Functional Analysis

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Lecture No. 41

L-p Spaces in Euclidean spaces - Part 1

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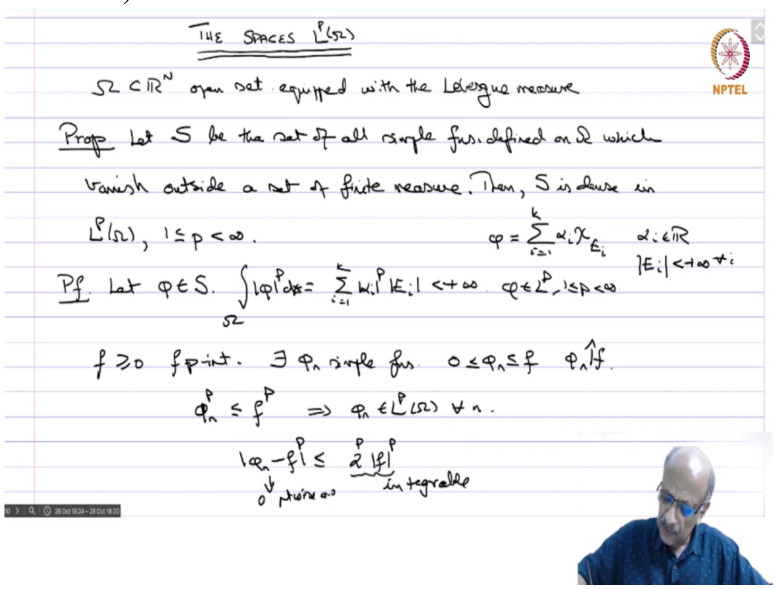

We will now talk about the spaces $L^p(\Omega)$. So, as I already said earlier Ω in R^N open set and equipped with the Lebesgue measure. So the set is Ω and then the sigma algebra is a Lebesgue sigma algebra and μ the measure is the Lebesgue measure. So, such spaces since the measure is fixed, so we concentrate on the domain instead and we call these spaces $L^p(\Omega)$, $1 \le p \le \infty$. So, we want to study the properties of these, these are very special. They occur in a lot of applications and therefore let us go ahead.

Proposition: Let S be the set of all simple functions defined on Ω, which vanish outside a set of finite measure. Then S is dense in $L^p(\Omega)$, $1 \le p < \infty$.

So, what do you mean by simple function? A simple function is a function ϕ of the form $\sum_{i=1}^{k} \alpha_i \chi_{E_i}$, where χ_E is the indicator function or the characteristic function of set E which is 1 if $x \in E$, 0 if $x \notin E$. So we are saying here, you can actually take the E_i disjoint and the α_i are all real numbers. So, we say that the function vanishes outside set of finite measures, means the measure of E_i, that is, $|E_i| <+ \infty$, the $|\cdot|$ symbol for the set will denote the Lebesgue measure and therefore the measure of E_i is finite for all x. That is what we mean by vanishing. So outside this, the union of E_i, $\phi = 0$ so it vanishes outside a set of finite measure. So, such functions we claim are dense in $L^p(\Omega)$.

Proof: So, let $\phi \in S$. Then ϕ vanishes outside a set of finite measure, then what is the value? So, $|\phi|^p$ is nothing but $\sum_{i=1}^{k} |\alpha_i|^p \chi_{E_i}$. So, if you take $\int_{\Omega} |\phi|^p dx$, this equal to $\sum_{i=1}^{k} |\alpha_i|^p |E_i|$ which is finite. So therefore, all these functions are in L^p. So, $\phi \in L^p, 1 \leq p < \infty$. In fact it is also in L^∞, because it takes only the values α_i or 0 and therefore it is also in L^∞ but that it is not relevant to this particular problem. So let f non-negative, f^p integrable. Now, any non-negative measurable function can be approximated by an increasing sequence of simple functions, so there exists ϕ_n simple functions, $0 \leq \phi_n \leq f$ and in fact ϕ_n is monotonic, so it increases to f. So this means that automatically, since $\phi_n^p \leq f^p$, everything is non-negative, so I do not have to put the modulus, so the integral is also finite, so this implies that $\phi_n \in L^p(\Omega)$ for all n. Furthermore, $\left|\phi_n - f\right| \leq 2|f|$ and therefore you put a power p, so you have $|\phi_n - f|^p \leq 2^p f^p$, $2^p f^p$ is integrable and $|\phi_n - f|^p \rightarrow 0$ point wise, almost everywhere.

