Functional Analysis
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Lecture No. 36
Exercises

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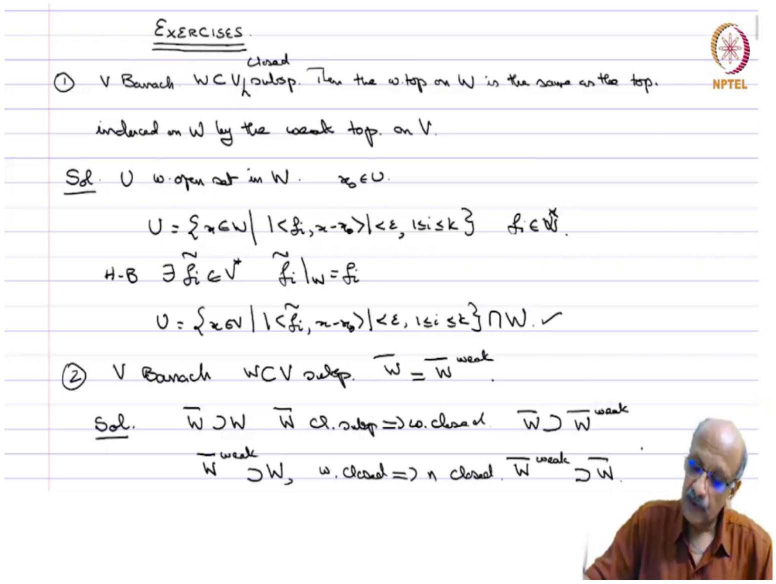

So, it is time to do some exercises. So the first one,

Problem 1: So V Banach and $W \subset V$ subspace. Then, so let us say closed subspace. So that it is also a Banach space. Then, the weak topology on W is the same as the topology induced on W by the weak topology on V.

Solution: So, let us take U, weakly open set in W. So, $x_0 \in U$. So, what is this going to be? So, U would be of the form, $\{x \in W: |\langle f_i(x - x_0)\rangle| < \epsilon, 1 \leq i \leq k\}$, so this is neighborhood of x_0 and $f_i \in W^\star$. So, this is how the neighborhood will look like. But then, by the Hahn Banach theorem, there exists $\tilde{f}_i \in V^\star$, such that $\tilde{f}_i|_W$ is the same as f_i and therefore $U = \left\{x \in V: |\langle \tilde{f}_i(x - x_0)\rangle| < \epsilon, 1 \leq i \leq k\right\} \cap W$. So, every weakly open set is, weakly open set in $V \cap W$ and therefore it is open in the induced topology. Conversely if you have weakly open set in the induced topology, then the restriction of, if you take any open set like this and then any $\widetilde{f}_i$ when restricted to W is a continuous linear function on W and therefore it will work out to be like this. Therefore you have the weakly open sets in W are precisely the weakly open sets in $V \cap W$ and therefore you have that it is the same as induced topology.

So next one,

Problem 2: V is Banach and $W \subset V$ is a subspace. Then the norm closure $\overline{W}$ is the same as the weak closure $\overline{W}^{weak}$. So, there is no difference between the weak closure of W and the norm closure of the W.

Solution: So, $\overline{W} \supset W$. $\overline{W}$ is a closed subspace and therefore it is weakly closed. So, it is weakly closed set containing W. Therefore $\overline{W}$ will also contain $\overline{W}^{weak}$, because that is the smallest closed subset in the thing. Similarly, $\overline{W}^{weak} \supset W$ and it is weakly closed, implies of course norm closed and therefore $\overline{W}^{weak} \supset \overline{W}$. So, you have both inclusions and therefore you have that the two closures are the same.

