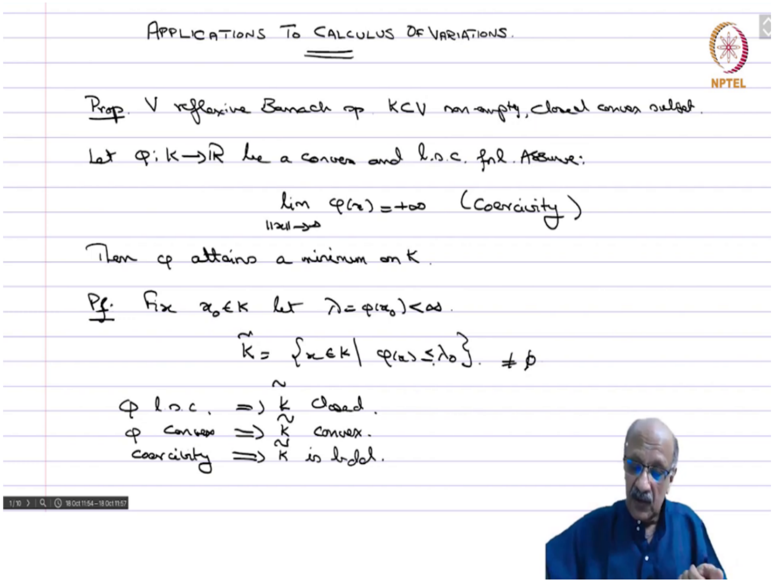
Functional Analysis
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No. 35
Applications

(Refer Slide Time: 00:17)

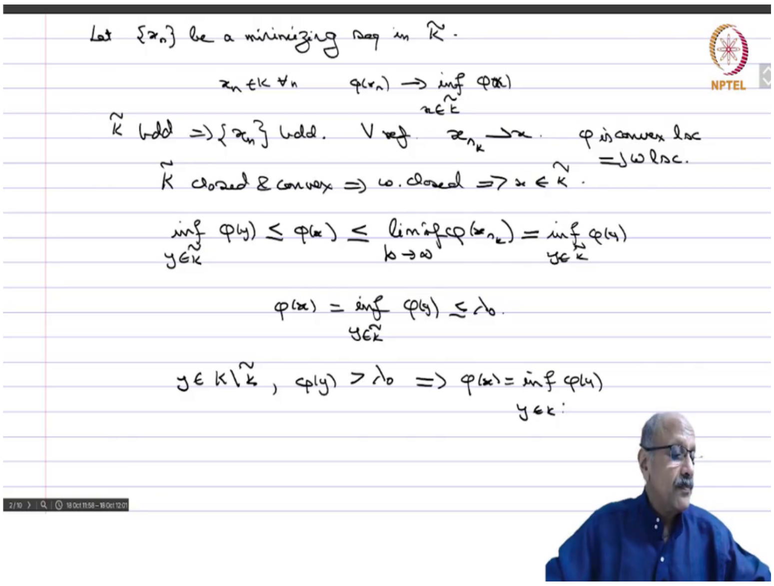


We will now look at some applications to calculus of variations. Calculus of variations can be thought of as doing maxima minima in infinite dimensional spaces. So, we are going to prove the following proposition.

Proposition: V reflexive Banach space and $K \subset V$ non-empty closed convex subset. Let $\phi: K \rightarrow R$. I am working with V as a real Banach space and be a convex and lower semi-continuous function functional. Assume, that $\phi(x)\ =+\ \infty$. So, this is called the coercivity property. Then ϕ attains a minimum on K. That means there exists a point which is the minimum value of ϕ and it is on K.

