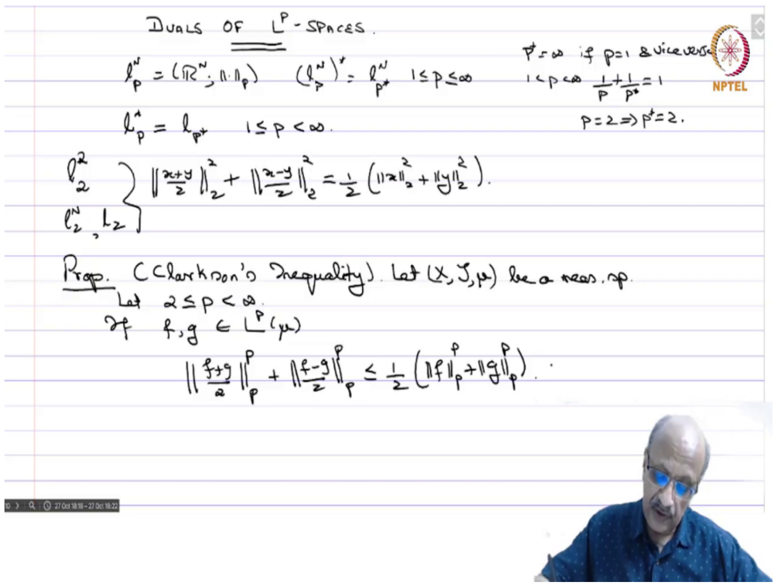
Functional Analysis

Professor. S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences

Lecture No. 40

Duality

(Refer Slide Time: 00:16)



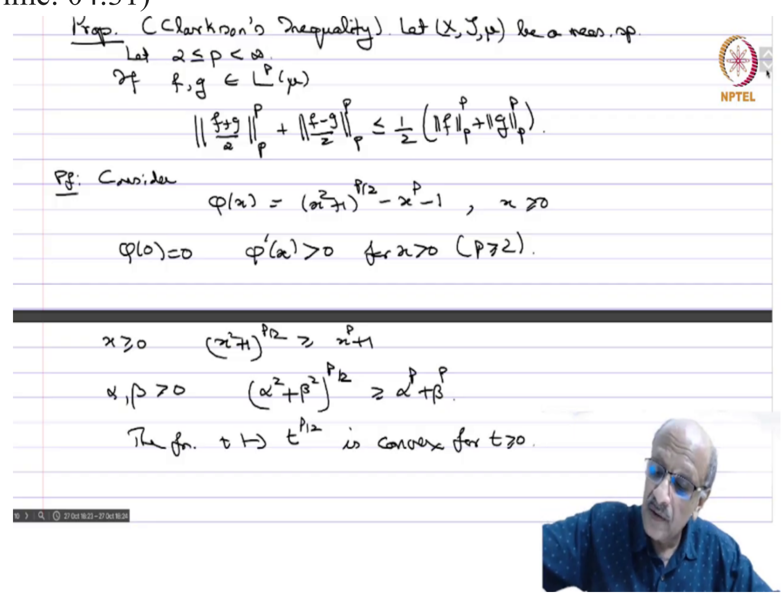
We will now study about Duals of L^p spaces. We have already done some of these competitions earlier. So, let us look at l_p^N which I recall is R^N RN with the $\|\cdot\|_p$ and so, this, $(l_p^N)^*$ is $l_{p^*}^N$ for $1 \le p \le \infty$. All these spaces being finite dimensional are reflexive and therefore this holds for the entire range of values of p where $p^* = \infty$ if $p = 1$ and vice versa and if $1 \le p < \infty$ then you have $\frac{1}{p} + \frac{1}{p^*} = 1$. So, in particular if $p = 2$, this implies $p^* = 2$ and then we have also computed l_p^* and this is equal to l_{p^*}, this is true for $1 \le p < \infty$ and we have seen that l_∞^* does not come from l_1. It is something bigger. So, these are the duals. So now, we will study for the L^p spaces given a measure space.

