$. So, we have the unit ball. Now, if K is any compact set which does not contain the origin, then of course it is okay because, so there, the function is continuous, therefore it is bounded and therefore it is integrable. So, we only have to look at neighborhoods of the origin where f is unbounded and therefore, we have to check if it is locally integrable. So, enough to check on $\overline{B(0; \epsilon)}$, $\epsilon > 0$ small. So, this is closure, so closed ball, that is a compact set which contains the origin and any other compact set near the, containing the origin can be put inside some of, inside ball so in fact ϵ small is not necessary, $\epsilon > 0$ is fine. So there, it is enough to check this. So, let us integrate this. So, $\int_{B(0;\epsilon)} |f| dx$ and that is equal to $\int_{B(0;\epsilon)} \frac{1}{|x|} dx$ and I am going to use polar coordinates so this equal to $\int_0^{\epsilon} \int_0^{2\pi} \cdot r dr\, d\theta$. That is what $dx_1\, dx_2$ which is dx will become and then you have to put $\frac{1}{r}$ and the beauty is that this r gets canceled so you just get $2\pi\epsilon$ which is always finite. So, this is the function which is unbounded and which is locally integrable.

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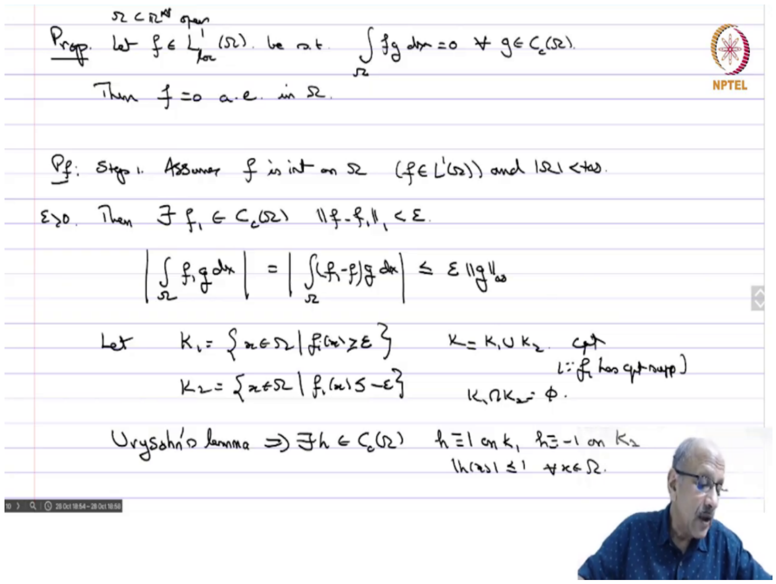

So now, we have the following proposition which will be useful for us in the future.

Proposition: So let $f \in L^1_{loc}(\Omega)$, so it is locally integrable, be such that $\int_{\Omega} fg\, dx = 0$ for all $g \in C_c(\Omega)$. So, $\Omega \subset R^N$ open set. Then, $f = 0$ almost everywhere.

So, this, in Ω. So, this is, if a function, when you integrate it multiply by any C_c, a continuous function with compact support, if you integrate it, if you get 0 as the integral then the function in fact has to be 0. So, we will prove this in some few steps.

Proof: So, Step 1: So, you assume f is integrable on Ω. That is f belongs not to L^1_{loc} but to $L^1(\Omega)$. So, that is why, as I said an example of a locally integrable function and $|\Omega$ is finite. So, let us assume both these properties. Then, there exists $f_1 \in C_c(\Omega)$ such that, so let $\epsilon > 0$, $\|f - f_1\|_1 < \epsilon$ because $C_c(\Omega)$ is dense in L^1. So then, $\left|\int_{\Omega} f_1 g\, dx\right| = \left|\int_{\Omega} (f_1 - f) g\, dx\right|$ because $\int_{\Omega} fg\, dx = 0$, so I have just subtracted 0 and $\left|\int_{\Omega} (f_1 - f) g\, dx\right| \leq \epsilon \|g\|_{\infty}$because this, norm, L^1 norm is less than ϵ and that is less than this. So now, you would define, let K_1 be the set of all $x \in \Omega$ such that $f_1(x) \geq \epsilon$ and K_2 equals set of all $x \in \Omega$ such that $f_1(x) \leq -\epsilon$. So, both of these sets are closed and $K = K_1 \cup K_2$. And they are closed subsets, and this is compact because f_1 has compact support. So, this is compact since f_1 has compact support. And then $K_1 \cap K_2 = \Phi$. So, we can have two disjoint closed sets and therefore we can use Urysohn's lemma from topology, implies there exists a h which is in $C_c(\Omega)$, such that $h \equiv 1$ on K_1 and $h \equiv -1$ on K_2, and also such that $|h(x)| \leq 1$ for all $x \in \Omega$. In fact it can be taken between -1 and 1. That is what we exactly are saying.