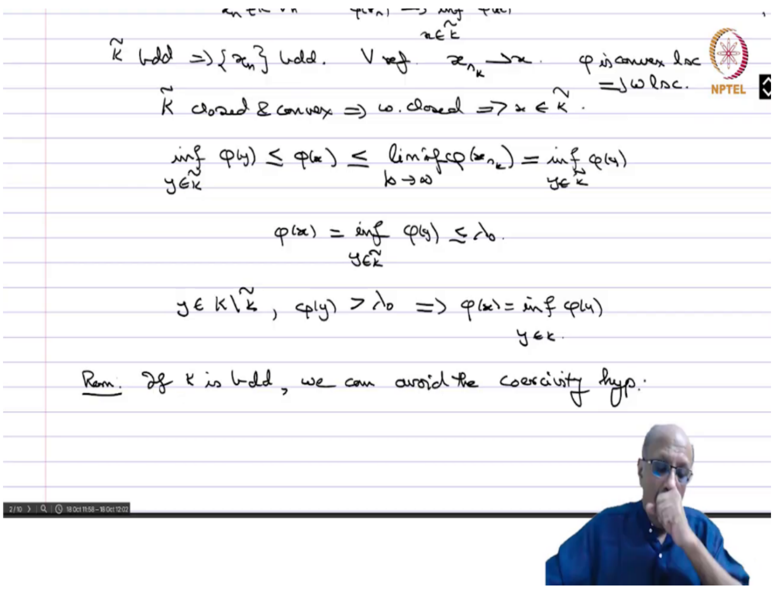
Proof: So, fix some $x_0 \in K$ and let $\lambda_0 = \phi\left(x_0\right) < \infty$. So, it could be some finite number and you define $\widetilde{K} = \{x \in K: \phi(x) \leq \lambda_0\}$. This is non empty because x_0 is already in that set. So, we are having that ϕ is lower semi continuous therefore $\widetilde{K}$ is closed. Then ϕ is convex so this implies that $\widetilde{K}$ is also convex. Then coercivity implies of course that $\widetilde{K}$ is bounded. Because, if you have an unbounded sequence inside $\widetilde{K}$ then $\phi(x)$ should go to $+\infty$. Whereas, you know $\phi(x) \le \lambda_0$. So, you have all these properties.

(Refer Slide Time: 04:04)



So, let $\{x_n\}$ be a minimizing sequence in $\widetilde{K}$. So, what does it mean? So, $x_n \in \widetilde{K}$ for all n and $\phi\left(x_n\right) \to \phi(x)$. So, that is such a sequence exists always because you are taking the infimum you can always find a sequence which goes to the infimum and such a sequence is called a minimizing sequence. So, $\widetilde{K}$ bounded implies $\{x_n\}$ is bounded. Now, V is reflexive and therefore we have shown that any bounded sequence has a weakly convergent subsequence. So, $x_{n_k} \rightharpoonup x$ in $\widetilde{K}$. So, there exists a weakly convergent subsequence. So, now because $\widetilde{K}$ is closed and convex; this implies weakly closed. And therefore, this implies that $x \in \widetilde{K}$. So, $\phi(y) \le \phi(x)$. Because, x is in $\widetilde{K}$ and ϕ is lower semi continuous and also ϕ is convex and lower semi-continuous implies weakly lower semicontinuous we have seen this also. And therefore, since $\{x_{n_k}\}$ converges takes weakly one of the characterizations, one of the properties of the lower semicontinuous function is $\phi(x) \le \phi(x_{n_k})$. And of course but we know $\{\phi(x_{n_k})\}$ converges to the minimum. So, this is limit in fact it is equal to the limit so this is equal to $\phi(y)$. So, all of them are equal so we have $\phi(x) = \phi(y)$. And if $y \in K\setminus\widetilde{K}$ then we have $\phi(y) > \lambda_0$ and $\phi(y)$ is of course less than or equal to λ_0. Because, all of the $\widetilde{K}$ is set of all y which is such that $\phi(y) \leq \lambda_0$. And therefore, we have that $\phi(x)$ is in fact equal to $\phi(y)$. All on the entire set so that proves the theorem so the existence of a minimum is there. So, what does this theorem say: a coercive lower semi-continuous convex functional defined on the non-empty closed convex set of reflexive Banach space always attains its minimum.