So before that, one preliminary result. So, in the case of l_2 or l_2^N, if you remember, so we proved the parallelogram law. So, namely $\|\frac{x+y}{2}\|_2^2 + \|\frac{x-y}{2}\|_2^2 = \frac{1}{2}(\|x\|_2^2 + \|y\|_2^2)$. So, this was the parallelogram law, the Apollonius Theorem in plane geometry which can be generalized also to in general to l_2^N, l_2 etc. All these spaces have this identity valid. So now, we are going to generalize this to L^p spaces. It is available, such inequality, if you will not get an equality, you will get an inequality but and it is available for all values of p in the range 1 to of ∞, but 1 to 2 is difficult. So, we will prove the easier portion of it.

So this proposition is called Clarkson's inequality.

Proposition(**Clarkson's inequality**): So, let (X, ζ, μ) be a measure space. Let $2 \le p < \infty$. So, if $f, g \in L^p(\mu)$, then you have $\|\frac{f+g}{2}\|_p^p + \|\frac{f-g}{2}\|_p^p \le \frac{1}{2}(\|f\|_p^p + \|g\|_p^p)$. So, this is the Clarkson's Inequality which generalizes the parallelogram law which we wrote earlier.

(Refer Slide Time: 04:31)

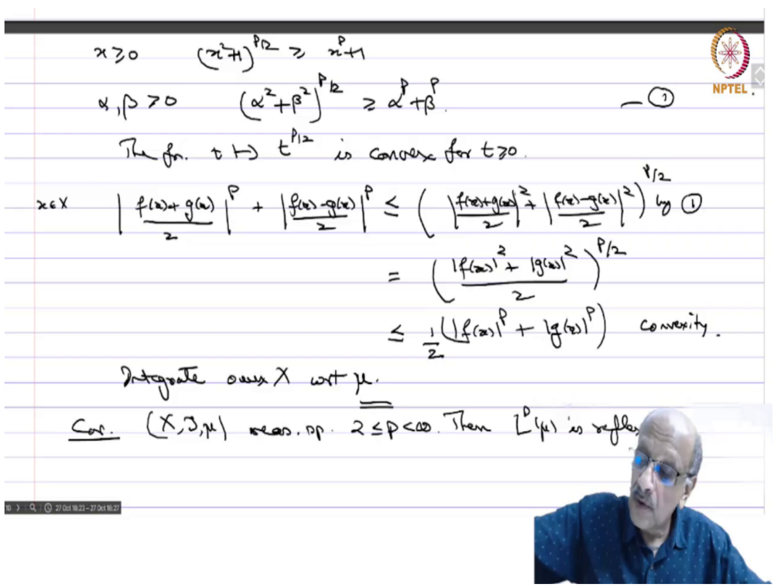


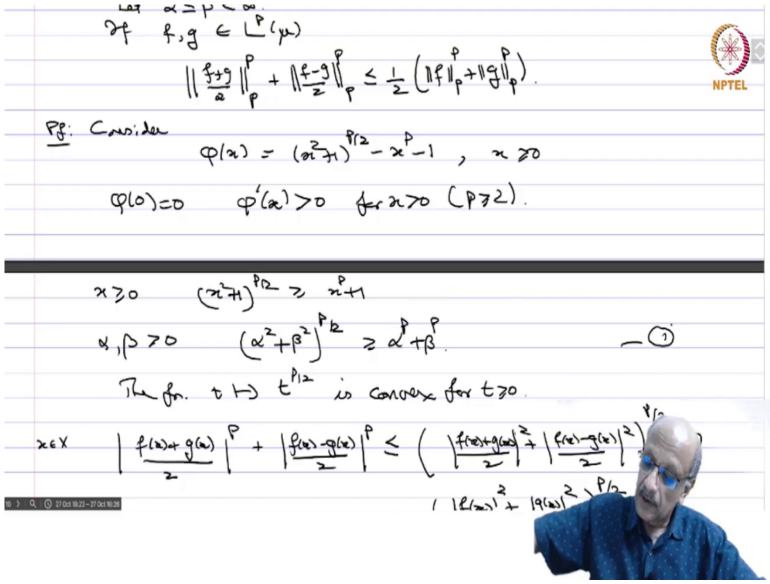
Proof: So consider the function $\phi(x) = (x^2 + 1)^{\frac{p}{2}} - x^p - 1$ for $x \ge 0$. So, then it is easy to see that $\phi(0) = 0$ and if you do the computation, $\phi'(x)$ will be positive for x positive and this is because $p \geq 2$. So, that is the reason why we need this range of values of p. So, you have this relationship, so it is just a straightforward competition of the derivative. So, ϕ is an increasing function, starts at 0 and therefore you have for $x \geq 0$, you have that $(x^2 + 1)^{\frac{p}{2}} \geq x^p + 1$ and so if α, β positive numbers, so then you have that