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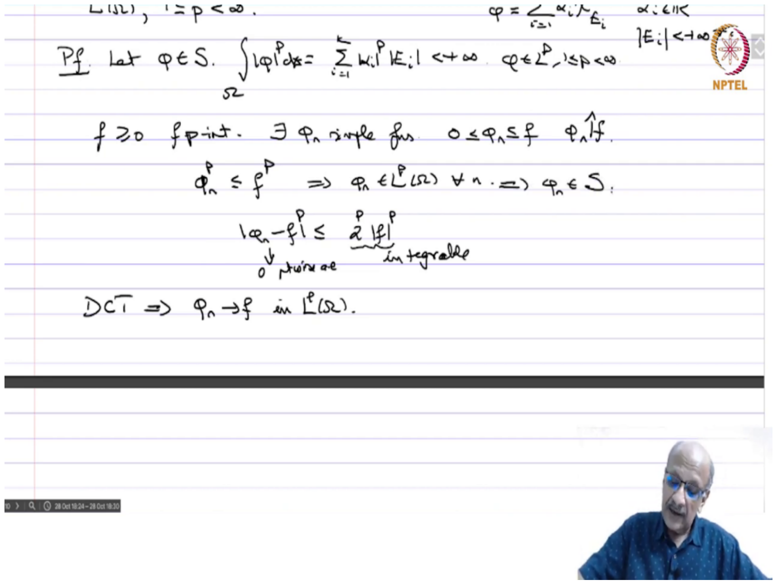

And therefore by the dominated convergence theorem implies that $\phi_n \to f$ in L^p. So, we have approximated f in fact by simple functions which vanish. Also, further, I forgot to say one thing. Since $\phi_n \leq f$, $\phi_n \in L^p$ and therefore this implies that ϕ_n has to belong to S also. Because automatically, because if it did not vanish outside a set of positive measure then it is a simple function, then the integral will blow up. So, ϕ_n automatically belongs to S and therefore we have approximated f by means of simple functions which vanish outside a set of finite measure.

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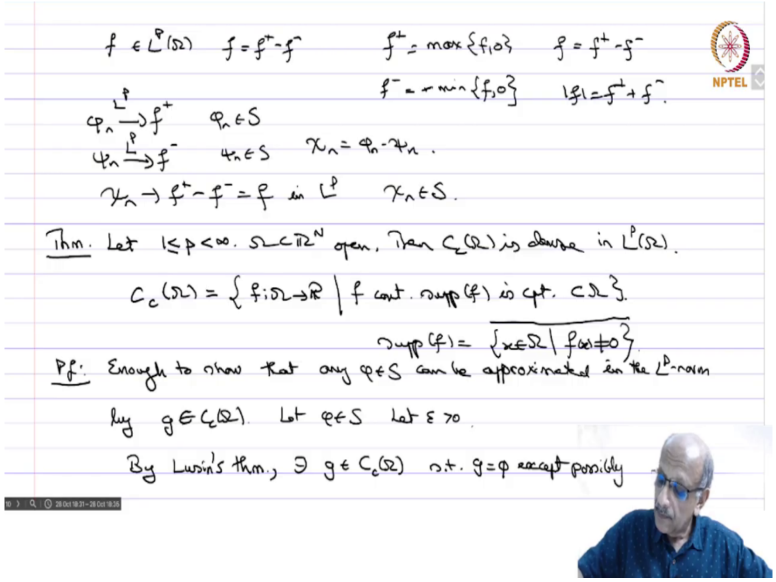