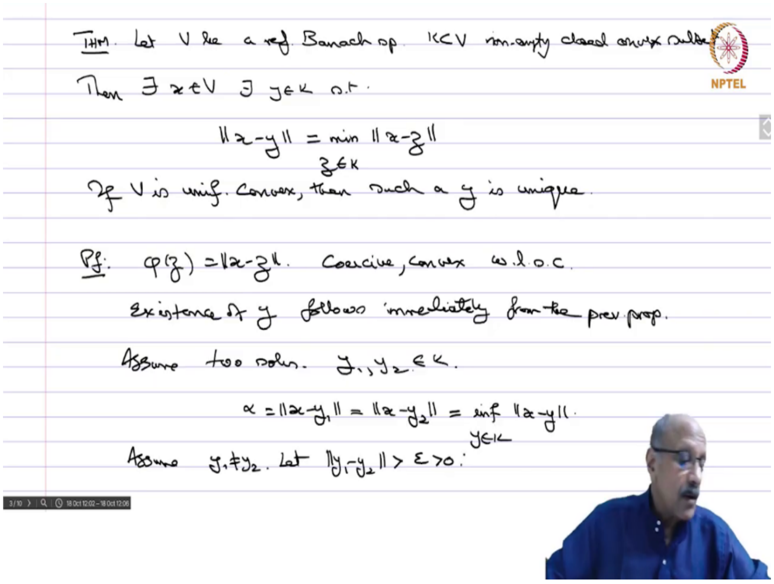
(Refer Slide Time: 07:46)



So, the remark

Remark: If K is itself bounded; we can avoid the hypothesis, avoid the coercivity hypothesis.

(Refer Slide Time: 08:07)



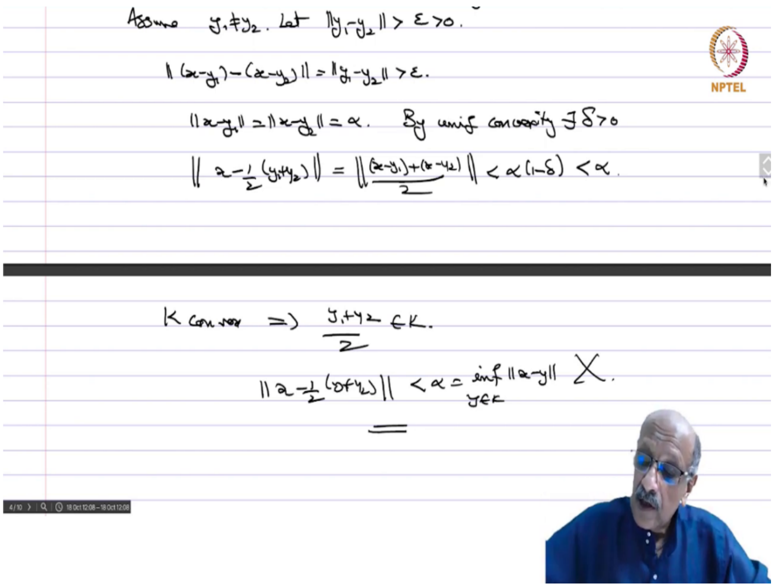
So, we now prove a theorem which is a direct consequence of this, it is a fairly important theorem.

Theorem: So, let V be a reflexive Banach space and K a closed, $K \subset V$ non-empty closed convex subset. Then for every $x \in V$ there exists a $y \in K$ such that $||x - y|| = ||x - z||$. And if V is uniformly convex then such a y is unique.

Proof: So, you take the function $\phi(z) = ||x - z||$. So, this is clearly coercive; it is convex and weakly lower semi continuous. It is lower semi continuous. It is continuous and then it is convex so it is weakly lower semi continuous. We have already seen this property of the norm. And therefore, existence of y follows immediately from the previous proposition. So, in the previous proposition we said ϕ is lower semicontinuous and convex and therefore it was weakly lower semicontinuous. Here we have ϕ is in fact continuous and convex and therefore it is weakly lower semicontinuous by that property. So, that is and this weak lower semicontinuous which is important you do not need other in fact weakly sequentially lower semicontinuous. That means it is enough if for sequences you have that if $x_n \rightarrow x$ then $\phi(x) \leq \phi(x_n)$ that is called sequential lower semi continuity. And that even is enough in a metric space that is equivalent to lower semi continuity. In the general topological space lower semicontinuous implies sequentially lower semicontinuous but not vice versa.

