Functional Analysis
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Lecture No. 39
Completeness

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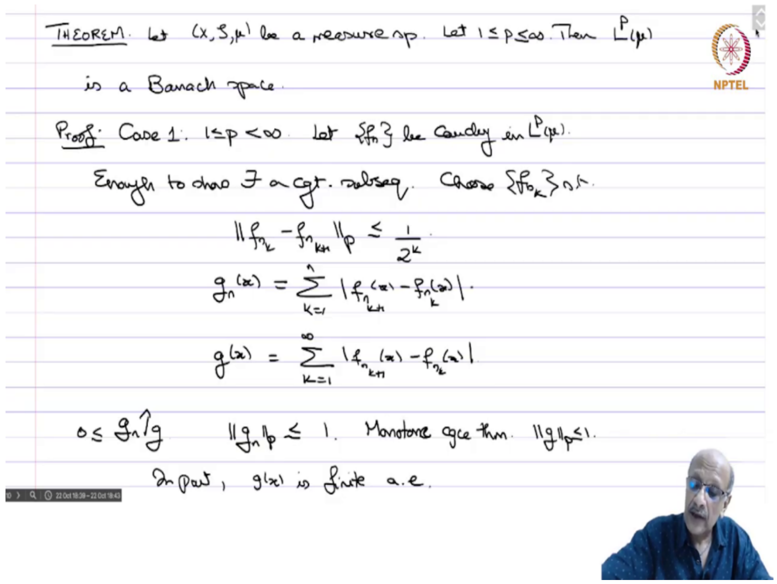

We will now prove an important theorem,

Theorem: Let (X, ζ, μ) be a measure space. Let $1 \leq p \leq \infty$. Then $L^p(\mu)$ is a Banach space. So, L^p is a space of equivalence classes with respect to relation of equality almost everywhere, with the p integrability norm and, or if it is essentially bounded, then it is a norm infinity and so each of these spaces is complete.

Proof: So first case we will take $1 \leq p < \infty$, because these involve integral, so the arguments are slightly different. So, let $\{f_n\}$ be Cauchy in $L^p(\mu)$. So, we have to show that this converges to some function in L^p. So, enough to show there exists a convergent subsequence because we have a Cauchy sequence and you have a convergence subsequence and the entire Cauchy sequence will converge to the same limit. So, it is enough to show there exists a convergence subsequence.

Now because you have a Cauchy sequence, you can always find a subsequence such that consecutive terms are as close to each other as you prescribe.

This we have done several times before. So, choose subsequence $\left\{f_{n_k}\right\}$ such that $\|f_{n_k} - f_{n_{k+1}}\|_p \leq \frac{1}{2^k}$. so you can always do this. So and we now define $g_n(x) = \sum_{k=1}^{n} \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|$ and then you define $g(x) = \sum_{k=1}^{\infty} \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|$. So then of course, you have that $g_n \uparrow g$. So, it is a monotonically increasing sequence for each x, because you are taking 1 to n and then you are adding positive terms more and more. So, this is a monotonically increasing sequence and you have that it increases to $g(x)$ and also by the triangle inequality, $\|g_n\|_p \leq \sum_{k=1}^{n} \|f_{n_{k+1}} - f_{n_k}\|_p \leq \sum_{k=1}^{n} \frac{1}{2^k}$, that is a geometric series and I can estimate it higher by 1 to ∞, so that is less than equal to 1. So, $0 \leq g_n$ and it increases to g, and $\|g_n\|_p \leq 1$, so the monotone convergence theorem, you have $\|g\|_p \leq 1$, because the $\int \left|g_n\right|^p$ will converge to $\int |g|^p$. That is the monotone convergence theorem and therefore $\int |g|^p \leq 1$. So in particular, we have $g(x)$ is finite almost everywhere. So, except on a set of measure zero. Because its p integral is less than equal to 1, so it cannot be infinity on a set of positive measure. So, it has to be finite everywhere.