Now, if $f \in L^p(\Omega)$, you write $f = f^+ - f^-$, the positive and negative parts. So, f^+ is nothing but the $\max\{f, 0\}$, f^- is the $-\min\{f, 0\}$. So, both are non negative functions and you have $f = f^+ - f^-$ and $|f|$ in fact is equal to $f^+ + f^-$. So, then you have ϕ_n, you have $\phi_n \to f^+$, $\phi_n \in S$. This convergence is an L^p and then you can have ψ_n which also convergence in L^p to f^-, $\psi_n \in S$. So you put $\chi_n = \phi_n - \psi_n$, then $\chi_n \to f^+ - f^- = f$ in L^p and of course $\chi_n \in S$ and this proves the theorem. So, we have found that anything can be approximated.

So, this leads us to an important theorem. Let, so which we will repeatedly use.

Theorem: $1 \le p < \infty$, $\Omega \subset R^N$ open. Then $C_c(\Omega)$ is dense in $L^p(\Omega)$. So what is $C_c(\Omega)$? So $C_c(\Omega) = \{f : \Omega \to R : f \text{ is continuous and } supp(f) \text{ is compact and contained in } \Omega\}$. So, what is $supp(f)$? The set of all $x \in \Omega$ a such that $f(x)$ is different from 0 and then you have to take the closure. So, it is a closed set always and that set, if it is compact, then you will say it is a continuous function with compact support. So, these functions vanish outside a compact set. In particular, they vanish outside a set of finite measure. So that is, these are function but they are not simple, they are continuous functions. So we want to show that these functions are dense in $L^p(\Omega)$.

Proof: So, enough to show S, that any $\phi \in S$ can be approximated in the L^p norm by g in $C_c(\Omega)$. Because any $f \in S$ can be approximated by ϕ, if every ϕ can be approximated by g, by the triangle inequality you have that any f can be approximated as closely as you like by function g. So, let $\phi \in S$ and let $\epsilon > 0$. So, then by Lusin's theorem, so this is a theorem in analysis, which say, in measure theory, so we say there exists $g \in C_c(\Omega)$. So, it is a continuous function with compact support such that $g = \phi$ except possibly on a set of measure less than ϵ.

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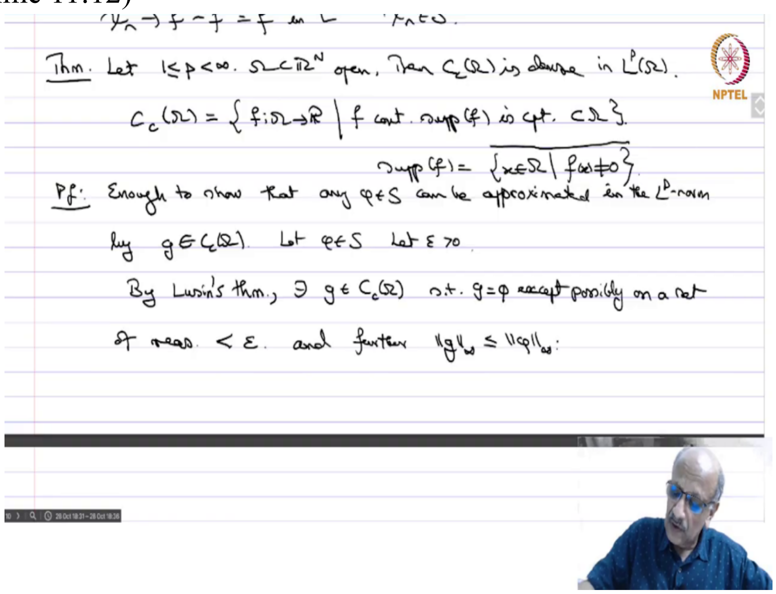

So, function, measurable function in S is almost like a continuous function with compact support. Namely, you can make it equal to a continuous function with compact support with failing except only where you fail that you be a set which is a very small measure and further $||g||_\infty$, sup norm is less than equal to $||\phi||_\infty$ which is also, ϕ is also in L^∞ as I said, remarked earlier and therefore, you have this.

