Functional Analysis
Professor. S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No. 37
Lp Spaces – Part 1

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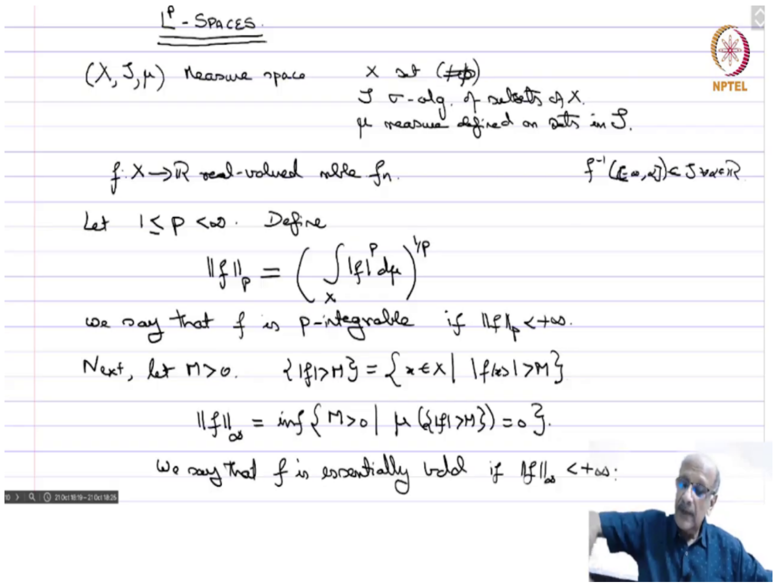

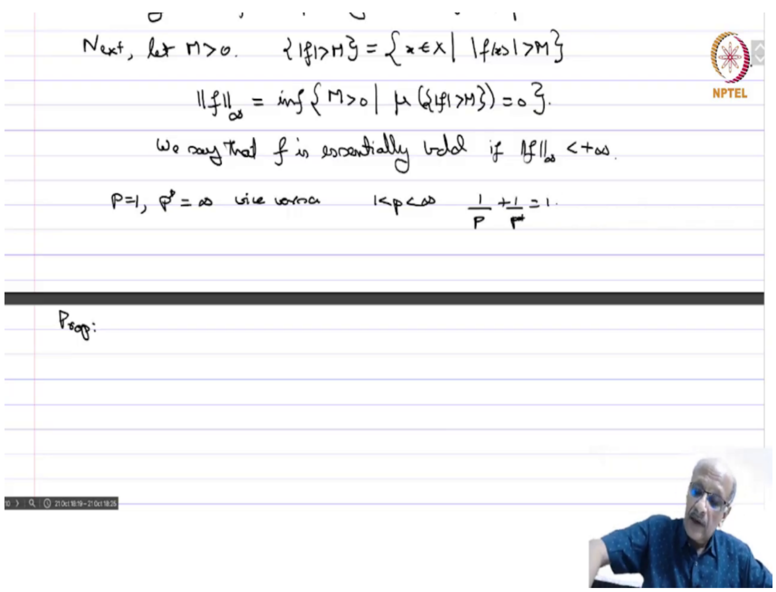

We will now discuss L^p Spaces. The L^p Spaces, L for Lebesgue constitutes a rich source of examples and counterexamples in functional analysis. They also form an important class of function spaces for the study of various topics in applied mathematics. For instance, to study partial differential equations, we use L^p Spaces and subspaces of these L^p Spaces, which are special spaces called Sobolev spaces and so on. So, L^p Spaces are very important, both in analysis and functional analysis.

And we will study in this chapter some important properties of L^p Spaces, from a functional analytic point of view. Of course in the course in measure theory, there will be other emphasis. But here we will look at some of the important things. So, what is the basic starting point? So, we have measure space. What does this mean? X is a set, non-empty always, though I may not say it and then ζ is a sigma algebra of subsets of X. That means X and empty set are in this.