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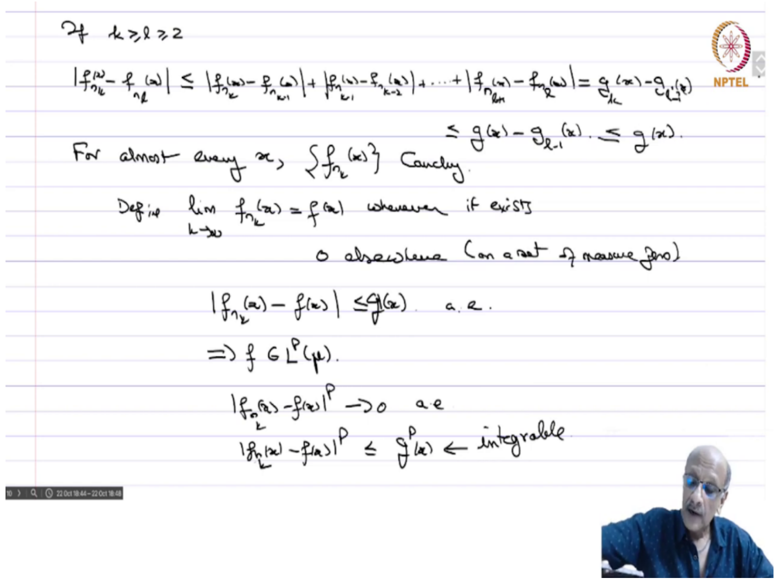

And further if $k \geq l \geq 2$, then you have

$$\left|f_{n_k}(x) - f_{n_l}(x)\right| \leq \left|f_{n_k}(x) - f_{n_{k-1}}(x)\right| + \left|f_{n_{k-1}}(x) - f_{n_{k-2}}(x)\right| + \cdots + \left|f_{n_{l+1}}(x) - f_{n_l}(x)\right|.$$

Now, this is, what is this, this is nothing but $g_k(x) - g_{l-1}(x)$ and therefore that is less than equal to $g(x) - g_{l-1}(x)$. But you know that $\{g_n\}$ converges to g point wise almost everywhere. Therefore for almost every x, we have $\left|f_{n_k}(x) - f_{n_l}(x)\right| \leq g_k(x) - g_{l-1}(x)$ which goes to 0 as l tends to infinity and therefore, $\left\{f_{n_k}(x)\right\}$ is Cauchy and therefore, you let f, so define $f_{n_k}(x) = f(x)$. Whenever it exists and 0 elsewhere and this is on a set of measure zero, because almost everywhere it converges and therefore you have this. So, we now have a candidate and we have to check if this candidate is in L^p and if you can converge in L^p. So $g(x) - g_{l-1}(x)$ is of course less than equal to $g(x)$ also, so that is you already have. So, now if you take l tending to infinity, in this thing, so you get $\left|f_{n_k}(x) - f(x)\right| \leq g(x)$, almost everywhere. Because except on a set of measure zero, there everything is zero and we do not have to worry. So, g is in L^p, because $\|g\|_p \leq 1$, f_{n_k} is in L^p because it is a finite sum of, it is anyway in L^p, that is given to you, so $\left|f_{n_k}(x) - f(x)\right| \leq g(x)$ implies that $f \in L^p$ and then what, you have $\left|f_{n_k}(x) - f(x)\right|^p \to 0$ almost everywhere of course, and $\left|f_{n_k}(x) - f(x)\right|^p \leq g^p(x)$, and this is integrable. Because $\int g^p \leq 1$.

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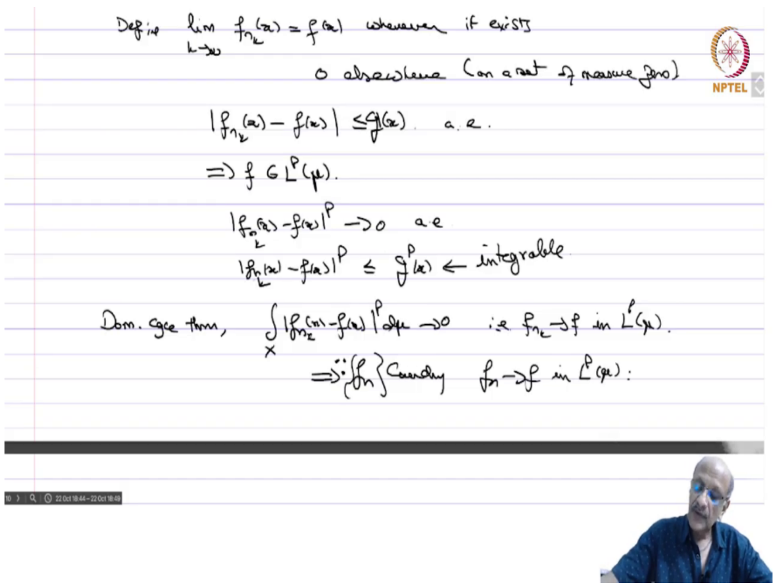