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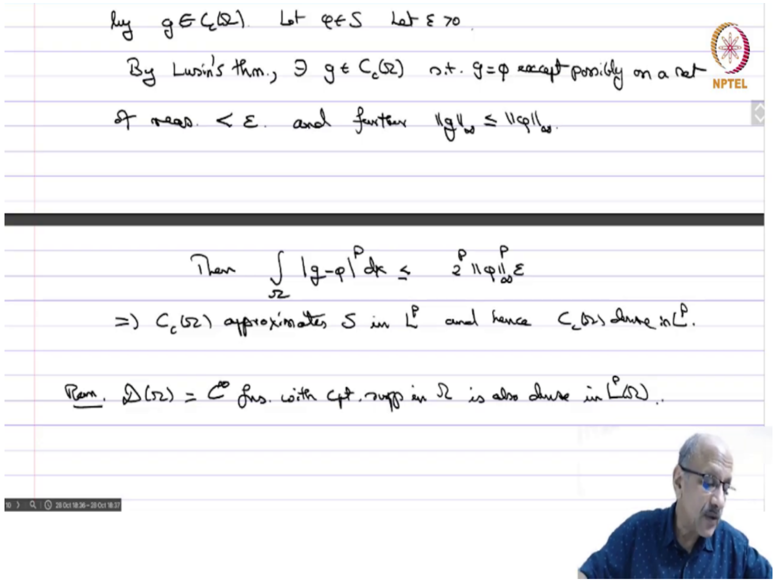

So then, what do you have? You have $\int_{\Omega} |g - \phi|^p dx$ is what? Is less than equal to, so $\|g\|_{\infty} \leq \|\phi\|_{\infty}$, so $\int_{\Omega} |g - \phi|^p dx \leq 2^p \|\phi\|_{\infty}^p \epsilon$, that will all come out. Then you have measure. You would not have measure of Ω because $g = \phi$ except on a set of measure 0. So, this function is 0 except it is nonzero only on a set of measure less than ϵ. So, that measure will come here so this is less than $2^p \|\phi\|_{\infty}^p \epsilon$ and therefore this shows that Cc omega is dense, approximates S in L^p in the sense I mentioned above and hence $C_c(\Omega)$ is dense in L^p.

Remark: In fact we can say $D(\Omega)$ which is C^{∞} functions with compact support in Ω, this is a much small set, $C_c(\Omega)$ is bigger, C^{∞} you want them to be infinitely differentiable, then this is also dense in $L^p(\Omega)$ but this requires more sophisticated tools from analysis like convolution and so on and we will not prove it in this place.

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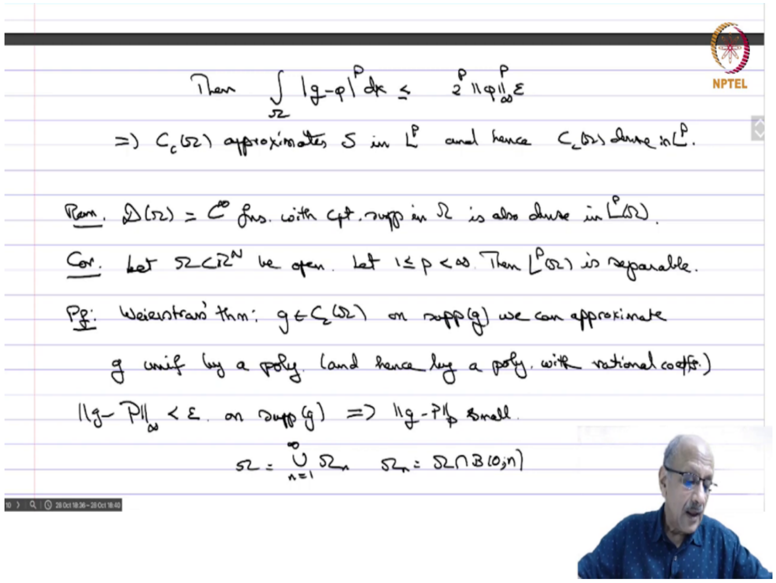

So next, corollary.