And if you have countable collection of elements here, then their union is also a member of ζ. So it is closed and a countable union and it is closed under complementation. If $A \in \zeta$, A^c is also in ζ. So, this is the basic collection on which one defines a measure. So, μ is a measure defined on sets in ζ. So this is a standard measure. So, those who are not completely familiar with measure theory, I recommend that you look at the first chapter in the book on functional analysis, Trim series 52 which I mentioned in the beginning of this course.

There, there is a rapid introduction to measure spaces, at least the important results which we will use are all stated there, though not proved. So now, you take $f: X \rightarrow R$, real value measurable function. So, if you know something on measure theory, you know what. So, inverse image of sets of the form, so $f^{-1}(-\infty, \alpha]$ are members of ζ, so for every $\alpha \in R$, that is a measurable function. So, f is a real value measurable function.

There are many equivalent forms of this. This one in particular. So, let $1 \leq p < \infty$ and then define, so I am going to very suggestively put the symbol $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}$. So we say that, f is p integrable if $\|f\|_p < \infty$. Now, so if you say $p = 1$ we say integrable, if $p = 2$, we say square integrable and so on. So next, let M be positive and we say that, we denote by $\{|f| > M\} = \{x \in X: |f(x)| > M\}$.

So then, $\|f\|_{\infty}$ is the smallest M, so $inf\{M > 0: \mu(\{|f| > M\})$, so this is set and because of the various properties of measurable functions, this will be measurable. That means you can define its measure, so $\mu(\{|f| > M\}) = 0\}$. So, M is such that it is almost an upper bound. If $|f(x)| \leq M$, for almost every point x, except it may be violated on a set of measure 0. Now you take the smallest such M, so you call the infimum.

So we say that, f is essentially bounded if $\|f\|_{\infty} < +\infty$. So essentially, why do you say essentially bounded, it may not be, it may take infinite values or it may be unbounded in some places, but that where it is so, is essentially, is a set of measures 0 and therefore negligible in the sense of measure theory. So now, we have the proposition. So, if $p = 1$, we say $p^{\star} = \infty$ and vice versa. If $p = \infty$, $p^{\star} = 1$ and if $1 < p < \infty$, then $p^{\star}$ is the conjugate exponent. So $\frac{1}{p} + \frac{1}{p^{\star}} = 1$.

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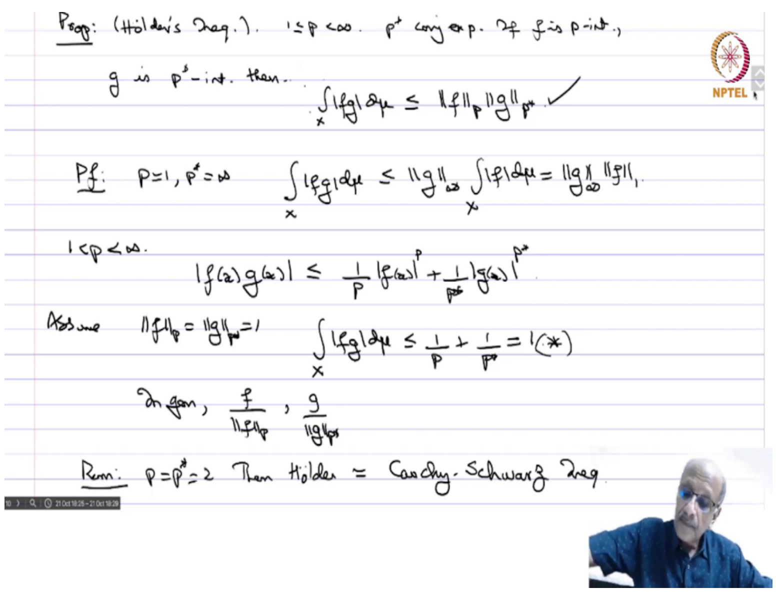

So, now we meet an old friend,

Proposition (Holder's inequality): So $1 \leq p < \infty$ and $p^{\star}$ conjugate exponent. If f is p integrable, g is $p^{\star}$ integrable or essentially bounded, does not matter, I already find $p^{\star}$, so that I do not have to go ahead and redo it again. Then, $\int_X |fg|\, d\mu \leq \|f\|_p \|g\|_{p^*}$. So this is very similar to the Holder inequality which we did before.