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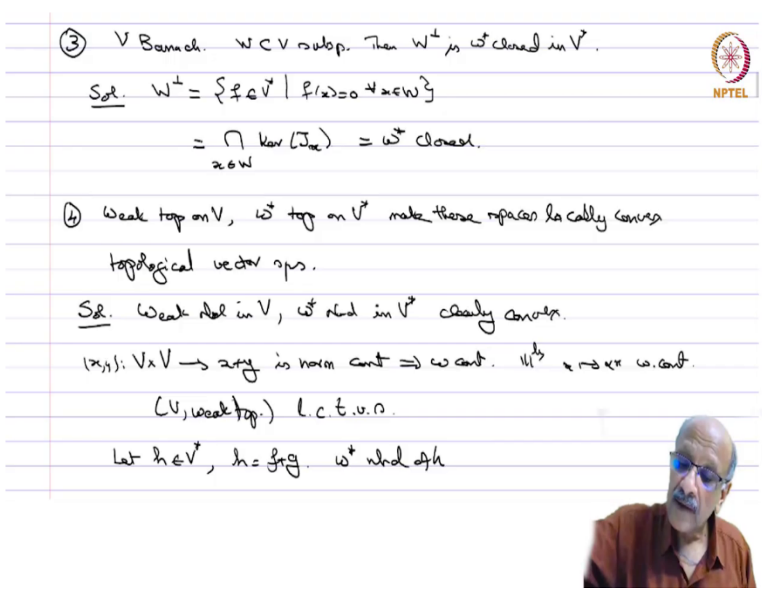

Problem 3: V is Banach and $W \subset V$ is a subspace, then $W^{\perp}$ is $W^{\star}$ closed in $V^{\star}$.

Solution: What is $W^{\perp}$? $W^{\perp} = \{f \in V^{\star} : f(x) = 0, \forall x \in W\}$. So, this is nothing but $\bigcap_{x \in W} Ker(J_x)$. But then J_x is $W^{\star}$ continuous, that is how the weak star topology is defined. So, $Ker(J_x)$ is $W^{\star}$ closed, arbitrary intersection of closed set is closed, so this is $W^{\star}$ closed.

Problem 4: Weak topology on V, weak star topology on $V^{\star}$ make the spaces locally convex topological vector spaces.

Solution: So, W neighborhood in V, $W^{\star}$ neighborhood in $V^{\star}$, clearly convex. So, every point has a convex neighborhood. If you just see the definition, you can check that it is, they are clearly convex sets. So, only to show addition and scalar multiplication are continuous. So, but if you take (x, y): $V \times V \mapsto x + y$ is norm continuous implies weakly continuous. Similarly, $x \mapsto \alpha x$ is weakly continuous. Therefore weak topology, so V with weak topology is a locally convex topological vector space. Now, what about V. So, let $h \in V^{\star}$, $h = f + g$ and then, you take $W^{\star}$ neighborhood of h. How does it look like?

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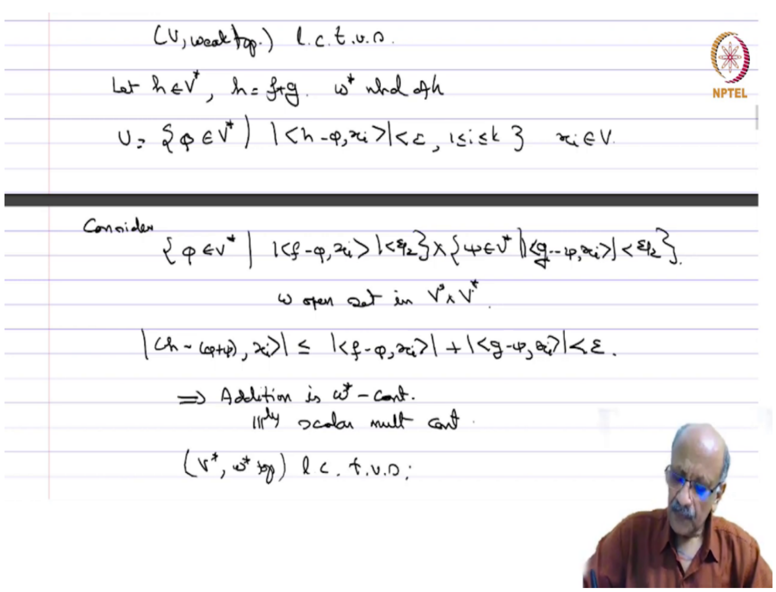

