Functional Analysis
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Lecture 4
Continuous Linear Maps - Part 1

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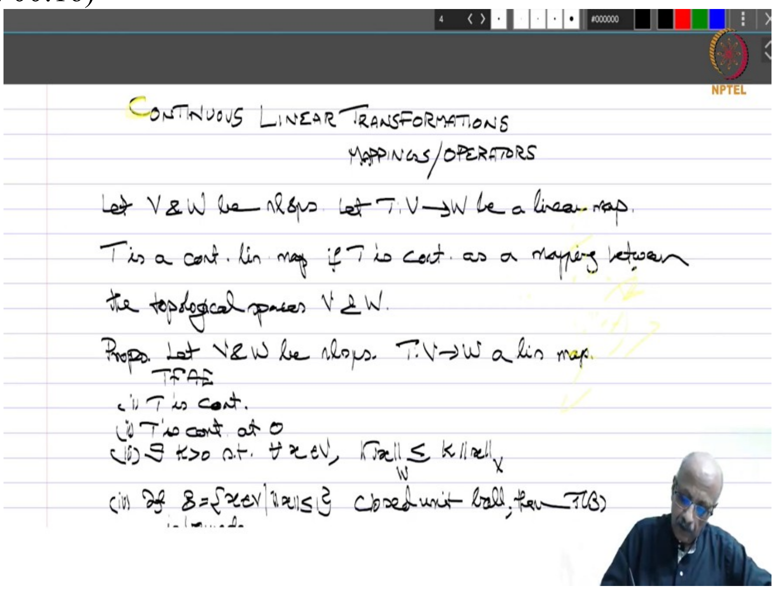

Continuous Linear Transformations. We will now look at continuous linear transformations. The word transformations could be replaced by mappings or operators. Let V and W be non-linear spaces. Let $T: V \mapsto W$ be a linear map (we know what is meant by a linear map between vector spaces). T is a continuous linear map if T is continuous as a mapping between the topological spaces V and W (V and W being normed-linear spaces, have their norm topology), i.e., if T is a map which is continuous with respect to these topologies, then you say T is a continuous map and if it is, in addition, a linear map, then we have a continuous linear map. Continuity is a local phenomenon, so you can also talk of continuous at a point or continuous, and a function is continuous on the space if it is continuous at every point.

We now have a very useful characterization of continuous linear mappings or transformations.

Proposition. Let V and W be norm-linear spaces. Let $T: V \mapsto W$ be a linear map. The following are equivalent.

(i) T is continuous (that means it is continuous at all points).

(ii) T is continuous at the origin.

(iii) There exists a constant $K>0$, such that $\forall x \in V$, we have $\|T(x)\|_W \leq K\|x\|_V$(since we are dealing with two spaces, let us specify the norms in each spaces. So, $\|.\|_V$ is the norm in V and $\|.\|_W$ is a norm in W).

(iv) If $B=\{x \in V : \|x\|_V \leq 1\}$ (this is called the closed unit ball), then $T(B)$ is bounded.

What we mean by a bounded set? A bounded set is something which is contained inside a ball. That means we have a constant such that $\|T(x)\|_W \leq C, \forall x \in B$. Then you say that it is a bounded set. So, let us prove this proposition.

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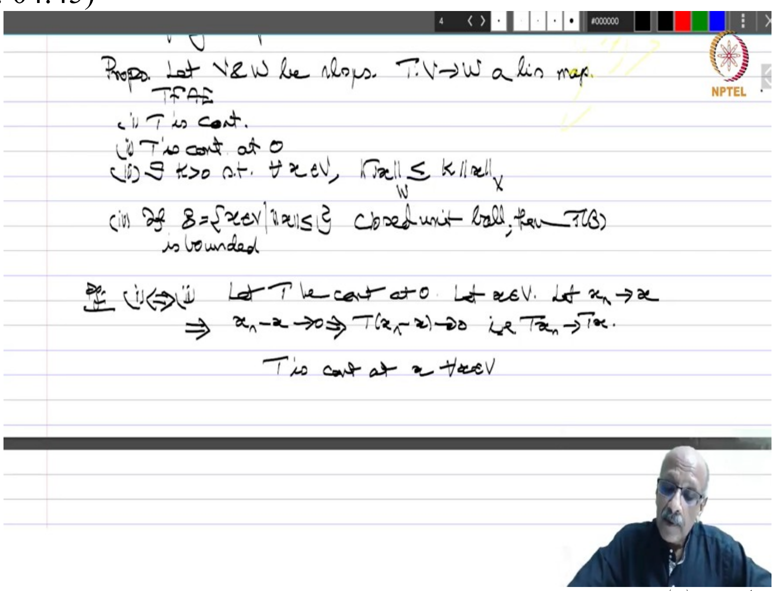