Proof: So if $p = 1$, then $p^* = \infty$. So, $\int_X |fg|\, d\mu$. g can be brought out by its essential supremum, so $\int_X |fg|\, d\mu \leq \|g\|_\infty \int_X |f|\, d\mu$ and that is equal to $\|g\|_\infty \|f\|_1$. So that is completely proved. So, now let us assume that $1 < p < \infty$. So, we can also assume that f and g are non-zero, because even if one of them is 0 then there is nothing to prove in this inequality. So, we have that, if you remember the very first lemma which we proved in this course, so $|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^{p^*}}{p^*}$. So, this is the generalized A.M. G.M. inequality which we have proved in the very first lecture in this series. So now you assume that $\|f\|_p = \|g\|_{p^*} = 1$, assume. So, if we integrate this inequality on both sides, you get $\int_X |fg|\, d\mu$ is less than equal to $\frac{1}{p}$ $\int_X |f(x)|^p$ (which is 1), plus $\frac{1}{p^*}$ $\int_X |g(x)|^{p^*}$ (which is also 1). And therefore $\int_X |fg|\, d\mu \leq \frac{1}{p} + \frac{1}{p^*} = 1$ and in the general case, so now in general, you take $\frac{f}{\|f\|_p}$ and $\frac{g}{\|g\|_{p^*}}$ and apply this inequality, then you will get straight away the Holder inequality.

Remark: If $p = p^* = 2$, then Holder equals to Cauchy Schwarz's inequality.

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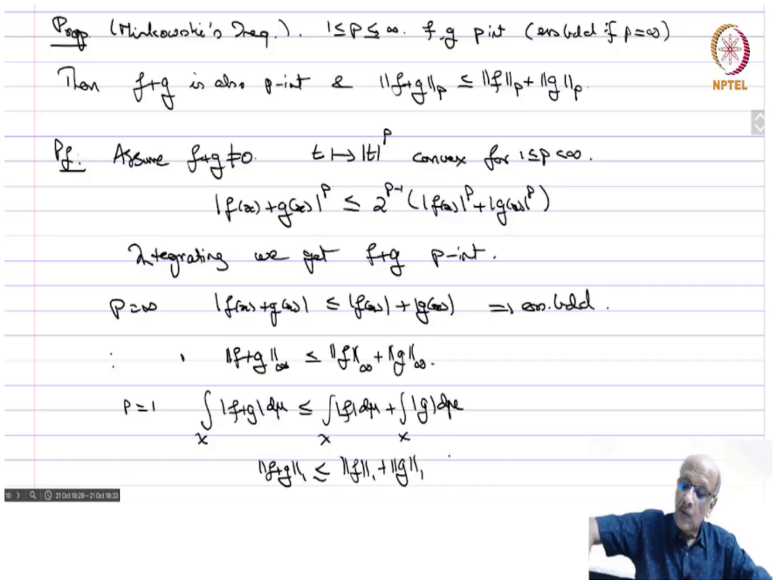

So, the next proposition is Minkowski's inequality. Again we are going to follow what we did earlier.