$U = \left\{ \phi \in V^{\star} : \left| \langle h - \phi, x_i \rangle \right| < \epsilon, 1 \leq i \leq k \right\}$ and x_i are arbitrary elements in V. So, now you consider the following set in the product space. The weak topology, because of the dual space is nothing but the product of the duals. Therefore the weak topology on the product space is nothing but the product to the weak topologies. So now let us consider the following, $\left\{ \phi \in V^{\star} : \left| \langle f - \phi, x_i \rangle \right| < \frac{\epsilon}{2} \right\} \times \left\{ \psi \in V^{\star} : \left| \langle f - \psi, x_i \rangle \right| < \frac{\epsilon}{2} \right\}$. So this product, so this is a weak open set in $V^{\star} \times V^{\star}$. So then you have $\left| \langle h - (\phi + \psi), x_i \rangle \right| \leq \left| \langle f - \phi, x_i \rangle \right| + \left| \langle g - \psi, x_i \rangle \right|$ and therefore this is less than $\frac{\epsilon}{2} + \frac{\epsilon}{2}$, that is ϵ. Therefore this shows, that addition is $W^{\star}$ continuous. Because we take any neighborhood, you can find a neighborhood of f and g, such that this is true. Therefore $\left\{ \phi \in V^{\star} : \left| \langle f - \phi, x_i \rangle \right| < \frac{\epsilon}{2} \right\}$ is the neighborhood of f, $\left\{ \psi \in V^{\star} : \left| \langle f - \psi, x_i \rangle \right| < \frac{\epsilon}{2} \right\}$ is the neighborhood of g and the product is contained in the neighborhood of h and therefore addition is continuous and similarly scalar multiplication. Therefore $V^{\star}$ with weak star topology also locally convex topological vector space. So, I have made a remark earlier, that we have a locally convex topological vector space, then all the Hahn Banach theorems can be reproduced and therefore the Hahn Banach theorems are all there.

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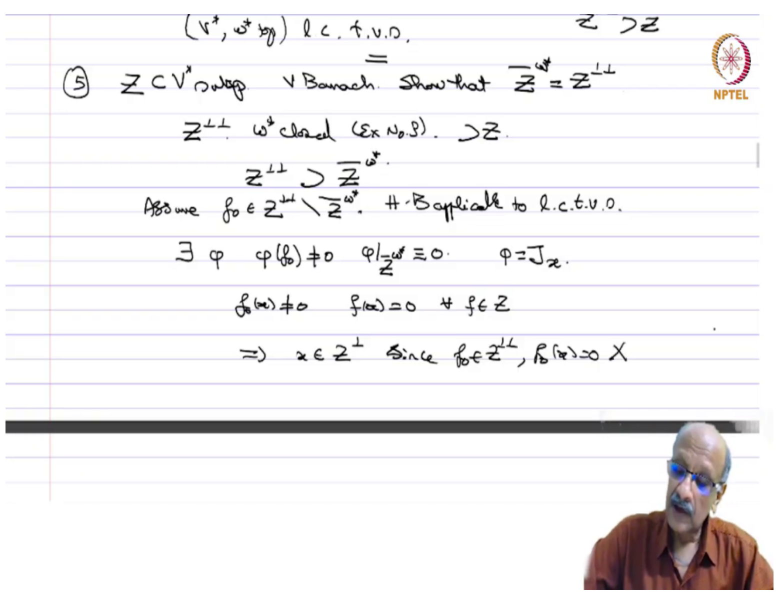

So now, let us do the following,

Problem 5: $Z \subset V^{\star}$ subspace, V Banach. Show that $\overline{Z}^{W^{\star}} = Z^{\perp\perp}$. So if you recall, we have shown that if $Z \subset V^{\star}$, then $Z^{\perp\perp} \supset \overline{Z}$, this is the norm closure. So in general you cannot say that they are equal. But it happens that $Z^{\perp\perp}$ is a $W^{\star}$ closure.