$$(\alpha^2 + \beta^2)^{\frac{p}{2}} \geq \alpha^p + \beta^p. \qquad (1)$$

All you have to do is to take $x = \frac{\alpha}{\beta}$ or $\frac{\beta}{\alpha}$ as you wish. The other thing which we want to, so this is one, the second thing which we want is the function $t \mapsto t^{\frac{p}{2}}$ is convex for $t \geq 0$. So, all you need to do again is to take the second derivative of this function and then because $p \geq 2$, you will get the second derivative is positive, non negative and therefore you have the convexity. So, using these two results we will now do the following.

(Refer Slide Time: 06:37)

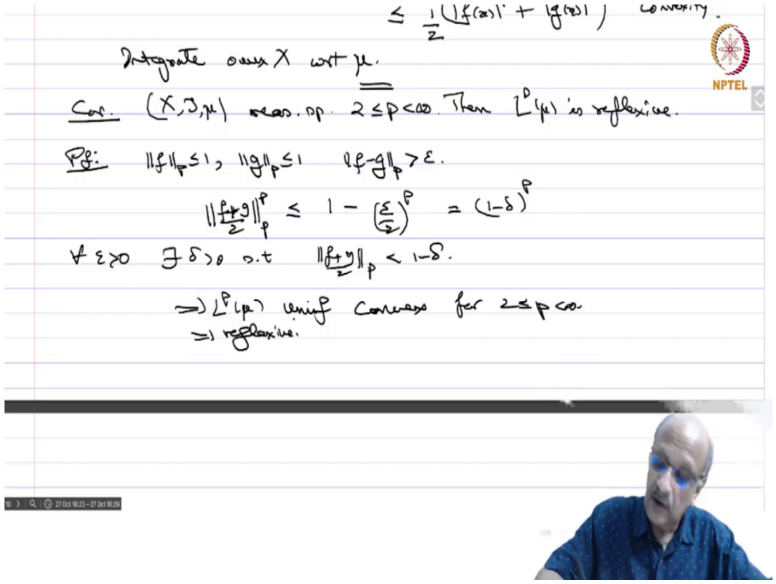




So, $\left|\frac{f(x)+g(x)}{2}\right|^p + \left|\frac{f(x)-g(x)}{2}\right|^p$, so this is $\alpha^p + \beta^p$. So, that is less than equal to $(\alpha^2+\beta^2)^{\frac{p}{2}}$. So, that is $\left(\left|\frac{f(x)+g(x)}{2}\right|^2 + \left|\frac{f(x)-g(x)}{2}\right|^2\right)^{\frac{p}{2}}$. Now, if you expand this thing, you will get f^2 two times, divided by 4, that will be f^2, $\frac{1}{2}$ of f^2, $\frac{1}{2}$ of g^2 and then you will, the cross terms will cancel. So you will get, $\left(\left|\frac{f(x)+g(x)}{2}\right|^2 + \left|\frac{f(x)-g(x)}{2}\right|^2\right)^{\frac{p}{2}} = \left(\frac{|f(x)|^2+|g(x)|^2}{2}\right)^{\frac{p}{2}}$. So now, you use the convexity of the function $t^{\frac{p}{2}}$, so this is the value at the midpoint and therefore that is less than equal to the average of the values at the end points, so this you have $\frac{1}{2}f^2$, evaluate power $\frac{p}{2}$ that will give you $\frac{1}{2}\left(|f(x)|^p + |g(x)|^p\right)$. So using, so here we have used the convexity and here you have used, (1) is this, this relationship here, we used (1). And so here, we used the convexity and therefore, so now, you just integrate over X with respect to μ and that will give you precisely whatever you want. So, this is the proof of Clarkson's Inequality.

Corollary: So if (X, ζ, μ) is a measure space and then $2 \leq p \leq \infty$, then $L^p(\mu)$ is reflexive.