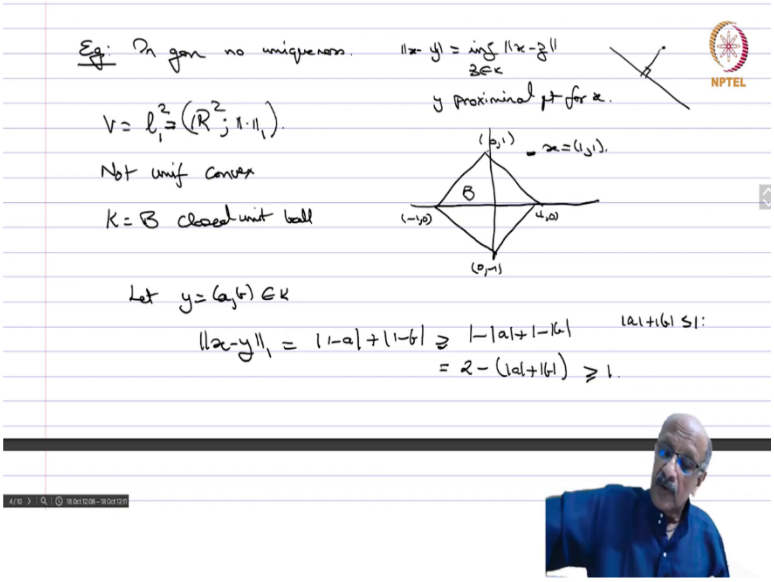
So, anyway we have this immediately so assume now two solutions y_1 and y_2 in K. That means you have $\alpha = \|x - y_1\| = \|x - y_2\| = \|x - y\|$. So, then assume $y_1 \neq y_2$ so let $\|y_1 - y_2\| > \epsilon$ which is strictly positive.

(Refer Slide Time: 12:19)



So, $\|(x - y_1) - (x - y_2)\| = \|y_1 - y_2\| > \epsilon$. And $\|x - y_1\| = \|x - y_2\| = \alpha$. Then by uniform convexity there exists a $\delta > 0$ such that $\|x - \frac{(y_1+y_2)}{2}\| = \|\frac{(x-y_1)+(x-y_2)}{2}\| < \alpha(1 - \delta)$. In the definition of uniform convexity we did it with $\alpha = 1$. Now, if we if you have any other α it is obvious that you should scale it out so and this of course is strictly less than α. Now, K is convex so this implies that $\frac{y_1+y_2}{2}$ is also in K and you have $\|x - \frac{(y_1+y_2)}{2}\| < \alpha$ which is the $\|x - y\|$ and that is a contradiction. Because, you contradict the minimality of the K. Therefore, you have only so the assumption $y_1 \neq y_2$ is wrong and therefore this proves the uniqueness. Later when we do Hilbert spaces this will be the cornerstone of that Hilbert space theory and we will see we will apply this there.

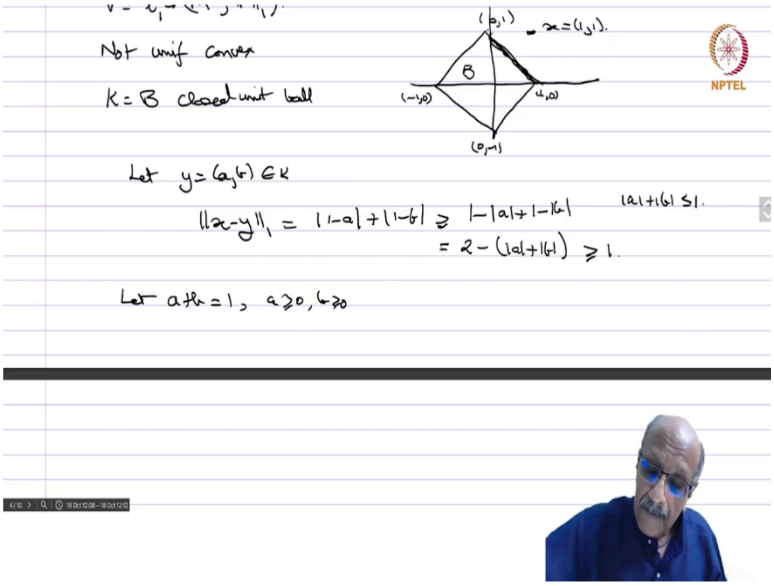
(Refer Slide Time: 14:19)



So, let us now look at an example.

