Functional Analysis
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Lecture No. 38
Lp Spaces – Part 2

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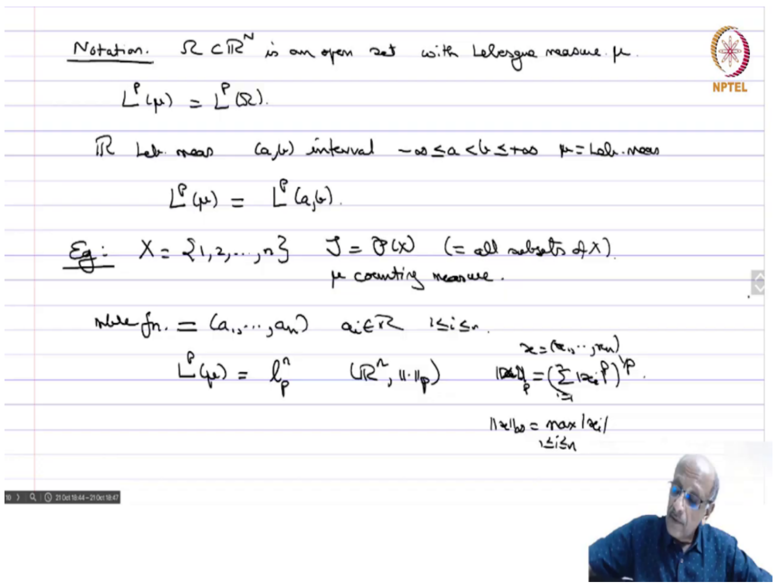

So now we will, one special notation.

Notation: So, if $\Omega \subset R^n$ is an open set with Lebesgue measure. So, this is the usual standard measure we have on R^n and its subsets, then $L^p(\mu)$, we will write as $L^p(\Omega)$, because we understand, we do not have to specify the measure, we know the measure is a Lebesgue measure. We want to know what set we are working on and so we put the space $L^p(\Omega)$.

Similarly, if you are working with R, with Lebesgue measure, and you have (a, b) as an interval, where $-\infty \leq a \leq b \leq +\infty$. There could be infinity intervals also. Then $L^p(\mu)$, so μ is the Lebesgue measure. $L^p(\mu)$, I will write $L^p(a, b)$, so this is just notation, specially when you are dealing with R or R^n. So now, let us look at some examples, familiar examples.

Example: So, now we take $X = \{1, 2, \cdot, \cdot, \cdot, n\}$. $\zeta = \wp(X)$, that means all subsets. And μ is the counting measure. That means $\mu(A)$ is equal to the number of elements in A and it will be infinite if A is an infinite set. Then what is integration with respect to μ, it is just a summation. So, it is just integration of a function, a function on, on X, is nothing but set of numbers $\{1, 2, \cdot, \cdot, \cdot, n\}$. So it is an n-tuple and if integration is nothing but adding the values. So, that is the integration with respect to the counting measure. So every measurable function, every function is measurable because we have taken all sets in the sigma algebra is identified with the n-tuple $(a_1, a_2, \cdot, \cdot, \cdot, a_n)$, $a_i \in R$. So, therefore in this case, $L^p(\mu)$ is nothing but our familiar l_p^n, namely R^n with $\|\cdot\|_p$ which we have defined here. So, $\|x\|_p$, $x = (x_1, x_2, \cdot, \cdot, \cdot, x_n)$ is nothing but $\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$. So $\|x\|_\infty$ is, and it is max, and so $\|x\|_\infty = |x_i|$. So, the infinite dimensional spaces which we have seen are precisely l_p spaces.

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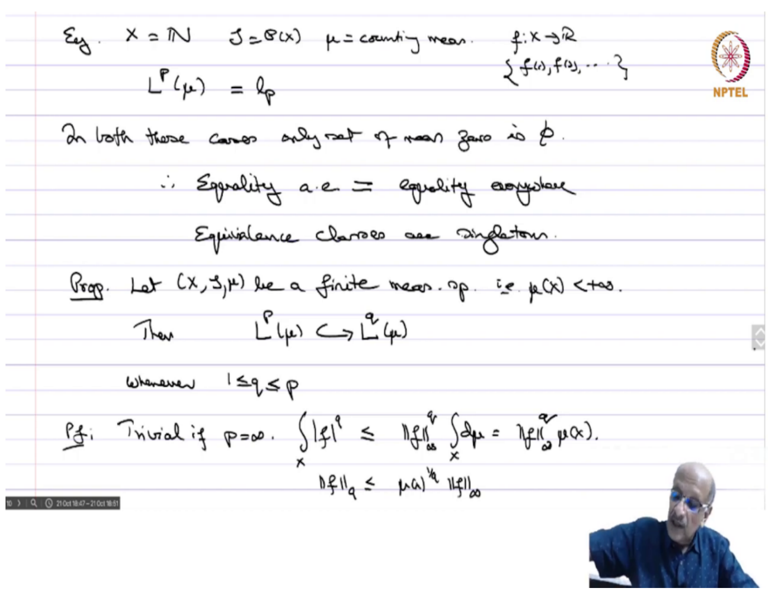