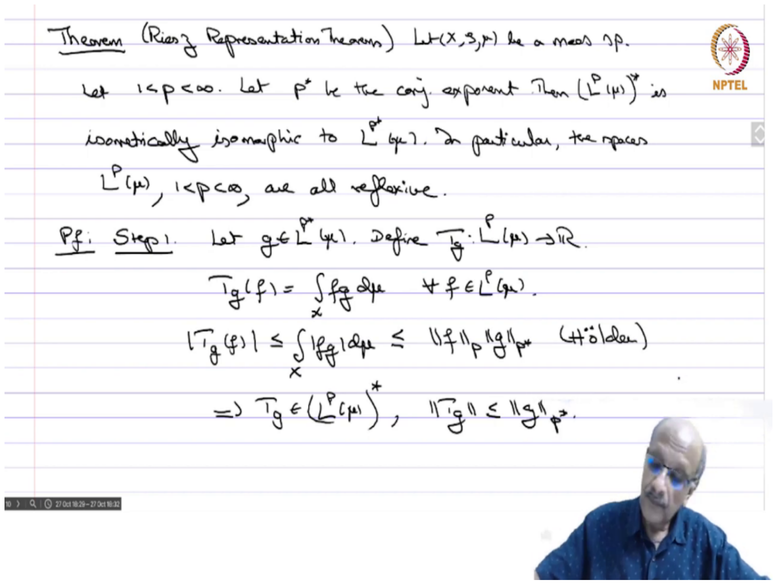
(Refer Slide Time: 09:26)



So, we are going to prove it.

Proof: So, we have $\|f\|_p \leq 1$, $\|g\|_p \leq 1$ and $\|f - g\|_p > \epsilon$, then what you get from Clarkson's Inequality is that $\left\|\frac{f+g}{2}\right\|_p^p \leq \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right) - \left\|\frac{f-g}{2}\right\|_p^p$, so that is each $\|f\|_p^p$, $\|g\|_p^p$ less than equal to 1, so 1 plus 1, 2 divided by 2 is just 1. So, that is less than 1 and then you have minus, $\|f - g\|_p > \epsilon$, so you get, so $\left\|\frac{f+g}{2}\right\|_p^p \leq \frac{1}{2} - \left(\frac{\epsilon}{2}\right)^p$. So, you can write this as $(1 - \delta)^p$ where you can compute what should be δ in view of this, you can write this like this and therefore you get that $\left\|\frac{f+g}{2}\right\|_p < 1 - \delta$. So, there exists δ such that for every ϵ positive and therefore $L^p(\mu)$ is uniformly convex for $2 \leq p \leq \infty$ implies reflexive. We have proved this theorem. Uniformly convex implies reflexive.

(Refer Slide Time: 11:18)



So now, we can prove the main theorem. So, this is the Riesz Representation Theorem. So, there is a whole family of theorems with the same name and they are all about computation of duals of some function or space or the L^p spaces, continuous functions, various things. So, these are all attributed mainly to Riesz who was a pioneer in this area of computing duals and also, representation.

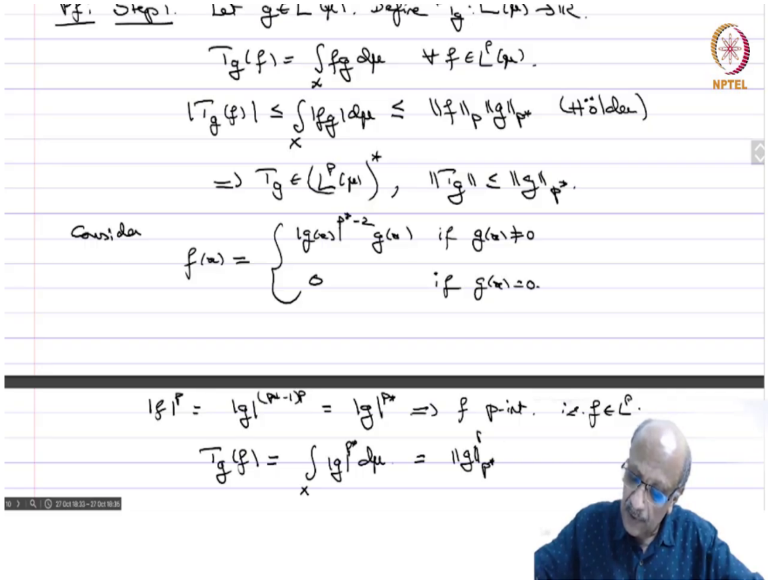
So, the abstract linear functional is given a concrete representation in terms of non objects. So, that is why it is called the Riesz Representation Theorem. So, there is a whole family of such theorems, this is just one of them.