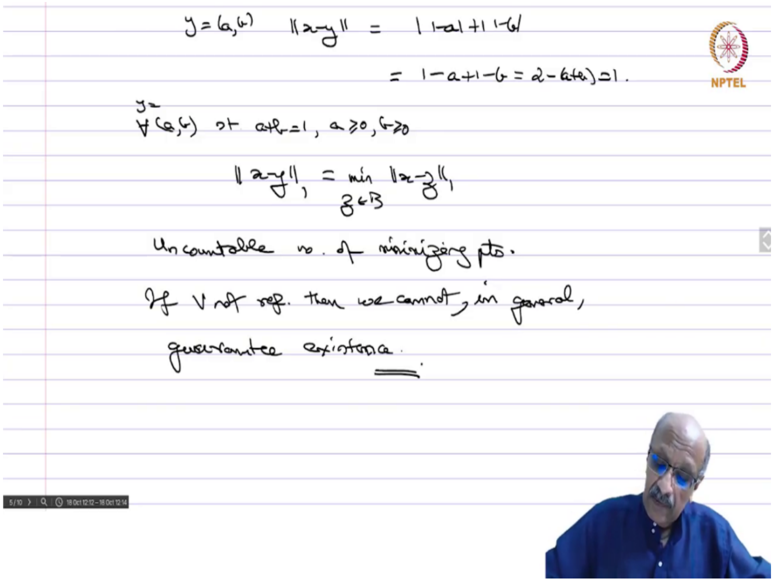
Example: So, in general no uniqueness. So this y such that $||x - y|| = ||x - z||$ y is called proximinal point. In the language of approximation theory this is a very important notion so this that means the closest point. So, if you look like for instance if you take K to be a straight line in the plane and you have a point so this is a closed convex set. Then you know that the closest point is a perpendicular projection so this is the kind of thing which we are looking at. So, now let us take $V = l_1^2$. That means R^2 with $|| \cdot ||_1$. So, this is not uniformly convex. We saw that the unit ball has several flat portions and therefore you do not have this. So, we have we take $K = B$ closed unit ball. So, we have this is B the closed unit ball and then so this will be the point $(1, 0)$, $(- 1, 0)$ and then $(0, - 1)$ and $(0, 1)$. So, this will be the closed unit ball. And now we are going to take the point x to be the point $(1, 1)$. Which is the point here. So, now let a let $y = (a, b) \in K$ in the unit ball. So, what is $||x - y||_1$ this equal to $|1 - a| + |1 - b|$ which is greater than or equal to by the triangulation $1 - |a| + 1 - |b|$ which is equal to $2 - (|a| + |b|)$. $|a| + |b| \leq 1$ so $2 - (|a| + |b|) \geq 1$. Since, $|a| + |b| \leq 1$ since (a, b) belongs to the ball. So, the distance of x from B is at least 1, but what about taking any point on the line?

(Refer Slide Time: 17:39)



So, let $a + b = 1, a \geq 0, b \geq 0$. That means we are looking at this particular line here.

(Refer Slide Time: 17:59)



Then what is if $y = (a, b)$, $\|x - y\|_1$ will be $|1 - a| + |1 - b|$. But, a and b are non negative and sum is 1 so each of these numbers is positive is equal to $1 - a + 1 - b$ which is equal to 1. Therefore, for all (a, b) such that $a + b = 1, a \geq 0, b \geq 0$. So, for every point on that line for every point in this line we have that we have $\|x - y\|_1$ so call $y = (a, b)$,

$\|x - y\|_1$ is the $\|x - z\|_1$. So, an uncountable number of minimizing points. So, you do not have uniqueness and this is because of the lack of uniform convexity.

Of course V is reflexive which is because it is not uniformly convex but V is reflexive because we are in finite dimensions that is why usually $\| \cdot \|_1$ in l^2 and I mean l^1 will not be reflexive. But, because we are in finite dimension every space is reflexive and therefore you have reflexive but it's not uniformly convex. Therefore, you do not have. So, if not reflexive then we cannot in general guarantee existence. So, there is no point talking about uniqueness, you do not even know if you have a minimum available at all. So, we will wind up this chapter with this and we will take up some exercises next time.