Solution: So, $Z^{\perp\perp}$ is $W^{\star}$ closed, we have already seen this in the third exercise, exercise number 3 and then it contains Z also. So, $Z^{\perp\perp} \supset \overline{Z}^{W^{\star}}$. So now, assume that this is not equal. Assume $f_0 \in Z^{\perp\perp} \setminus \overline{Z}^{W^{\star}}$. So, by Hahn Banach theorem, so I told you that Hahn Banach theorem is applicable, so Hahn Banach applicable to locally convex topological vector spaces and therefore there exists a continuous linear function such that $\phi\left(f_0\right) \neq 0$ and $\phi|_{\overline{Z}^{W^{\star}}} \equiv 0$. So we want to get a contradictory. But what is ϕ, a continuous linear functional in the $W^{\star}$ topology, we saw the only possibility is that it gives the form of J_x. So we said, $W^{\star}$ topology is formed in terms of the J_x's, but we later proved a proposition that every continuous linear function with respect to the weak star topology has to be of the form some J_x. So, we have $f_0(x) \neq 0$ and $f(x) = 0$ for all $f \in Z, \overline{Z}^{W^{\star}}$, so in particular it is true for all Z. So, this implies that $x \in Z^{\perp}$ and this implies since $f_0 \in Z^{\perp\perp}$, we have $f_0(x) = 0$ which is a contradiction, because we assume that it is not 0. Therefore this is a contradiction and that proves the problem.

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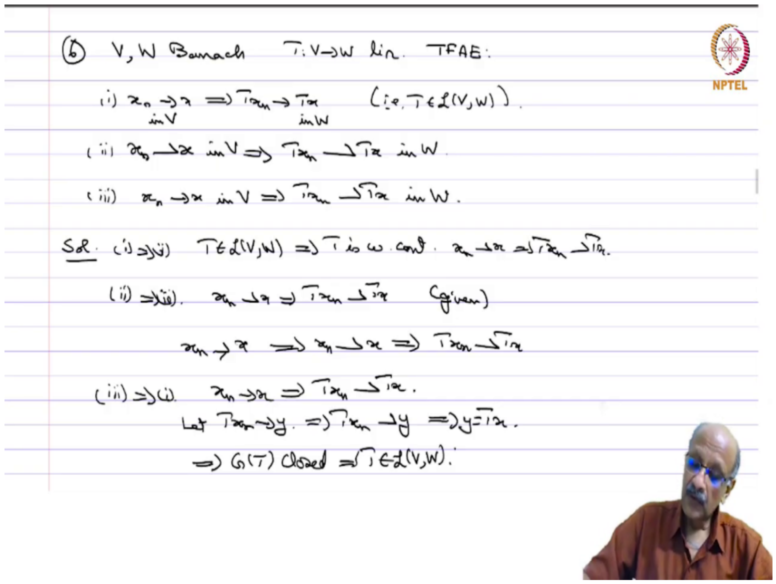

Problem 6: V, W Banach and $T: V \rightarrow W$ is linear. This is linear. The following are equivalent.

1. $x_n \rightarrow x$ in $V \Longrightarrow Tx_n \rightarrow Tx$ in W. So this is, that is $T \in L(V, W)$.
2. $x_n \rightharpoonup x$ in $V \Longrightarrow Tx_n \rightharpoonup Tx$ in W and
3. $x_n \rightarrow x$ in $V \Longrightarrow Tx_n \rightharpoonup Tx$ in W.

So here, all these three statements are equivalent.