Theorem (Riesz Representation theorem): So, let (X, ζ, μ) be a measure space and let $1 < p < \infty$. Let p^* be the conjugate exponent. Then, $(L^p(\mu))^*$ the dual is isometrically isomorphic to $L^{p^*}(\mu)$. In particular, this space is $L^p(\mu)$, $1 < p < \infty$ are all reflexive.

Proof: <u>Step 1</u>: So let $g \in L^{p^*}(\mu)$ and you define as we have done earlier in the case of sequence spaces $T_g: L^p(\mu) \to R$. So, $T_g(f) = \int_X fg\, d\mu$ and this is for all $f \in L^p(\mu)$. Therefore, $\left|T_g(f)\right| \le \int_X |fg|\, d\mu$. $f \in L^p$, $g \in L^{p^*}$, so by the Holder inequality,

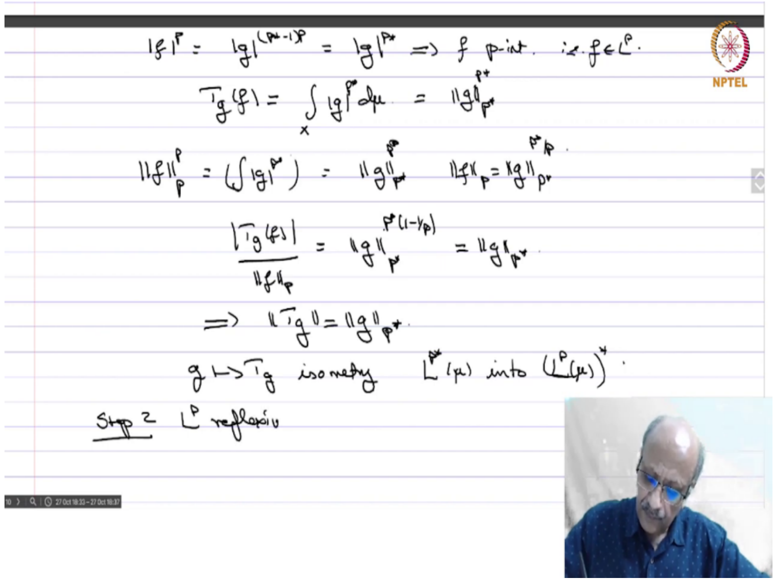
$\int_X |fg|\, d\mu \le \|f\|_p \|g\|_{p^*}$. So, this is Holder and therefore, $T_g \in L^{p^*}(\mu)$, and $\|T_g\| \le \|g\|_{p^*}$.

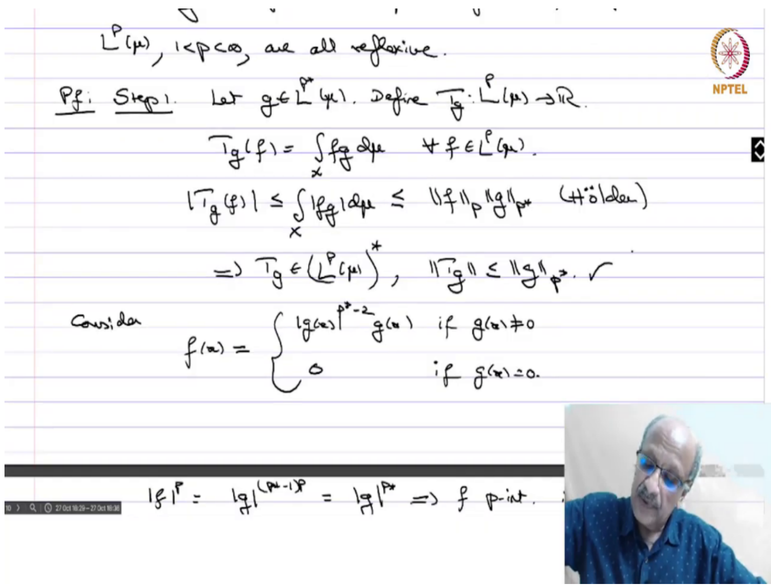
(Refer Slide Time: 15:00)