So next example,

Example: X is the set of natural numbers and again $\zeta = \wp(X)$ all subsets. And μ is again the counting measure. So now, you can immediately relate to what I am going to say, so $L^p(\mu)$ is nothing but l_p. Because the measurable function is nothing but a sequence, so of numbers, real numbers. So $f: X \to R$ means it is a sequence $\{f(1), f(2), \cdot, \cdot, \cdot, \}$. So it is nothing but a sequence and integration with respect to counting measure is nothing but summation and therefore $L^p(\mu)$ is nothing but l_p.

So, in both these cases, only the set of measure zero is the empty set. No other set has measure zero. Therefore equality almost everywhere is the same as saying equality everywhere. So, equivalence classes are single tasked. So, there is no real fuss. So, we do not, this time we are really talking about functions only. We do not have any of this.

Proposition: So, let, (X, ζ, μ) be a finite measure space. That means, $\mu(X)$ is finite. Then,

$$L^p(\mu) \hookrightarrow L^q(\mu).$$

That means this subset, this notation means it is a subset and the inclusion map is a continuous linear map, with whenever you have $1 \leq q \leq p$. So, when $p \geq q$, in a finite measure space, L^p the bigger p is always contained in the smaller L^q.

Proof: So, trivial if p is ∞, because you have $\int_X |f|^q \, d\mu \leq ||f||_\infty^q \int_X d\mu$ which is equal to $||f||_\infty^q \mu(X)$ which is finite and therefore you have $||f||_q \leq \mu(X)^{\frac{1}{q}} ||f||_\infty$ and therefore this shows that the inclusion map is a continuous linear operative map and of course that L^∞ is contained in every L^q. So now, let us prove it.

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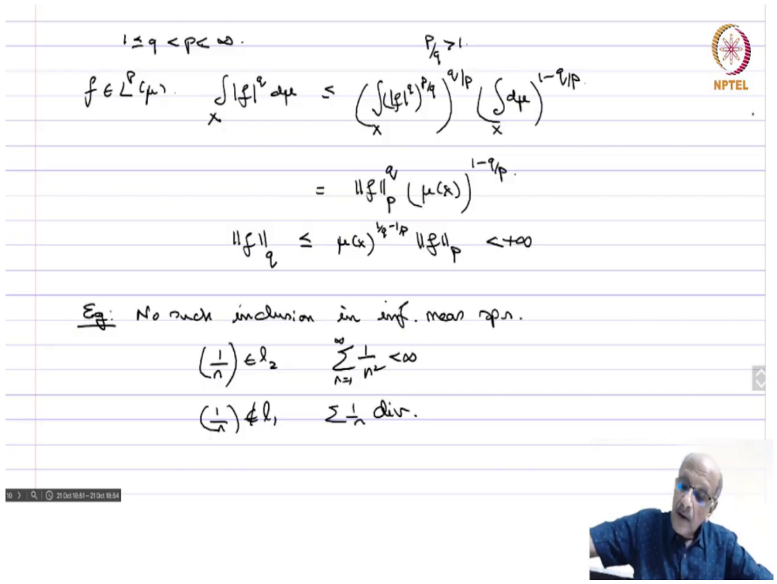

For any, so $1 \leq q \leq p < \infty$. So, $f \in L^p(\mu)$. So now we apply, so $\int_X |f|^q \, d\mu$. I am going to calculate, see if this is a finite quantity. So I am going to apply this Holder's inequality to this with the first function is $|f|^q$, the second function is 1. So if I apply that, so I will get this is less than equal to. And since $q \leq p$, $\frac{p}{q} \geq 1$. So, we can apply Holder's inequality with the p by q as the index. So, $\int_X |f|^q \, d\mu \leq \left(\int_X (|f|^q)^{\frac{p}{q}} \, d\mu \right)^{\frac{q}{p}} (\mu(X))^{1-\frac{q}{p}}$. Now, this is nothing but, $\int_X (|f|^q)^{\frac{p}{q}} \cdot \int_X |f|^p$ is nothing but $\|f\|_p$, so this is $\|f\|_p$. Yeah. $\|f\|_p^q (\mu(X))^{1-\frac{q}{p}}$. So we have, $\|f\|_q \leq (\mu(X))^{\frac{1}{q}-\frac{1}{p}} \|f\|_p$ and that is finite and also this shows that the inclusion map is a continuous linear operator.