And consequently by the dominated convergence theorem, we have that $\int_X \left|f_{n_k}(x) - f(x)\right|^p d\mu \to 0$ and that is $\left\{f_{n_k}\right\}$ converges to f in $L^p(\mu)$. So, we have subsequence which converges in L^p. Therefore in the original sequence being Cauchy, so since $\left\{f_n\right\}$ Cauchy, we have $\left\{f_n\right\}$ itself converges to f in L^p. Therefore $L^p(\mu)$ is complete. So now that completes the case where p is strictly less than ∞.

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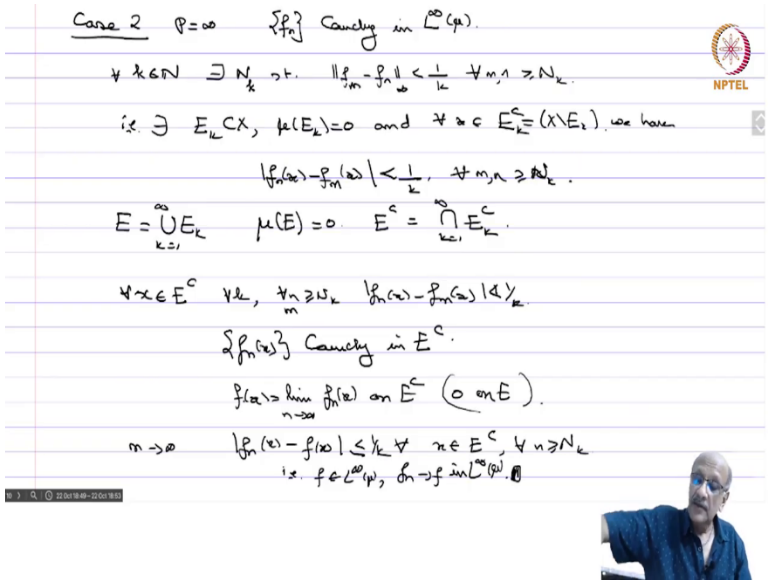

Case 2: $p = \infty$. So, you have $\{f_n\}$ is Cauchy in $L^{\infty}(\mu)$. So, for every k, natural number or positive integer, there exists N_k such that $\|f_m - f_n\|_{\infty} < \frac{1}{k}$, for all $m, n \geq N_k$. Now what does this mean? So, that is, there exists $E_k \subset X$, $\mu(E_k) = 0$ and for every $x \in E_k^c$, this is equal to $X \setminus E_k$, we have $|f_n(x) - f_m(x)| < \frac{1}{k}$. That is what we mean by saying that because this is essential supremum and therefore it is valid except on the set of measure 0 and this is true for all $m, n \geq N_k$. So now, you take $E = \cup_{k=1}^{\infty} E_k$ and then $\mu(E)$ is a countable union of sets of measure zero and therefore $\mu(E)$ is also 0 and what is E^c, so $E^c = \cap_{k=1}^{\infty} E_k^c$. So, if you take every, for, so every $x \in E^c$, is in every E_k^c. So, for all $x \in E^c$ you have, which is the complement of the set of measure zero, you have for every k and for all $n, m \geq N_k$, we have $|f_n(x) - f_m(x)| < \frac{1}{k}$ and therefore $\{f_n(x)\}$ is Cauchy, uniformly Cauchy in fact in E^c. So, you take $f(x) = f_n(x)$ on E^c and then you can take in fact 0 on E for instance if you like. So then you have, if you allow m to tend to infinity, you have $|f_n(x) - f(x)| \leq \frac{1}{k}$ for all $x \in E^c$ and for all $n \geq N_k$. That is $f \in L^{\infty}$ and $f_n \to f$ in L^{∞}. So, that completes the proof of this theorem that all the L^p spaces are in fact Banach

spaces and then we will say, see later, when they are reflexive, when they are separable and all these things in certain cases.