So now, you consider, consider the function $f(x)$ to be defined in the following way. This is $|g(x)|^{p^*-2}g(x)$, if $g(x)\neq 0$ and 0 if $g(x) = 0$. See, if $p^* \leq 2$, then $p^* - 2$ is negative, so $g(x)$ will go to the denominator so you will have trouble if you have $g(x) = 0$. So, we avoid that and then we put it as 0 if $g(x) = 0$. So essentially, it is seem to, you can still say it is this only $|g(x)|^{p^*-2}$ is not defined with $g(x) = 0$. So then, what is $|f|^p$? if you take $|f|$ you get $|g|^{(p^*-1)p}$, $p^*p - p$ is nothing but p^*, so this is $|g|^{p^*}$. So, this implies and this is g, this is integrable because $g \in L^{p^*}$ and therefore f is p integrable. That is, $f \in L^p$. And what is $T_g(f)$? So, I can apply f to this, so $T_g(f)$, if you recall is $\int_X fg\, d\mu$, so you have to multiply f by g, so you get g^2 and $p^* - 2$, so we will get $\int_X |g|^{p^*} d\mu$. So, this is nothing but $\|g\|_{p^*}^{p^*}$.

(Refer Slide Time: 17:11)

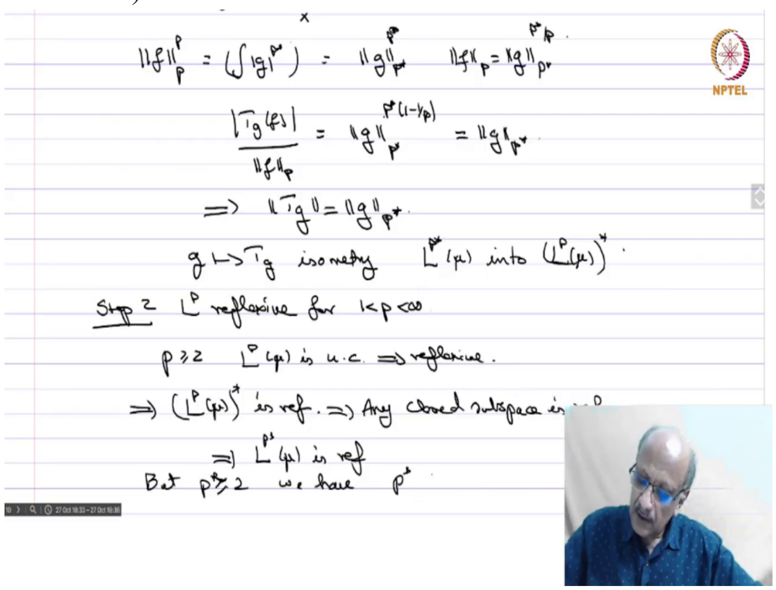




And what is $\|f\|_p$? $\|f\|_p^p$ is nothing but $\int_X |f|^p d\mu$, so that is nothing but $\int_X |g|^{p^*} d\mu$ and that is equal to $\|g\|_{p^*}^{p^*}$ and therefore $\|f\|_p = \|g\|_{p^*}^{\frac{p^*}{p}}$. So, if you substitute $\frac{|T_g(f)|}{\|f\|_p}$ you will get exactly $\|g\|_{p^*}^{p^*\left(1-\frac{1}{p}\right)}$ which is, just 1 and so this will give you $\|g\|_{p^*}$. So this, supremum of all $\frac{|T_g(f)|}{\|f\|_p}$ is norm $\|T_g\|$ and that is less than equal to $\|g\|_{p^*}$ because we know $\|T_g\| \leq \|g\|_{p^*}$ but we have actually found an f for which it is actually acting so this implies that $\|T_g\| = \|g\|_{p^*}$. So, so you have that $g \mapsto T_g$ is an isometry from $L^{p^*}(\mu)$ into $(L^p(\mu))^*$. So the image is a closed subspace of this. Our aim is to show in fact the image, is the whole thing. So,