Solution: $1 \Longrightarrow 2$: So, $T \in L(V, W)$ implies T is weakly continuous. Therefore, $x_n \rightharpoonup x$ implies $Tx_n \rightharpoonup Tx$, so that implies $1 \Longrightarrow 2$. Now $2 \Longrightarrow 3$: So $x_n \rightharpoonup x$ implies $Tx_n \rightharpoonup Tx$, that is given. So, if $x_n \rightarrow x$ in V, this implies $x_n \rightharpoonup x$ and therefore this implies that $Tx_n \rightharpoonup Tx$. So, that is just the statement. So, then $3 \Longrightarrow 1$: So, what is now given? $x_n \rightarrow x$ implies $Tx_n \rightharpoonup Tx$. So, we want to show T is continuous. So, let $\{Tx_n\}$ converge to some y. So this implies $Tx_n \rightharpoonup y$ and the weak topology is Hausdorff, so $y = Tx$. So $x_n \rightarrow x$, $Tx_n \rightharpoonup y$ implies $y = Tx$. So, that means $G(T)$ is closed. Implies $T \in L(V, W)$. So, all the three are equivalent.

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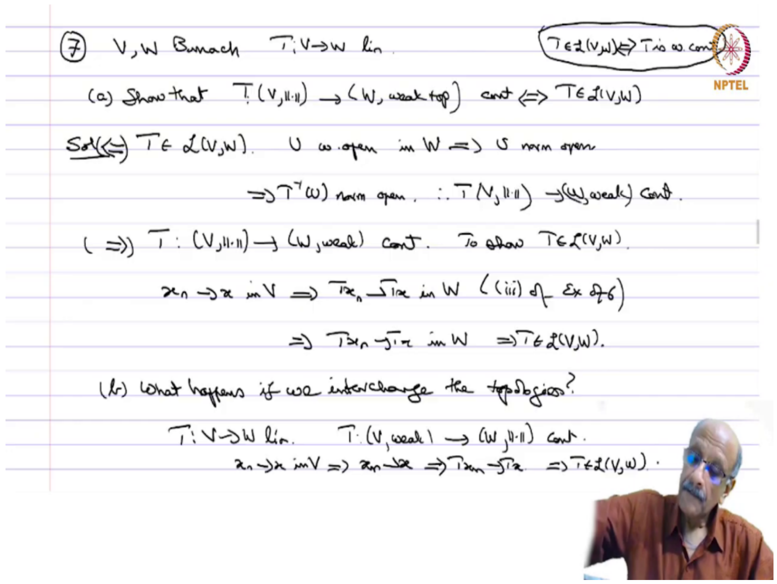

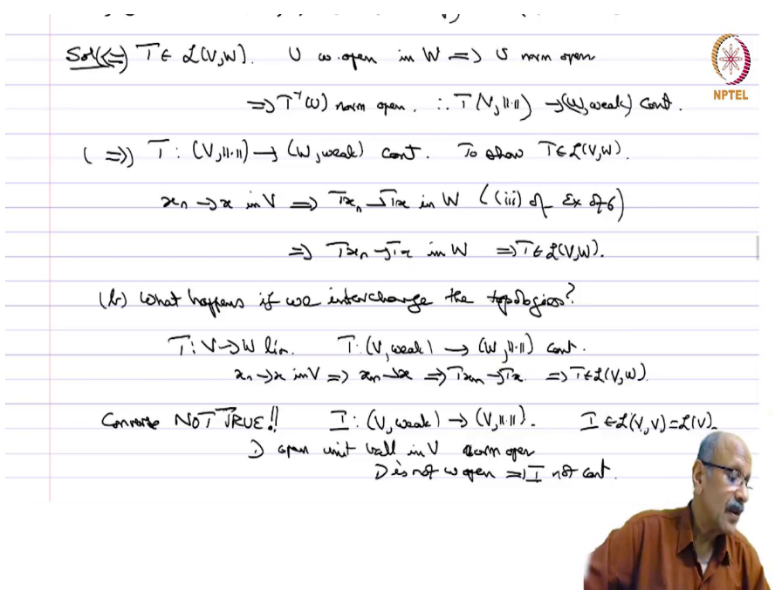

Problem 7: So V, W Banach and $T: V \to W$ linear. Then show that $T \in L(V, W)$, if and only if T is weakly continuous. So, this is the theorem which we have already proved. So, now we are going to say,