Proof $(i) \Longleftrightarrow (ii)$ If T is continuous, obviously T is continuous at 0. So$(i) \Longrightarrow (ii)$ is obvious. What about $(ii) \Longrightarrow (i)$? So, let T be continuous at 0. Let x belong to V. Since we are in a metric topology, continuity can be studied via sequences. So, let (x_n) converges to x. Then, $x_n - x \to 0$. T is continuous at 0, so $T(x_n - x) \to 0$. Thus, by linearity, $T(x_n) \to T(x)$. So, T is continuous at x for all x in T. So, this implies that $(i) \Longleftrightarrow (ii)$.

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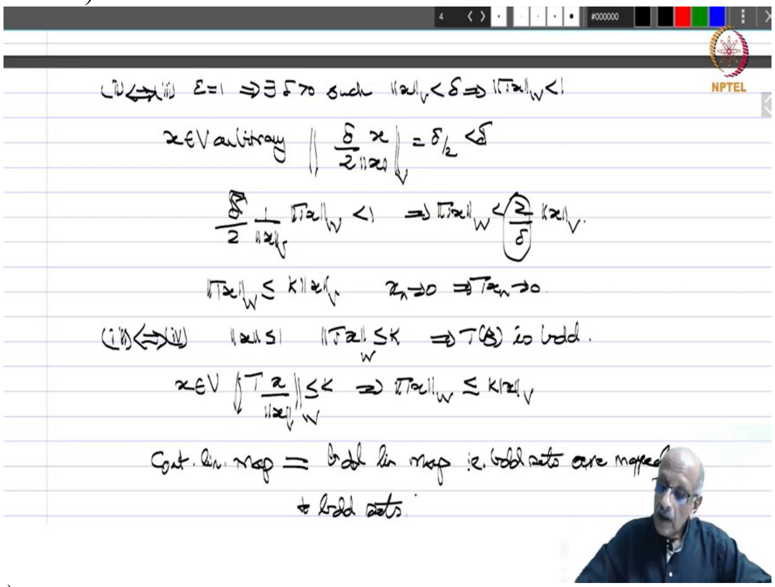

$(ii) \Longleftrightarrow (iii)$: If (ii) is true, then we have continuity at the origin. Let us look at the $\epsilon - \delta$ definition. Take $\epsilon = 1$. Then there exists a δ such that $\|x\|_V < \delta$ implies $\|T(x)\|_W < 1$. Now, let $x \in V$ be arbitrary and look at the vector $\frac{\delta x}{2\|x\|_V}$. Then what is the norm of this? $\left\| \frac{\delta x}{2\|x\|_V} \right\| = \frac{\delta}{2} < \delta$, and therefore, $\left\| T\left(\frac{\delta x}{2\|x\|_V} \right) \right\|_W = \frac{\delta}{2\|x\|_V} \|T(x)\|_W < 1$ (by linearity all the constants will come out). This means that $\|T(x)\|_W \leq \frac{2}{\delta} \|x\|_V$.

Conversely, if $\|T(x)\|_W \leq K\|x\|_V$, then $x_n \to 0$ automatically implies that $T(x_n) \to 0$ and therefore you have continuity at the origin. So, this shows that $(ii) \Longleftrightarrow (iii)$.

$(iii) \Longleftrightarrow (iv)$: B is the unit ball, so if $\|x\|_V \leq 1$, then by the inequality we have, $\|T(x)\|_W \leq K$. So, this means that $T(B)$ is bounded.