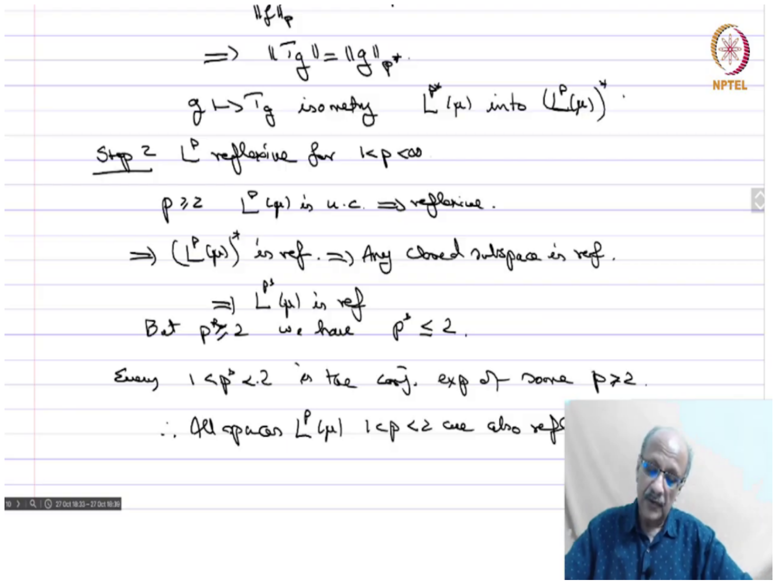
Step 2: So, L^p reflexive for $1 < p < \infty$.

(Refer Slide Time: 19:37)



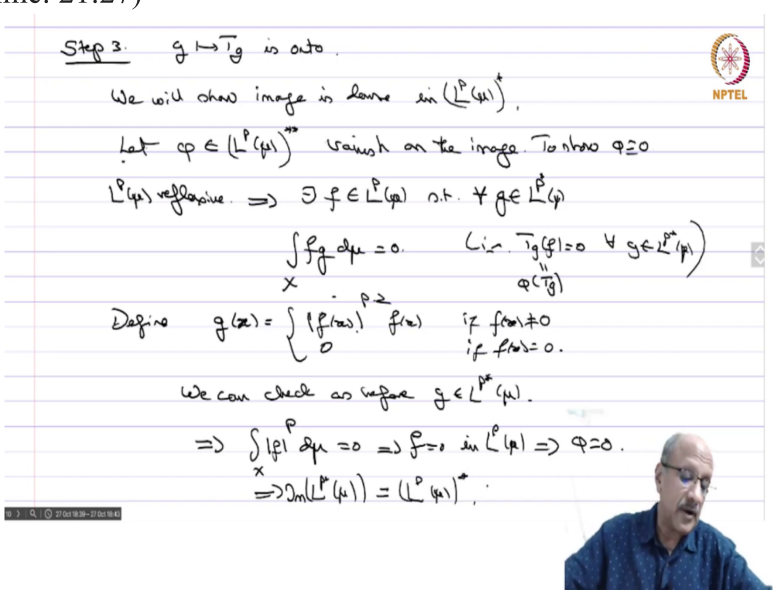
So, if $p \geq 2$, then you have that $L^p(\mu)$ is uniformly convex and therefore reflexive. So, this is already done in the corollary. So we now look at, so $L^p(\mu)$ is reflexive implies $(L^p(\mu))^*$ is reflexive, implies any closed subspace is reflexive, implies $L^{p^*}(\mu)$ is reflexive since it is isometrically isomorphic to sub, closed subspace of $(L^p(\mu))^*$. So this is, but if $p \geq 2$, we have $p^* \leq 2$.

(Refer Slide Time: 20:49)



and every $1 < p^{\star} < 2$ is the conjugate exponent of some $p\geq2$. Therefore all spaces $L^{p}(\mu)$, $1 < p < 2$ are also reflexive. So, this way we have proved the reflexivity of all the functions.