(a) Show that T from V with the norm topology to W with the weak topology continuous if and only if $T \in L(V, W)$. So, we are really showing that it does not matter as far as linear maps are concerned, the continuity in the standard sense is the same, whatever the other topologies we are going to put.

Solution: So, $T \in L(V, W)$. So we want to show, it is continuous. So, we are proving the reverse part. So, we want to show that this is continuous. So, let us say U weakly open in W. So implies U is norm open and that implies $T^{-1}(U)$ is norm open. Therefore, $T: (V, \|\cdot\|) \to (W, weak)$ is continuous. What about the other way around? So, $T: (V, \|\cdot\|) \to (W, weak)$ is given to be continuous. So to show, $T \in L(V, W)$. So, let us take $x_n \to x$ in V, then T is continuous, therefore it implies that $Tx_n \rightharpoonup Tx$ in W. And this is, if I see statement 3 of exercise 6 and therefore that implies statement one, so $Tx_n \to Tx$ in W by the closed graph theorem, that is what we saw and therefore we have this implies to $T \in L(V, W)$.

(b) What happens if we interchange topologies?

Solution: So, let us say $T: V \to W$ linear and $T: (V, weak) \to (W, \|\cdot\|)$ continuous. So, let us take $x_n \to x$ in V. So, this implies that $x_n \rightharpoonup x$ in V and therefore that implies as $T: (V, weak) \to (W, \|\cdot\|)$, so $Tx_n \to Tx$. So, this implies that $T \in L(V, W)$. Converse not true. So, this is the only case where you have a problem. So, you take identity map $I: (V, weak) \to (V, \|\cdot\|)$. So, $I \in L(V, V) = L(V)$. But you take D open unit ball in V, which is norm open, but D is not weakly open as we saw, therefore I is not continuous. So, you have a norm continuous linear map, but it will not be continuous from the weak topology to the norm topology. So, that is.

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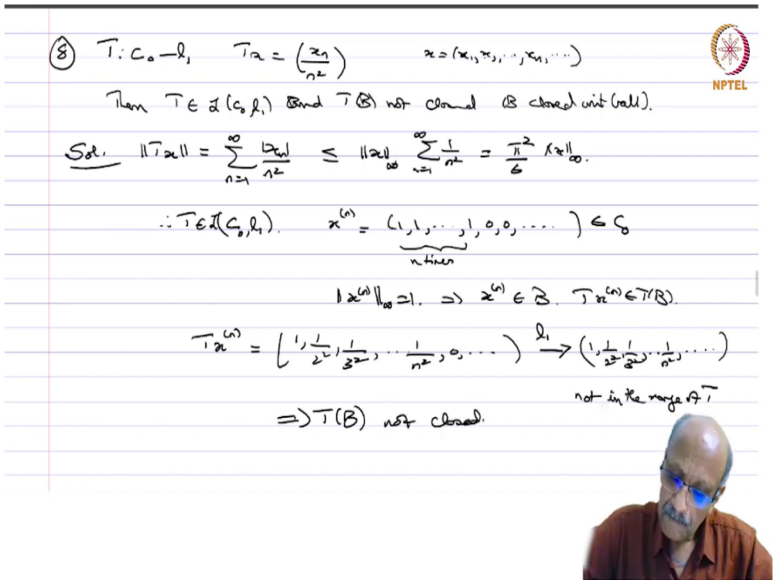

Problem 8: Let T: $C_0 \to l_1$, we define by $Tx = \frac{x_n}{n^2}$. So $x = (x_1, x_2, \cdot, \cdot, \cdot, x_n, \cdot, \cdot, \cdot)$. So you take, coordinate wise I am going to define. So then, $T \in L(C_0, l_1)$, and $T(B)$ is not closed. So, B is the closed unit ball. So, it is not necessary for closed sets to go to closed sets if you have this.