Proposition (Minkowski's inequality): So, $1 \le p \le \infty$. f and g are p integrable (essentially bounded if $p = \infty$) and then $f + g$ is also p integrable and $\|f + g\|_p \le \|f\|_p + \|g\|_p$. So, this is the triangle inequality which we proved for vectors, now we are proving it for functions. So again, we can assume,

Proof: Assume $f + g \neq 0$. Because otherwise the result is trivially true. There is nothing to do. Now if you take the function $t \mapsto |t|^p$, then this is convex for $1 \le p \le \infty$. So, $|f(x) + g(x)|^p \le 2^{p-1}(|f(x)|^p + |g(x)|^p)$. So, this just comes from the convexity, the midpoint. So, if you had by 2, you have power p less than one half of this. So, the 2^p will cross multiply and you will get this. So, this is just the definition of convexity of this function and you will get this. So, if you integrate both sides, so integrating, we get $f + g$ is p integral. If $p = \infty$, then of course $|f(x) + g(x)| \le |f(x)| + |g(x)|$ and each is essentially bounded, so implies essentially bounded. So, there is nothing to group.

So, now if $p = 1$ or ∞, so you get norm, so if $p = \infty$, you will get from this that $\|f + g\|_\infty$ is trivially less than $\|f\|_\infty + \|g\|_\infty$ and if $p = 1$, the same thing tells,

$\int_X |f + g|\, d\mu \leq \int_X |f|\, d\mu + \int_X |g|\, d\mu$, that is $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$. So, all that is fine. So, we now only have to look at the case $1 < p < \infty$.

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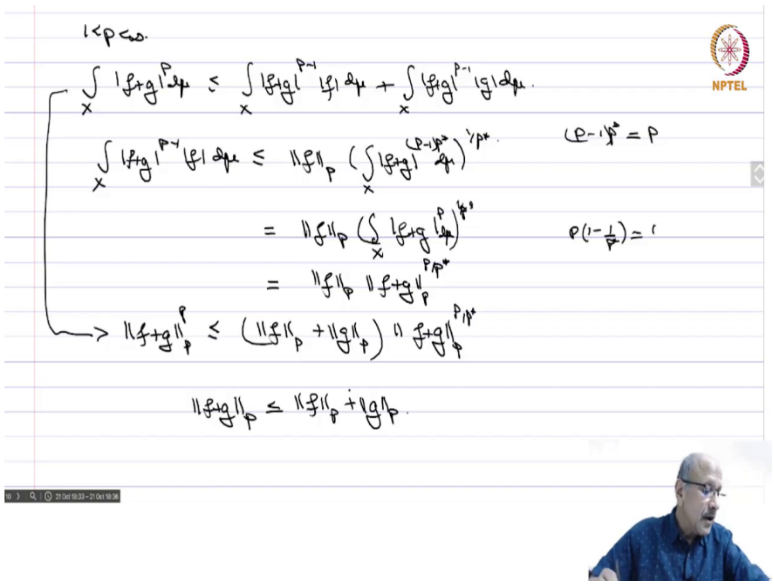

So, we now assume that $1 < p < \infty$. So, we are now going to just as we did in the earlier, we are going to apply Holder inequality. So, you write $\int_X |f + g|^p\, d\mu \leq \int_X |f + g|^{p-1}|f|\, d\mu + \int_X |f + g|^{p-1}|g|\, d\mu$. So, we are now going to apply Holder's inequality to each one of the terms on the right hand side. So what is, so let us take $\int_X |f + g|^{p-1}|f|\, d\mu$. So by Holder's inequality this will be less than equal to $\|f\|_p \, \| \cdot \|_{p^\star}$ of whatever follows. So, $\left(\int_X |f + g|^{(p-1)p^\star}\, d\mu\right)^{\frac{1}{p^\star}}$. But $(p - 1)p^\star$ is $pp^\star - p^\star$, which is just p, so this is equal $\|f\|_p\left(\int_X |f + g|^p\, d\mu\right)^{\frac{1}{p^\star}}$. But that is nothing, but $\|f\|_p \, \|f + g\|_p^{\frac{p}{p^\star}}$. So, you get that norm, the left hand side, $\|f + g\|_p^p$ is less than equal to, apply this to each of the terms, I will get

$(\|f\|_p + \|g\|_p)\|f + g\|_p^{\frac{p}{p^*}}$. So now, since it is not 0, I can divide this here. So, now $p\left(1 - \frac{1}{p^*}\right)$ and that is nothing but 1. $1 - \frac{1}{p^*}$ is $\frac{1}{p}$ and so that is equal to 1. So, if I divide it, so I get $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. And that is exactly the Minkowski's inequality.