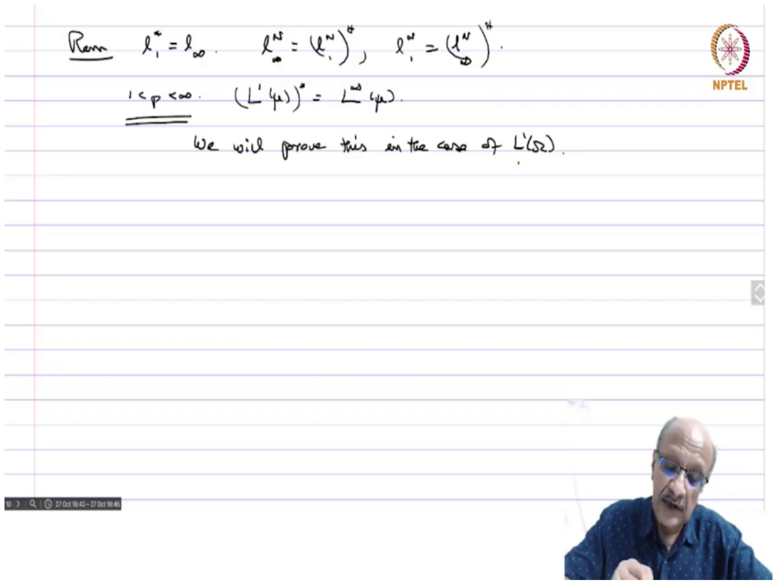
(Refer Slide Time: 21:27)



Step 3: So now, we will show that $g\mapsto T_{g}$ is onto. So we have shown that it is isometric to a closed subspace, so we need to show that it is a whole space. So, we will show image is dense in $(L^{p}(\mu))^{\star}$. So, how do you show that? By the Hahn- Banach Theorem. So if not,

so or you just, you do not need to do the contrary, so let $\phi \in (L^p(\mu))^{**}$ vanish on this image. So to show, $\phi \equiv 0$. But $L^p(\mu)$ is reflexive, so this implies that there exists a $f \in L^p(\mu)$ such that for all $g \in L^{p^*}(\mu)$, we have that $\int_X fg \, d\mu = 0$. Because, it vanishes on the image means T_g, $T_g(f) = 0$, $T_g(f)$ is nothing but $\phi(T_g)$, and for all $g \in L^{p^*}$. This is the meaning of this statement here. So now, again, we will do the same trick. So now, we define that, take $g(x) = |f(x)|^{p-2} f(x)$, if $f(x) \neq 0$ and $g(x) = 0$ if $f(x) = 0$. Once again, for the same reason like this and then you can show, we can check as before, g is now in $L^{p^*}(\mu)$ and then if you now substitute this, so this, so now, we can use this for g in this relationship. So, and therefore you will get $\int_X |f|^{p-2} f f \, d\mu = 0$ so that will give you $\int_X |f|^p \, d\mu = 0$. This will imply that f is 0 in $L^p(\mu)$ and that implies that $\phi = 0$. And therefore, you have that the image is dense but the image is already closed, so the image is equal to it's closure which is equal to $(L^p(\mu))^*$ and therefore, so this implies that image of $L^{p^*}(\mu)$ is nothing but $(L^p(\mu))^*$ and therefore it is isometric isomorphism and that completely proves the Riesz Representation Theorem in this case.

(Refer Slide Time: 25:48)



So earlier, so remark.

Remark: We have seen that $l_1^\star = l_\infty$ but $l_\infty^\star$ is not l_1 and so on. We have also seen that $l_\infty^N = \left(l_1^N\right)^\star$ and $l_1^N = \left(l_\infty^N\right)^\star$. These, either it is exercises or we have seen it. So now, we have not studied anything about $p = 1$ or ∞ in this theorem. We have only looked at $1 < p < \infty$. But if you want to show that $\left(L^1(\mu)\right)^\star = L^\infty(\mu)$, this can also be done. Now, this involves very complicated long measure theoretic arguments. In fact it is a package deal. This is proved and simultaneously $\left(L^p(\mu)\right)^\star$ is $L^{p^\star}(\mu)$ is also proved. So, the whole thing is proved but it is a fairly long proof and involves measure theoretic arguments whereas here using functional analytic arguments we have very easily proved, using reflexivity concepts about one for the, case $1 < p < \infty$. So this has not been done, but on the other hand, we will prove this result. So, we will prove this in the case of $L^1(\Omega)$. So, we will show that $L^1(\Omega)$ dual is $L^\infty(\Omega)$. So, I recall the definition of this, so we take Ω in R^N and equipped with the Lebesgue Measure. So, that is the space $L^1(\Omega)$. So, in that case we will show that this is indeed true.