Solution: So, let us take $\|Tx\|$. So you are taking l_1, so $\|Tx\| = \sum_{n=1}^{\infty} \frac{|x_n|}{n^2}$ and that is less than equal to maximum of the x_n's that is $\|x\|_\infty \sum_{n=1}^{\infty} \frac{1}{n^2}$ and I am sure all of you know what this is, so $\|x\|_\infty \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\|x\|_\infty$. Therefore, $T \in L(C_0, l_1)$. Now you take in B, so you take $x^{(n)} = (1, 1, \cdot, \cdot, \cdot, 1, 0, 0, \cdot, \cdot)$. So $= (1, 1, \cdot, \cdot, \cdot, 1, 0, 0, \cdot, \cdot) \in C_0$ because it is terminally 0 and then $\|x^{(n)}\|_\infty = 1$. Therefore $x^{(n)} \in B$. So, now you look at $Tx^{(n)} = \left(1, \frac{1}{2^2}, \frac{1}{3^2}, \cdot, \cdot, \cdot, \frac{1}{n^2}, \cdot, \cdot, \cdot\right)$ and what this is converge to in l_1. So this converges in l_1 to $\left(1, \frac{1}{2^2}, \frac{1}{3^2}, \cdot, \cdot, \cdot, \frac{1}{n^2}, \cdot, \cdot, \cdot\right)$. So this, so the image, so $Tx^{(n)} \in T(B)$ and it converges to something in l_1, but $\left(1, \frac{1}{2^2}, \frac{1}{3^2}, \cdot, \cdot, \cdot, \frac{1}{n^2}, \cdot, \cdot, \cdot\right)$ is not in the range. Because if it has to be in the range of T, there is only one particular candidate, namely

$(1, 1, 1, \cdot, \cdot, \cdot)$ and that is not an element of C_0. So this is, this vector is not in the $R(T)$,, therefore $T(B)$ is norm closed.

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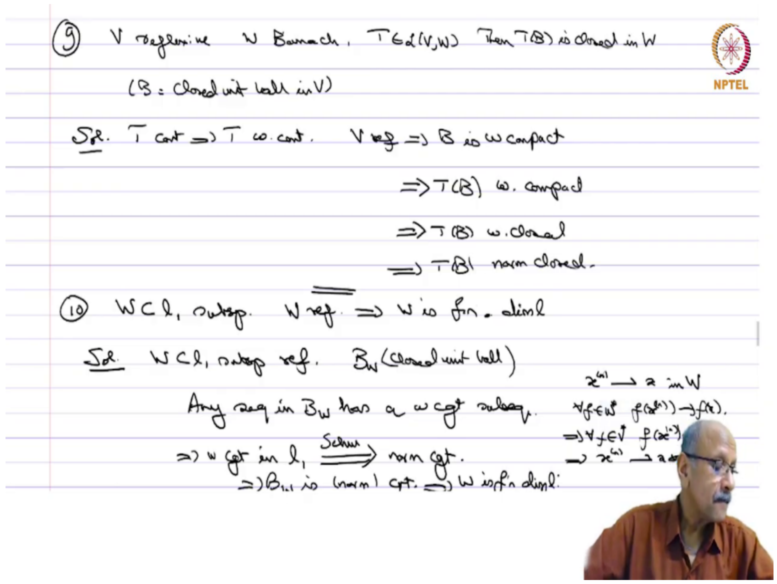

However if,

Problem 9: Suppose V is reflexive and W is a norm linear space, W is a Banach space and $T \in L(V, W)$, then $T(B)$ is closed. So, what is B? Closed unit ball and I hope I said that in the previous exercise also. Yeah, closed, B is a closed unit.