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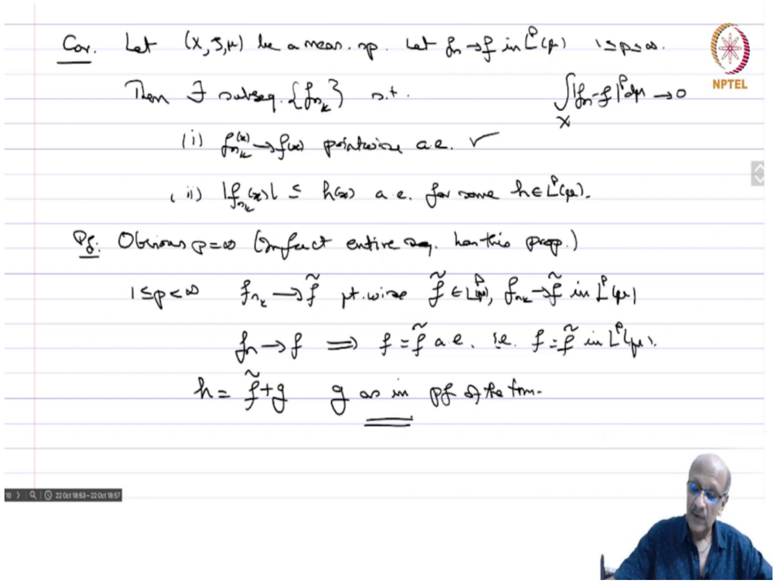

So, what is an important corollary, we have that,

Corollary: So let (X, ζ, μ) be measure space and let $f_n \to f$ in $L^p(\mu)$, $1 \leq p \leq q$. Then, there exists a subsequence $\left\{f_{n_k}\right\}$ such that

1. $f_{n_k} \to f$ point wise, that means $f_{n_k}(x) \to f(x)$ almost everywhere, that is except on a set of zero for all x this happens, and
2. you have that $\left|f_{n_k}(x)\right| \leq h(x)$ almost everywhere for some $h \in L^p$.

Proof: Obvious if $p = \infty$, in fact, the entire sequence has this property. Because we have shown that the entire sequence converges and that we did not even take a subsequence. So, if $1 \leq p < \infty$, then we saw there exists an $\left\{f_{n_k}\right\}$ which converges to $\tilde{f}$ point wise and $\tilde{f} \in L^p$ and $f_{n_k} \to \tilde{f}$ in $L^p(\mu)$ of course, we saw this. But then you are given that $f_n \to f$ and therefore this implies $f = \tilde{f}$

almost everywhere that is $f = \tilde{f}$ in L^p. Therefore the first one is proved, namely you have a subsequence which converges. So, this is a very important property, very useful property in convergence in L^p spaces. So, if you have L^p convergence, then for a subsequence, you have point wise convergence. So, L^p convergence is some convergence of some integral to 0. What is $\{f_n\}$ going to L^p? So, it means that $\int_X |f_n - f|^p d\mu \rightarrow 0$. So, this is something known about the integrals. Whereas I am saying then there you can find a subsequence which converges point wise except on a set of measure zero.

Now for the second part, you just take $h = \tilde{f} + g$, where g is the function that infinite series which we defined in the proof of the theorem, g as in proof of the theorem. Then h is in L^p and in fact $\left|f_{n_k}(x)\right| \leq h(x)$. This is triangle inequality and that will tell you this. So now, so we now have the Banach, L^p is a Banach space and then we will see various other properties, what are the duals of the L^p spaces, what happens when it is separable when it is reflexive and what are some other properties of this.