So, now some other examples.

Example: No such inclusion in infinite measure spaces. So, you have $\left\{\frac{1}{n}\right\}$, the sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \cdot, \cdot, \cdot\right\}$ belongs to l_2, because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is finite, that is $\frac{\pi^2}{6}$, we have already seen. But $\left\{\frac{1}{n}\right\}$ is not in l_1. Because $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. So, if something is in l_2 does not imply it is in l_1, if you have an infinite dimensional space.

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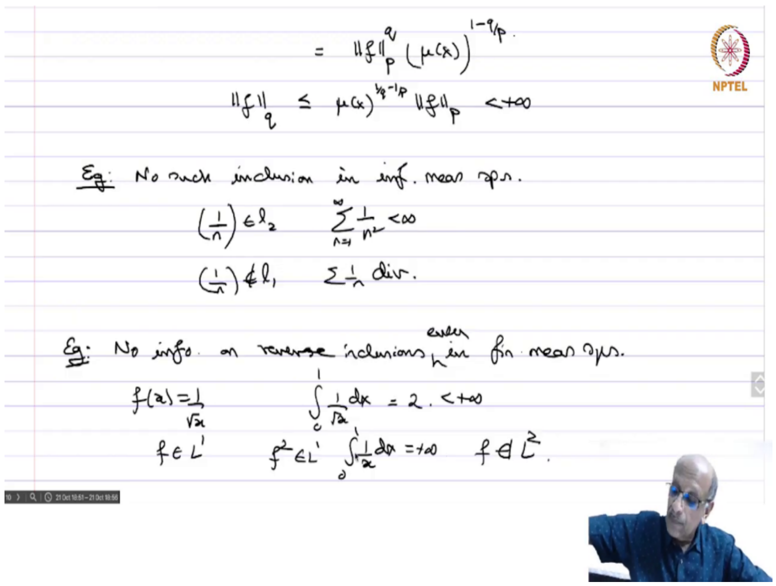

Example: No info, information on reverse, reverse inclusions. Even infinite dimension, infinite measures. Even infinite measure spaces. So you have that, take the function $f(x) = \frac{1}{\sqrt{x}}$. This is integrable, because if you can any, what is $\int_{0}^{1} \frac{1}{\sqrt{x}}$ is a non-negative function, so you do not have to take the modulus, so this is nothing but is equal to 2. So this is finite. So $f \in L^1$. But f^2 does not belong to L^1, that is $\int_{0}^{1} \frac{1}{x}$ is $+\infty$ and therefore, you have $f \notin L^2$. f^2 is not integrable, so $f \notin L^2$. So, you do not have any reverse inclusion. l_p does not imply L^p.

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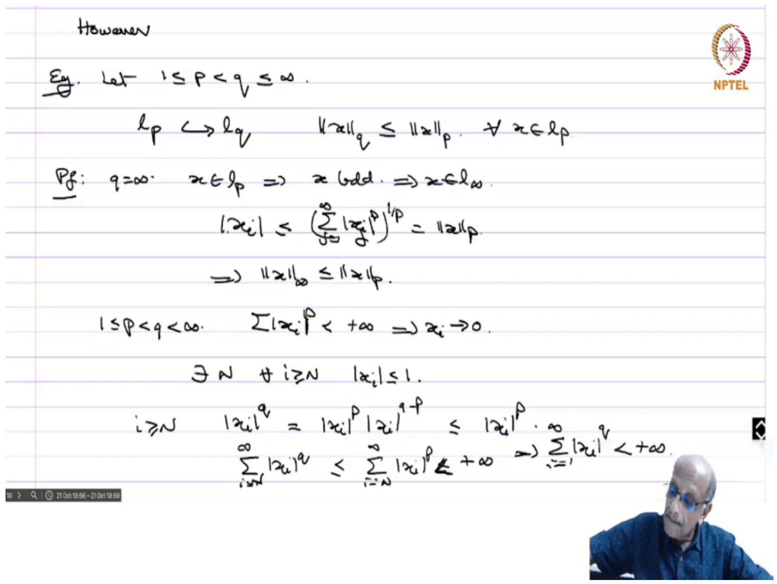

However, example again,

Example: Let $1 \leq p \leq q \leq \infty$. Then $l_p \hookrightarrow l_q$. So, you do have the reverse inclusion in the l_p case alone, in finite, though this is, there is nothing that can be said. It is completely free. There is no method by which you can estimate these things. So it happens sometimes, it will not happen sometimes. So, even in finite measure space, you do not have the reverse inclusions, namely l_p does not belong to a L^p. But in this infinite measure space, l_p and l_q, you do have this thing and not only that, you have that $\|x\|_q$, the bigger one is less than $\|x\|_p$, for all $x \in l_p$.