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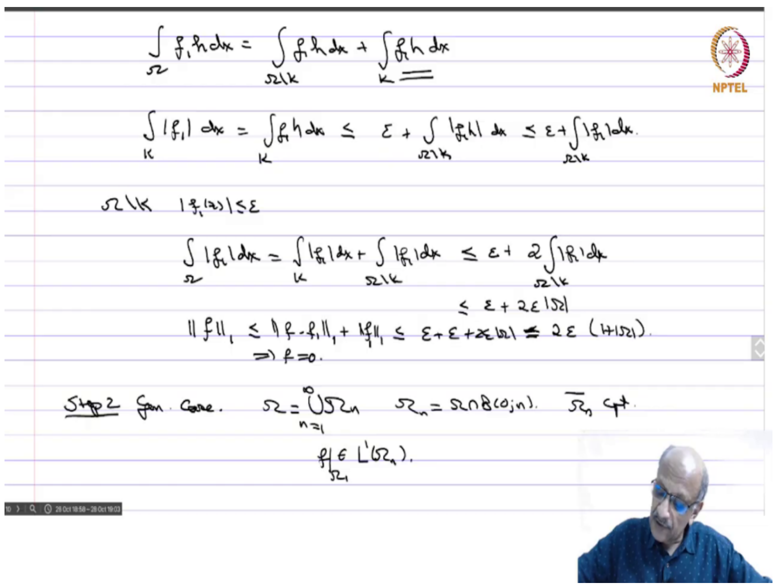

So now, if you have such a h. So, you have $\int_{\Omega} f_1 h\, dx = \int_{\Omega \setminus K} f_1 h\, dx + \int_{K} f_1 h\, dx$. I am just repeating the integral. Now, integral, so you have, integral of, what is on K? $h = +1$ on K_1, -1 on K_2 and therefore if you multiply, on K_1 it is non negative, K_2 it is negative and therefore if you multiply, you will get $|f_1|$ on K. That this is term, $f_1 h$ is equal to that. $\int_{K} |f_1| dx = \int_{K} f_1 h\, dx$ and $f_1 h \leq \epsilon \|h\|_{\infty}$, that is what we just saw. Where did we see that? $\int_{\Omega} f_1 g\, dx \leq \epsilon \|g\|_{\infty}$ for any C^{∞} function with compact support. So, this is less than $\epsilon \|h\|_{\infty}$ and then, plus the other term, $\int_{\Omega \setminus K} \left| f_1 h \right| dx$ and that is less than equal to $\epsilon + \int_{\Omega \setminus K} \cdot, |h|$ is again less than equal to 1, so $\left| f_1 \right| dx$. Therefore, but what happens on $\Omega \setminus K$. On $\Omega \setminus K$, $\left| f_1(x) \right| \leq \epsilon$ because on, what are K_1 and K_2? This is where it is bigger than epsilon so here it has to be less than ϵ.

Therefore, $\int_{\Omega} \left| f_1 \right| dx = \int_{\Omega \setminus K} \left| f_1 \right| dx + \int_{K} \left| f_1 \right| dx$, we have already seen, this is less than equal to

$\epsilon + 2 \int_{\Omega \setminus K} \left| f_1 \right| dx$. And on $\Omega \setminus K$, $\left| f_1 \right| \leq \epsilon$, so $\int_{\Omega} \left| f_1 \right| dx \leq \epsilon + 2\epsilon |\Omega \setminus K|$ and I can make $|\Omega \setminus K| \leq |\Omega|$. So, there is no problem, that is finite. So, $\|f\|_1 \leq \|f - f_1\|_1 + \|f_1\|_1$. $\|f - f_1\|_1 < \epsilon$, $\|f_1\|_1$ we have just computed, it is also less than $\epsilon + 2\epsilon|\Omega \setminus K|$. So, $\|f - f_1\|_1 + \|f_1\|_1 \leq \epsilon + \epsilon + 2\epsilon|\Omega \setminus K|$, this is less than equal to $2\epsilon(1 + |\Omega|)$ which can be made arbitrarily small and therefore this implies that $f = 0$. So, that is the first step where we assumed.

So now, Step 2: General case. So, you write again, $\Omega = \cup_{n=1}^{\infty} \Omega_n$. Ω_n is $\Omega \cap B(0; n)$. Then you know that $\overline{\Omega_n}$ is compact. So, f is locally integrable, therefore $f|_{\Omega_n} \in L^1(\Omega_n)$. And further, $|\Omega_n|$, which is the measure is also finite because it is contained in a ball.

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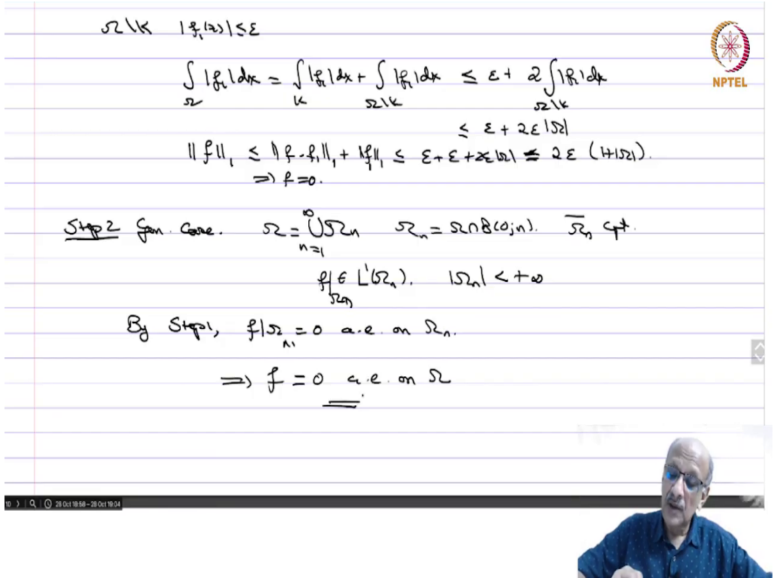

And therefore, by the previous, by Step 1, we have that $f|_{\Omega_n} = 0$ almost everywhere on Ω_n. So, there is a set E_n of zero measure, except on that, $f|_{\Omega_n}$ is 0. So, then if you take $f = 0$, almost everywhere on Ω because we are, only sets where it is not 0 are the E_n's which have measure 0 and countable union of sets of measure 0 is again, a set of measure 0 and therefore we have, this proposition is completely proved.