Corollary: Let Ω contained in R^N be open. Let $1 \leq p < \infty$. Then $L^p(\Omega)$ is separable. That means there exists a countable dense set.

Proof: So, Weierstrass Theorem says, so Weierstrass, what does it say? So, if $g \in C_c(\Omega)$ then on support of g, which is a compact set, we can approximate g uniformly that is in L^∞ norm by a polynomial and hence, by a polynomial with rational coefficients. So, if P is such a polynomial with rational coefficients and you have $\|g - P\|_\infty < \epsilon$ is as small as you like, then on support of g, outside the support g is 0, you put P also to be 0, so you get L^p function because it is continuous in the support and outside it is a closed set and outside it is 0, that is fine, support is of compact and therefore it has finite measure and therefore this function P is in L^p, so this will also imply that $\|g - P\|_p$ is small. Less than ϵ times the measure of something and so on so it will be, it is also small and therefore and notice that polynomials with rational coefficients on a compact set is a countable set. So, now we will write $\Omega = \cup_{n=1}^{\infty} \Omega_n$, where Ω_n is $\Omega \cap B(0; n)$, the ball in R^N which center origin and radius n.

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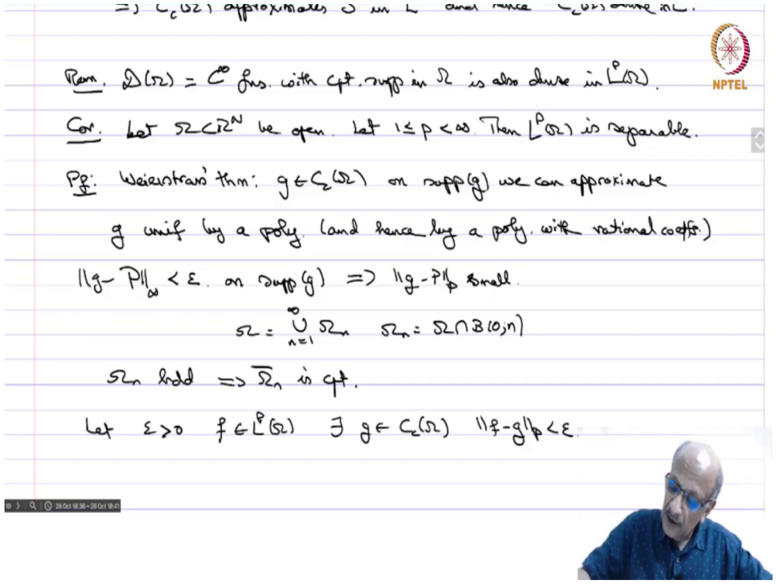

Then, you have that Ω_n is bounded and therefore $\overline{\Omega_n}$ is compact. Now, let $\epsilon > 0$ and $f \in L^p(\Omega)$. That means we take a representative which is a p-integrable function. Then, you have, there exists a $g \in C_c(\Omega)$ such that $\|g - f\|_p < \epsilon$.

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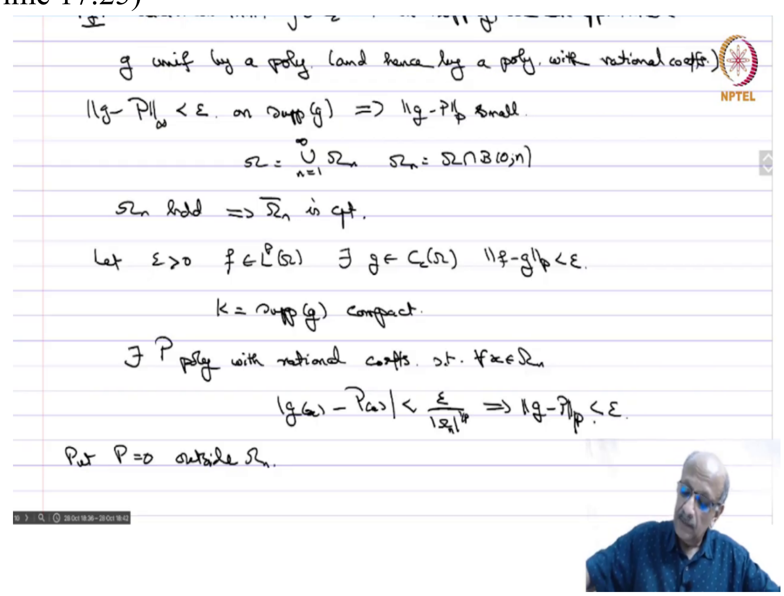