Conversely, let $T(B)$ be bounded, that means $\|x\|_V \leq 1$ implies that , $\|T(x)\|_W \leq K$, then for any, $x \in V$, you look at $\frac{x}{\|x\|_V}$, whose norm is 1. So $\left\| T\left(\frac{x}{\|x\|_V} \right) \right\|_W \leq K$ and this implies that

$\|T(x)\|_W \le K\|x\|_V$. So, this shows that all these conditions are equivalent to each other. So, this proves our proposition.

Now, because from this third condition, namely the closed unit ball is mapped to a bounded set, we can immediately see that every bounded set is mapped to a bounded set by T. Therefore, continuous linear transformations are also called bounded linear transformations. So, continuous linear transformation or map is the same as bounded linear map. That is, it takes bounded sets are mapped to bounded sets.

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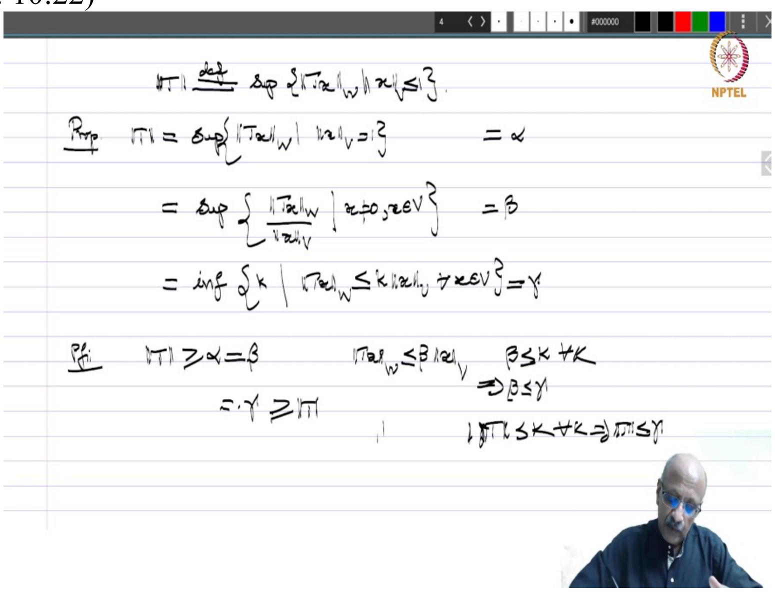

Because it takes bounded sets to bounded set, I am going to make a definition now. With some foresight, I am going to call it norm of T, which is defined as

$$\|T\| := \{\|T(x)\|_W, \|x\|_V \le 1\}.$$

This is well defined because you know that $T(B)$ is a bounded set. We are going to study this and we will eventually show that this in fact defines a norm on the space of all continuous linear transformations.

We have another proposition which allows us to calculate the $¿ \vee T \vee ¿$ in various ways.

Proposition. $\|T\| = \{\|T(x)\|_W, \|x\|_V = 1\} \; ¿ \left\{\frac{\|T(x)\|_W}{\|x\|_V}, x \neq 0, x \in V\right\} = inf\ \{K \vee \|T(x)\|_W \le K\|x\|_V, \forall x \in V\}$.

Proof. Let $¿\{\|T(x)\|_W, \|x\|_V = 1\} = \alpha$ and $up\left\{\frac{\|T(x)\|_W}{\|x\|_V}, x \neq 0, x \in V\right\} = \beta$

$inf\{K \| \|T(x)\|_W \le K\|x\|_V, \forall x \in V\} = \gamma$. So, we have to show that $\|T\| = \alpha = \beta = \gamma$. First of all $\|T\|$ is the supremum of $\|T(x)\|_W$ taken over $\|x\|_V \leq 1$ and α is the supremum of the same thing taken over $\|x\|_V = 1$. So, this a much smaller set, so we have $\|T\| \geq \alpha$.

Now, you look at these two sets A=$\{\|T(x)\|_W : \|x\|_V = 1\}$ and B=$\{\frac{\|T(x)\|_W}{\|x\|_V} : x \in V\}$. Every member of A can be witnessed as a member in B. Coverse is also true. Thus, these two sets are in fact equal, therefore the supremum are also equal and therefore$\alpha = \beta$.

Now, since we have β is the supremum, we get that$\|T(x)\|_W \leq \beta \|x\|_V$ (by definition). And therefore,β is one of the K in the third set. Therefore, it should be bigger than the infimum. So $\beta \geq \gamma$.