Solution: So T is continuous implies T is weakly continuous, V reflexive, so this implies B is weakly compact. T is weakly continuous, continuous image of compact, since B is compact, so $T(B)$ is weakly compact. Weak topology is Hausdorff, so $T(B)$ is weakly closed implies $T(B)$ is norm closed.

So, the reflexivity makes all the difference.

Problem 10: Let $W \subset l_1$ subspace. Then W reflexive implies W is finite dimensional. So, the only reflexive subspaces of l_1 are the finite dimensional subspaces. There are no other subspaces.

Solution: $W \subset l_1$ subspace which is reflexive. So, that means B_W closed unit ball is weakly compact. B_W is the closed unit ball. So then, any sequence in B_W has a weakly convergent subsequence. Because we saw in the reflexive space, any sequence contains, it is a converse of this was the Eberlein Scholian theorem. So, we proved of course that if you have a reflexive space, any bounded sequence has a weakly convergence subsequence. So, now weakly convergent means what? So $f\left(x^{(n)}\right) \to f(x)$, that means what, for every $f \in W^\star$, $f\left(x^{(n)}\right) \to f(x)$. But then, this implies for every $f \in V^\star$, restriction of $V^\star$ functions are also continuous linear functions in W. So, we have $f\left(x^{(n)}\right) \to f(x)$. Therefore $f\left(x^{(n)}\right) \to f(x)$ in V also. So implies, weakly convergent in l_1 and by Schur's lemma, this implies norm convergence. So, any sequence has a norm convergent subsequence and this implies that B_W is norm compact and that implies W is finite dimensional.

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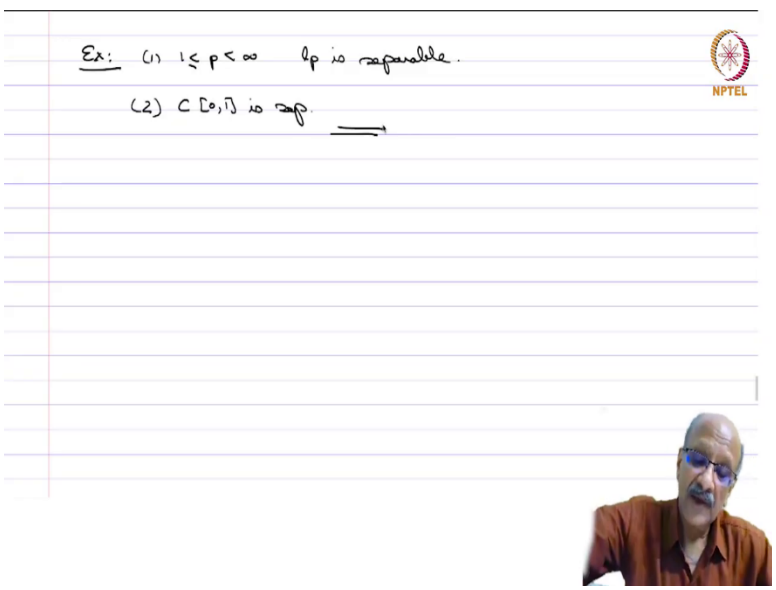

So, I would like you to, these are very simple exercises again, so I would like you to try it yourself.

Exercise: (1) So $1 \le p < \infty$ 1, then l_p is separable.

You might have even used this. So, all you have to do is take finitely supported sequences, that is 0 after some time, those are dense l_p and then show that rational sequence are dense in that and that finite rational sequences are countable and that will be dense. So, that will be the solution for that.

(2) Similarly $C([0,1])$ is separable. So, any continuous function can be uniformly approximated by a polynomial that is the Weierstrass theorem. So, then you approximate that uniformly by polynomial with rational coefficients. So, you just have to write down the arguments correctly and that is it. So with this, we wind up this chapter. Thank you.