So, let K be the support of g compact. So, then we can find a polynomial, so there exists P polynomial with rational coefficients such that for all $x \in \Omega_n$, we have

$|g(x) - P(x)| < \frac{\epsilon}{|\Omega_n|^{\frac{1}{p}}}$ and therefore, this implies that $\|g - P\|_p$, now we put $P = 0$, outside Ω_n, then P will become L^p function as I explained earlier and you will have $\|g - P\|_p < \epsilon$, by the choice because outside Ω_n everything is 0 and inside it is less than ϵ by $|\Omega_n|^{\frac{1}{p}}$ so just a measure get multiplied.

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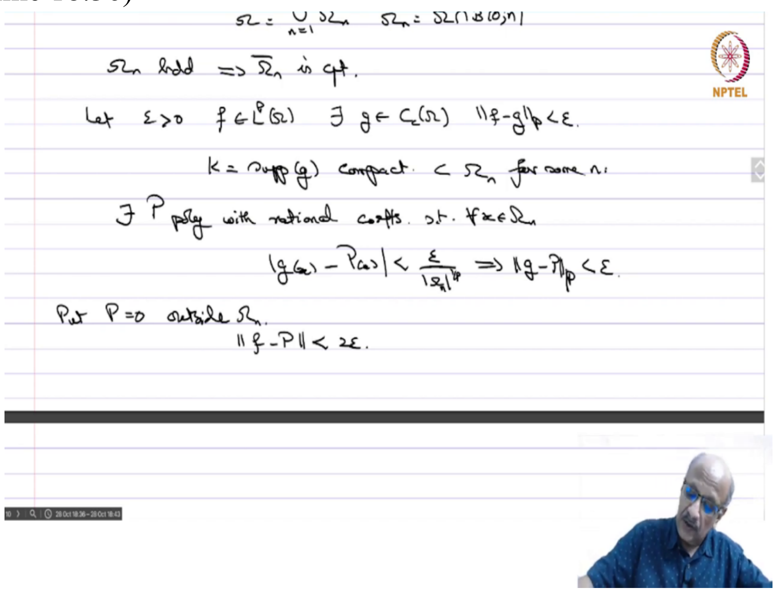

So, therefore you have $\|g - P\|_p < 2\epsilon$. So, you can always find a polynomial with rational coefficients with, on a compact set such that this is, so what is n here? So, this K is support, then this is contained in Ω_n for some n. Because K is a compact set, it is closed and bounded and therefore it has to be inside one of the Ω_n's. So, that is the n which we are taking here.

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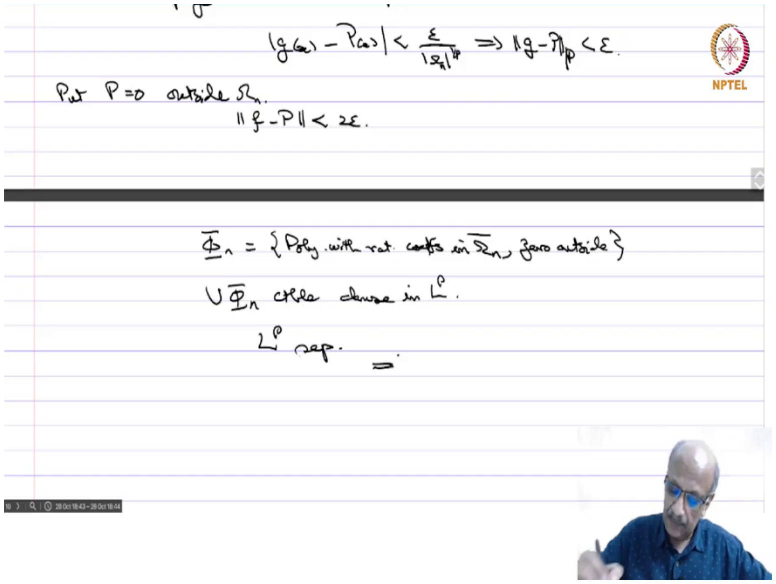