On the other hand, if we have any K such that $\frac{\|T(x)\|_W}{\|x\|_V} \leq K$, then the supremum will also be less than equal to K, that means $\beta \leq K$. If $\beta \leq K$ for every such$K$, then β is less than or equal to infimum, so beta $\beta \leq K$ for all K and this implies that $\beta \leq \gamma$. Therefore $\beta = \gamma$.

Finally, take set of all x such that $\|x\|_V \leq 1$. Then$\|T(x)\|_W \leq K \|x\|_V$ i.e., $\|T(x)\|_W \leq K$. . This means that the supremum over all elements in the unit ball is also less than or equal to k, that means $\|T\| \leq K$ and this is true for every K. Therefore, $\|T\| \leq K$for all K. This implies that $\|T\| \leq \gamma$.

So, we have a nice sandwich here. We have $\gamma \geq \|T\| \geq \alpha = \beta = \gamma$. Therefore, $\|T\| = \alpha = \beta = \gamma$. So, that gives you the proof of this proposition.

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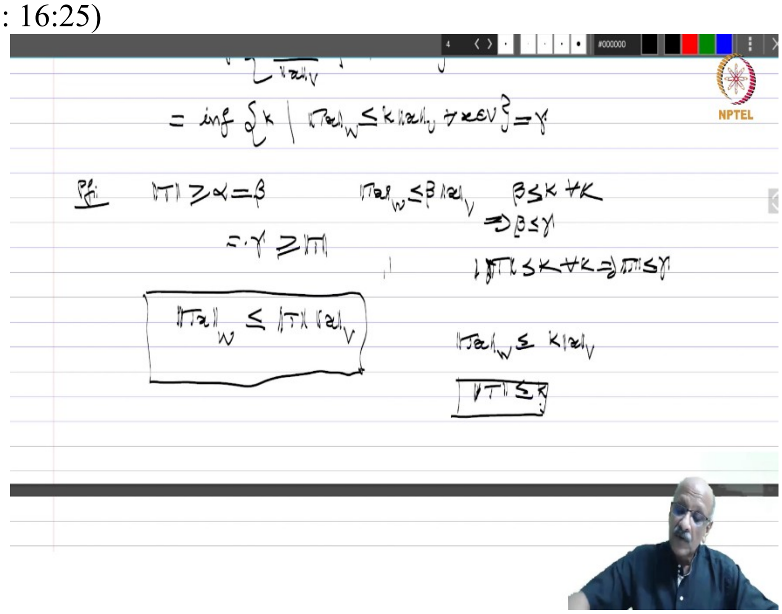

Now let us take this further. In particular we saw that $\|T(x)\|_W \le \beta \|x\|_V$. Therefore, we have that, if T is a bounded linear transformation, then $\|T(x)\|_W \le \vee |T| \vee \|x\|_V$. This is a very important, very useful inequality. The moment we know that T continuous linear transformation, we have $\|T(x)\|_W \le \vee |T| \vee \|x\|_V$. Further, if you have a linear transformation and you can show that $\|T(x)\|_W \le K \|x\|_V$, then it automatically means that T is continuous at the origin. Also from the first three conditions which we proved in the first proposition, it follows that T is continuous, and of course we have that $¿ \vee T \vee ¿ \le K$. So, whenever we produce such an inequality, we have this estimate for the norm. So, it is good to remember these two things.

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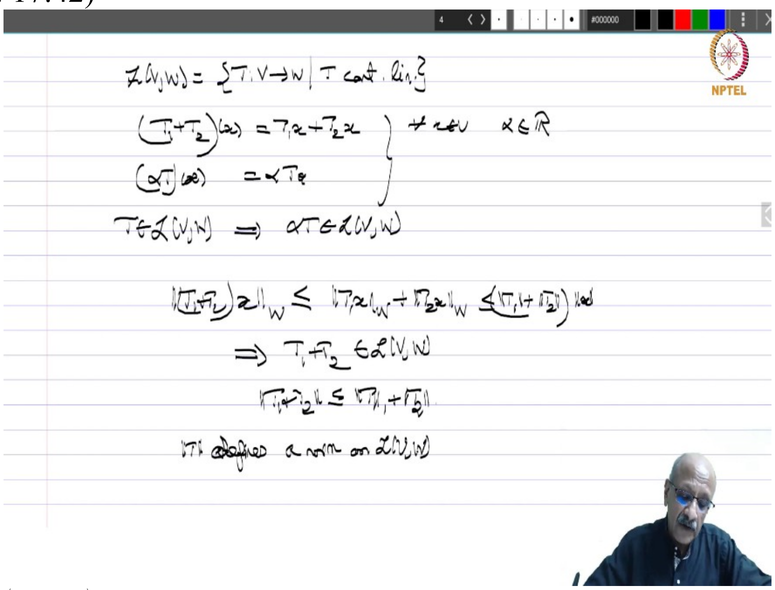