Proof: So let us take $q = \infty$. So $x \in l_p$ implies x is bounded. Because $\sum_{i=1}^{\infty} |x_i|^p$ is convergent. That means, the convergent series, the terms are all bounded and therefore $x \in l_\infty$. And for any i, $|x_i| \leq \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}} = \|x\|_p$ because this is only one term in this thing and therefore it is, you have this and therefore, if you take the sup, you get $\|x\|_\infty \leq \|x\|_p$. So now let us assume, $1 \leq p < q < \infty$. So, again $\sum_{i=1}^{\infty} |x_i|^p$ is convergent. So, this means that $x_i \to 0$. Therefore there exists an N such that for all $i \geq N$, you have $|x_i| \leq 1$. So, $|x_i|^q$ if $i \geq N$ is equal to $|x_i|^p |x_i|^{q-p}$ and since $\left|x_i\right| \leq 1$, so $|x_i|^p |x_i|^{q-p} \leq |x_i|^p$. So, $\sum_{i=N}^{\infty} |x_i|^q \leq \sum_{i=N}^{\infty} |x_i|^p$, that is less than the full series, so that is less than strictly less than $+\infty$. And therefore this implies $\sum_{i=1}^{\infty} |x_i|^q$ after all you are only adding a finite number of terms is less than $+\infty$ and therefore, you have that $l_p \subset l_q$.

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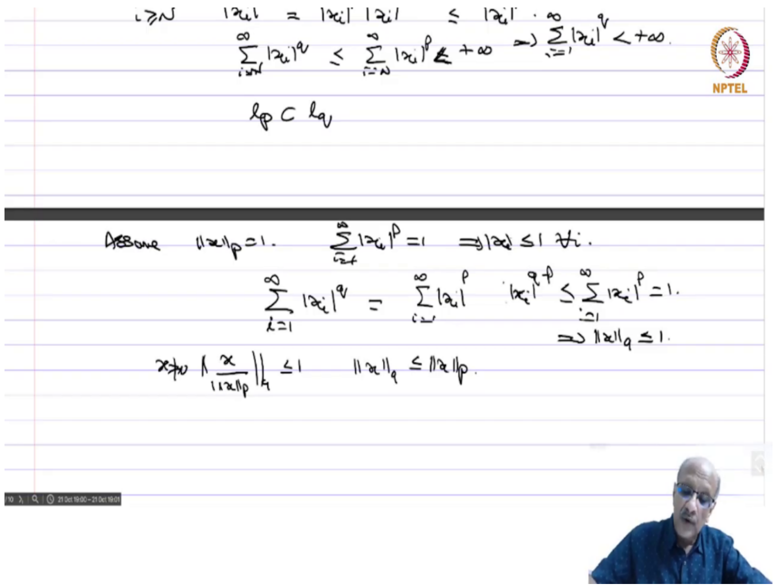

So now for the inequality, so assume norm $\|x\|_p = 1$. So this is $\sum_{i=1}^{\infty} |x_i|^p = 1$. So this implies that $\left|x_i\right| \leq 1$ for all i. So, $\sum_{i=1}^{\infty} |x_i|^q = \sum_{i=1}^{\infty} |x_i|^p |x_i|^{q-p}$. This is all less than equal to 1. So, $\sum_{i=1}^{\infty} |x_i|^p |x_i|^{q-p} \leq \sum_{i=1}^{\infty} |x_i|^p$ and that is equal to 1. So, you have this is true. So now, given any x (which is not 0 obviously), you take $\frac{x}{\|x\|_p}$. It's norm is 1 and therefore by whatever we get therefore, that $\|x\|_q$. So $\sum_{i=1}^{\infty} |x_i|^q \leq 1$ implies that $\|x\|_q \leq 1$. So, $\|\frac{x}{\|x\|_p}\|_q \leq 1$, that is $\|x\|_q \leq \|x\|_p$. So, this proves, so these are some set of examples which we have. So, now we will look at some further properties important points about L^p spaces in this.