So now, you take Φ_n equals polynomials with rational coefficients in $\overline{\Omega_n}$ and 0 outside. So, this is a countable set and therefore $\cup\ \Phi_n$ is countable and we have just seen it is dense in L^p and therefore L^p is separable. So, this is the poof of this important corollary which says that all the L^p spaces other than L^∞ are separable. So what about L^∞ itself?

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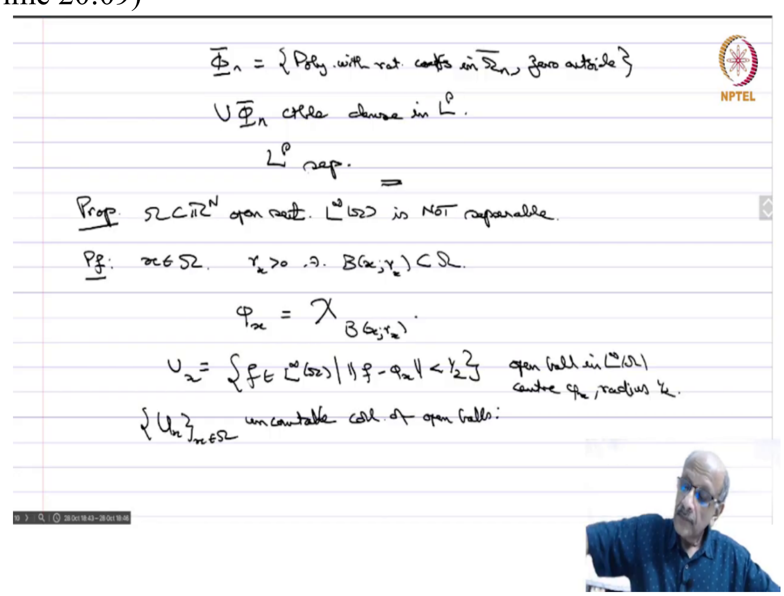

Proposition: Ω contained in R^N open set. Then $L^\infty(\Omega)$ is not separable. In fact we will show there is no countable dense set, so that is what we have to show.

Proof: So, let $x\in\Omega$. So then, you have r_x positive such that $B(x; r_x)$ is also contained in Ω. There will be a ball, which will be contained in Ω. So, let us take ϕ_x to be the function, which is the characteristic function of this ball and now, I am going to define U_x, so this is equal to the set of all $f\in L^{\infty}(\Omega)$ such that $\|f - \phi\|_{\infty} < \frac{1}{2}$. So this is open ball in $L^{\infty}(\Omega)$ center ϕ_x, radius $\frac{1}{2}$. So, these are all open sets. So, you have a collection, $\{U_x\}_{x\in\Omega}$, is an uncountable collection of open balls.