ow let us define $L(V,W) = \{T : V \mapsto W \vee T \, \text{is continuous linear}\}$.

I am going to define $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ and then $(\alpha T)(x) = \alpha T(x)$ for all $x \in V, \alpha \in R \vee C$.

Now clearly if $T \in L(V,W)$ then $\alpha T \in L(V,W)$. There is no difficulty about this, because you are just multiplying by a number α and therefore, the continuity property does not change. But what about the sum? Let us see

$$\|(T_1 + T_2)(x)\|_W \le \|T_1(x)\|_W + \|T_2(x)\|_W \le (\|T_1\| + \|T_2\|)\|x\|_V.$$

This implies that $T_1 + T_2$ is also in $L(V,W)$. So this space is closed under the vector addition and scalar multiplication. So it is a vector space and we have also shown from this inequality that $\|T_1 + T_2\| \le \|T_1\| + ¿ |T_2| \vee ¿$. This proves the triangle inequality, it is absolutely trivial to check the other things. Therefore, $\|T\|$ defines a norm on $L(V,W)$.

So, $L(V,W)$ has a natural norm linear space structure on it. So, you take the set of all continuous linear maps, you also get a norm linear space out of it. Now, we can go a little further.

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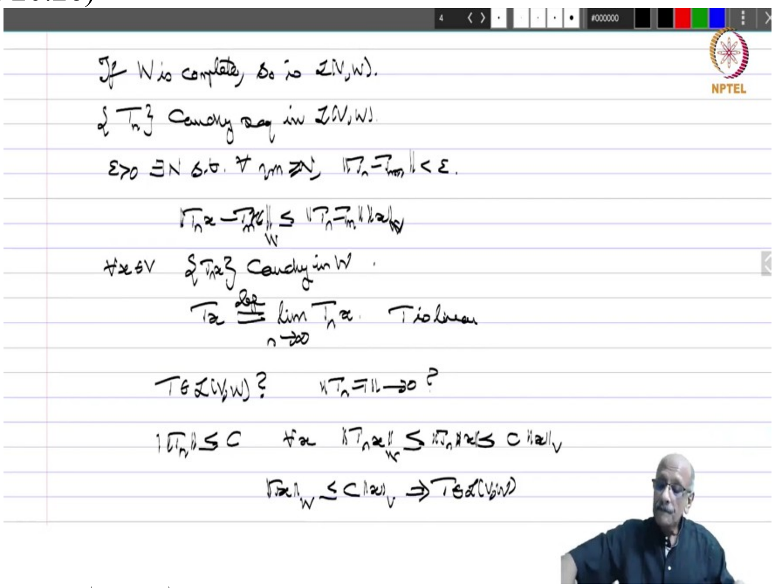

If W is complete, so is $L(V,W)$. So, note the importance here. We are not saying anything about V; V is just a norm linear space. If W is a complete norm linear space i.e., a Banach space, then $L(V,W)$automatically becomes complete. Let us check that. So, we have to show that every Cauchy sequence converges. Let (T_n) is a Cauchy sequence in $L(V,W)$. What does it mean? Given $\epsilon>0$, there exists N such that for$n,m\geq N$, we have $\|T_n-T_m\|<\epsilon$. But what do you know? $\|T_n(x)-T_m(x)\|\leq\|T_n-T_m\|\vee|x|\vee$¿. I am no longer writing V and W in the subscript, you can keep them if you like but I think now, from the context you can understand in what space we are working. Thus, for every $x\in V$, $(T_n(x))$ is Cauchy. Now, $(T_n(x))$ belongs to W, so it is Cauchy in W. Since W is complete, $(T_n(x))$ converges. Now, define $T(x)=\lim_{n\to\infty} T_n(x)$. Then, clearly, T is linear. Now have a candidate for the limit. So again, as usual, we have to check the eligibility of this candidate, so we want to show that $T\in L(V,W)$ (First question) and $\|T_n-T\|\to 0\,?$ (Second question).