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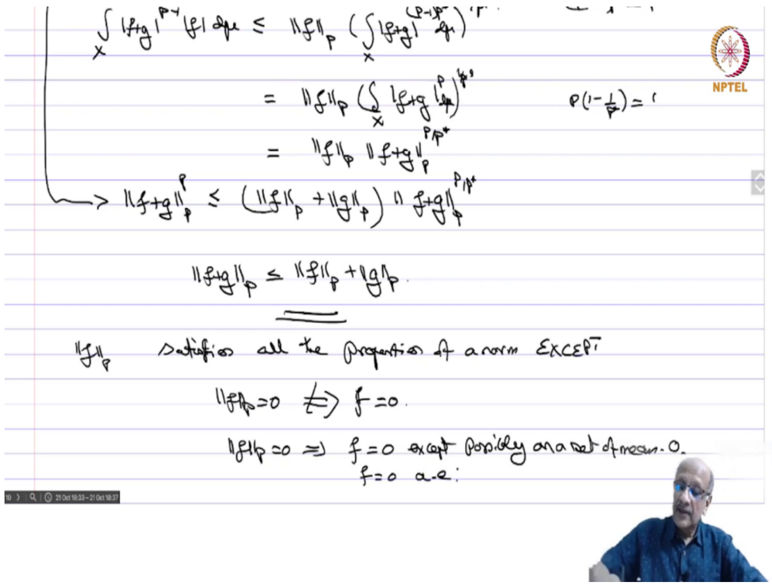

So, now what have we seen, so if you take this $\|f\|_p$. Then, it satisfies all the properties of a norm except, there is only one property which it does not follow. Norm $\|f\|_p = 0$ does not imply $f = 0$. So, it only implies $\|f\|_p = 0$ implies $f = 0$ except possibly on a set of measure 0. That is we say $f = 0$ almost everywhere. That means except on a set of measure 0, that is why it is true. So, you do not have this, so you do not have a norm of this. So what do we do? Whenever we have such a difficulty, we question things out.

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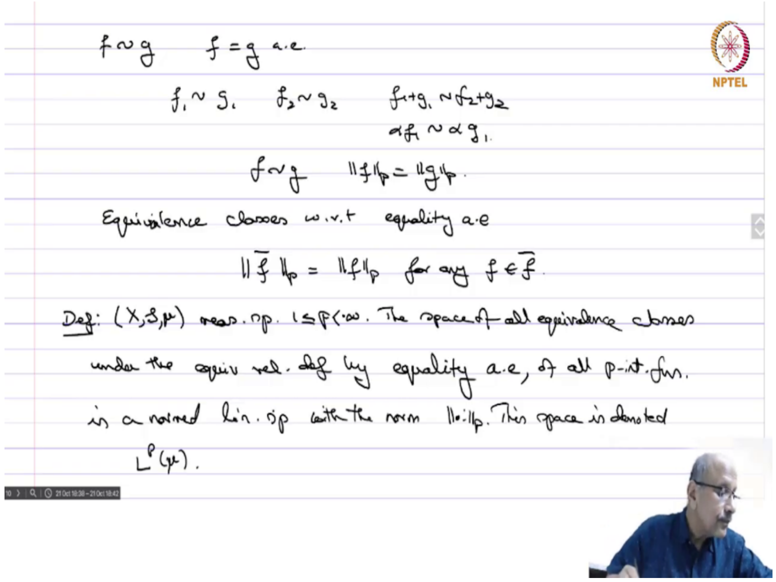