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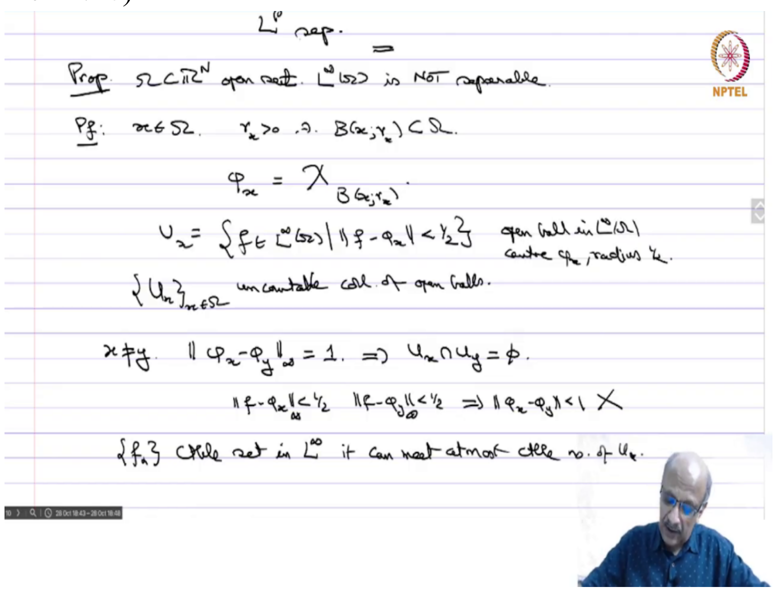

Now, what if $x\neq y$? Then you look at $\phi_x - \phi_y$. What are the possible values it can take? It can take the values 0 of course, if x, if the point value, evaluating it like outside both these balls, $B(x; r_x)$ and $B(y; r_y)$ or it can take 1 or -1, because if it is in one ball and not in another, if it is in both the, intersection of the two balls then it will be 0. So it will be 0, 1 or -1, therefore $\|\phi_x - \phi_y\|_{\infty}$ is always 1. And therefore if, so this implies that $U_x \cap U_y = \Phi$. Because if $f\in U_x$ and $f\in U_y$ then $\|f - \phi_x\|_{\infty} < \frac{1}{2}$, $\|f - \phi_y\|_{\infty} < \frac{1}{2}$ and by the triangle inequality $\|\phi_x - \phi_y\|_{\infty} < 1$, less than equal to these two which is strictly less than 1 which is a contradiction because we know it is equal to 1. So, $U_x \cap U_y = \Phi$.

So, if $\{f_n\}$ is a countable set in L^{∞}, it can meet at most countable number of U_x's because a particular f_n can belong to only one of the U_x, it cannot belong to two of them. So, each f_n will belong to one U_x, so if you take all the U_x's which contain these f_n's, they will be countable in number.

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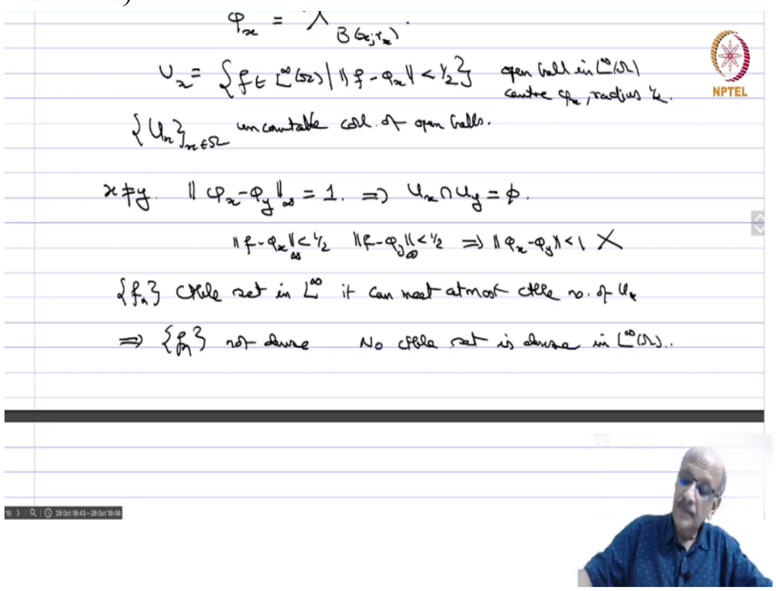

And therefore, there will be several sets which do not meet U_x, so this implies $\{f_n\}$ is not dense. Because several U_x's, that is an uncountable collection will be left out, they will not meet this set at all and therefore no countable set is dense in L^{∞} and therefore L^{∞} cannot be separable. So that is, we have shown, studied the separability of the thing, we saw the reflexivity of some of the L^p spaces and so now, we are going to.