These are the two questions which we want to answer. The first thing is, every Cauchy sequence is bounded, that means $\|T_n\|\leq C$ (because it is Cauchy). Therefore, for every x, we have $\|T_n(x)\|\leq\|T_n\|\|x\|\leq C\|x\|$.

Since norm is a continuous function, letting n tend to infinity we get $\|T(x)\| \le C\|x\|$. And this implies that T is continuous so $T \in L(V, W.)$

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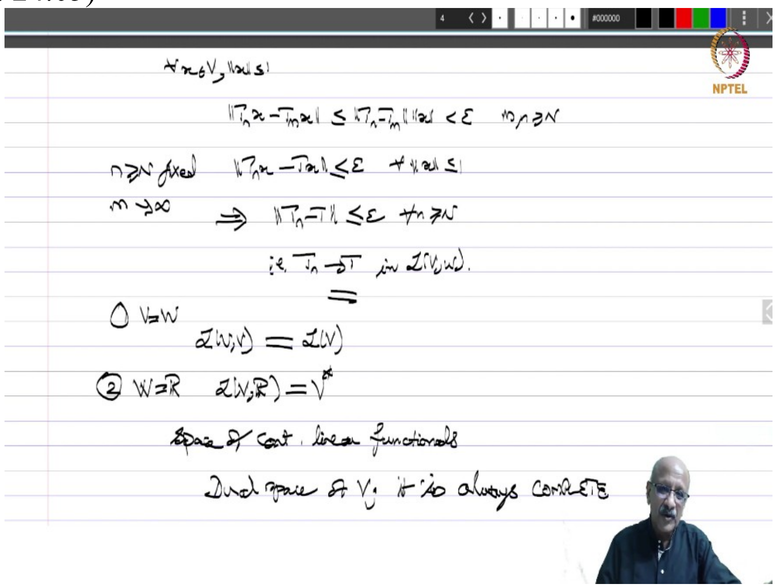

Next, we go back to the Cauchy condition. So, for all $x \in V$ with $\|x\| \le 1$, we have $\|T_n(x) - T_m(x)\| \le \|T_n - T_m\| \vee |x| \vee ¿$. Now, $\|T_n - T_m\| < \epsilon$, $\|x\| \le 1$, so $\|T_n(x) - T_m(x)\| \le \epsilon$. Now, keep n fixed bigger than N, and m tends to infinity and therefore we have norm $\|T_n(x) - T(x)\| \le \epsilon$ for all $x \in V$ with $\|x\| \le 1$. Then, by definition of the norm, the supremum overall over the unit ball is the norm. Therefore, $\|T_n - T\| \le \epsilon, \forall n \ge N$ i.e., $T_n \to T$ in $L(V, W)$. Therefore, every Cauchy sequence does converge and therefore if W is complete, then $L(V, W)$ is also complete. There are two important cases, so first case is when V equals W. So then you have the space $L(V, V)$, so it is silly to write V two times, so we will only write $L(V)$.

So, if V is Banach, then $L(V)$is also Banach. If the spaces are the same i.e., $V = W$, generally we use the word operator. Well, it is interchangeable with mapping or transformation as I already said, but this is space of bounded linear operators.

Then the second one is very important. So, if we $W = R \vee C$, then we have $L(V, R)$or $L(V, C)$ and $V^{¿}$. This is the space of continuous linear functionals. Whenever we have a mapping into the base field, we call it a functional. So, this is a continuous linear mapping into the base field and therefore this is the space of continuous linear functionals and it is called the dual space of V.

And, it is always complete, whatever may be V, because R is complete, so whatever we saw just now, the dual space is always a Banach space.

So dual space is one of the important areas of research in functional analysis. Given a space, we would like to know what is the space of all continuous linear functionals because we will see later that the more information you have on the dual, you will also have information on the base space. So, this is a very important thing.

So we have done, so now let us... [video ends]