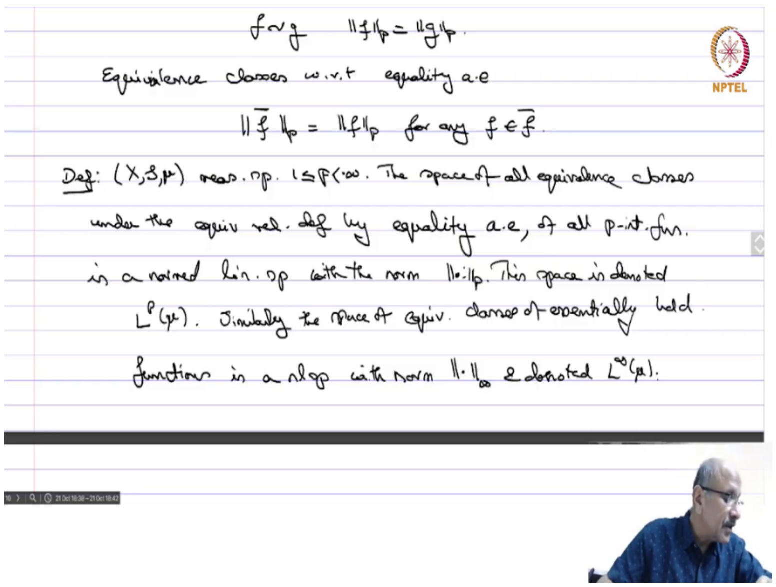

So we say that $f{\sim}g$ if $f\ =\ g$ almost everywhere. That means $f\ =\ g$ except on possibly on a set of measure 0. So, this is certainly reflexive. This is transitive and symmetric and therefore you have this in equivalence relation and it partitions the set of measurable p integrable functions into equivalence classes and also if you have that $f_1{\sim}g_1$, $f_2{\sim}g_2$ and then you have $f_1 + f_2{\sim}g_1 + g_2$ and $\alpha f_1{\sim}\alpha g_1$. And, so and also $f{\sim}g$, then you also have $\|f\|_p = \|g\|_p$. Therefore, you take any equivalence, so take equivalence classes with respect to equality almost everywhere. So, this relationship tilde is equality almost everywhere, so with respect to this relation, you take the equivalence classes and you define, so if I take $||\bar{f}||_p$, you define as $||f||_p$ for any f belonging to the equivalence class $\bar{f}$. So, it does not matter which representative you take, so all of them will be the same. Then of course it becomes a norm, because if $||\bar{f}||_p = 0$, then $||f||_p = 0$, then of course f is the function 0 almost everywhere and therefore you take it as it belongs to the 0 equivalence class and therefore it becomes a norm. Therefore we make the following definition.

Definition: So, (X, ζ, μ) measure space. $1 \leq p \leq \infty$. So this, let me say first, $p < \infty$. So, the space of all equivalence classes under the equivalence relation defined by equality almost everywhere of all p integrable functions is a norm linear space with the norm $|| \cdot ||_p$. How is it defined? In given any equivalence class, then you take any representative, and then evaluate the $|| \cdot ||_p$ for that. So, this space is denoted $L^p(\mu)$. Similarly, the space of equivalence classes of essentially bounded functions is a norm linear space with norm $|| \cdot ||_\infty$ and is denoted $L^\infty(\mu)$.

So, these are the spaces. So when we talk of course, we will not really make a fuss and talk about equivalence classes. We will say a function is in L^p. What do we mean? We mean that we are, this function which we are talking of is a representative, is p integrable, or is essentially the bound depending on what the value of p is and it is representative of an equivalence class in that space.

Now, however we will not, we will just say it is a function, because we are going to work all our computations via representatives only and it will not matter which representative we are taking. Because any two will be equal almost everywhere and they will not make a difference in all our calculations which are mostly integration and therefore, if it is 0 except on the set of measure 0, the integral is automatically 0. So.