Functional Analysis

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Lecture No. 34

Uniformly Convex Spaces

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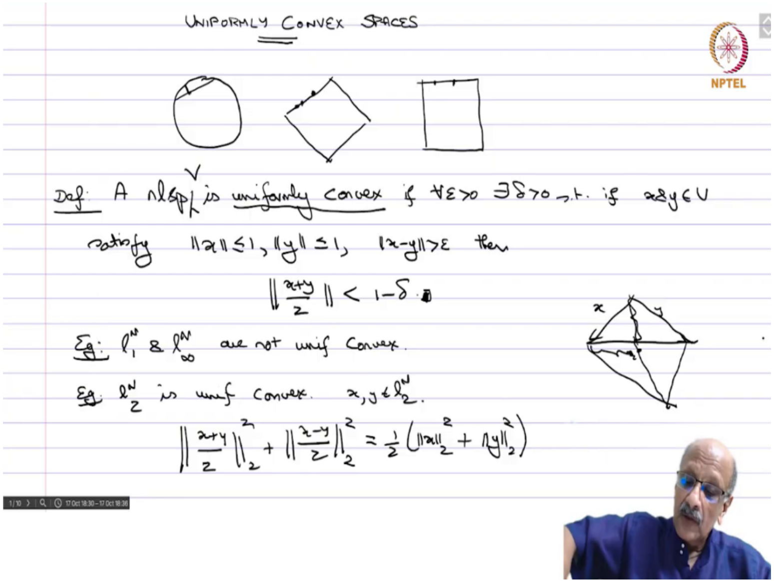

We will now talk about uniformly convex spaces. So, uniform convexity is a condition on the norm so it tells you something about the geometry of the space. So, if you look at the ball with the two norms in the plane then it bulges uniformly in all directions. And if you look at the ball in the one norm then it is something like this.

It is a diamond and if you look at it in the infinity norm then of course you get a square. So, these have flat portions on the boundary whereas this one bulges uniformly. Uniform convexity in some sense analytically quantifies what is this uniform bulging in all directions that is the thing.

So, we have the following definition.

Definition: A non-linear space V is uniformly convex if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x, y \in V$ satisfy $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x - y\| > \epsilon$. Then $\|\frac{x+y}{2}\| < 1 - \delta$. So, this is the definition so let us understand it briefly. So, if you look what it is essentially though we have said $||x||\leq1,\ ||y||\leq1$ actually the important thing happens near on the boundary.

Suppose, I have two points on the boundary of the unit ball so that means $||x|| = 1,\ ||y|| = 1$. Then it says that the midpoint must be away from the boundary in a uniform fashion irrespective of the positions of the two points x and y. If you have x and y as long as the relative distance is bigger than the bigger than ϵ then the midpoint should be sufficiently far away from the boundary.

So, that is what is given by this. So, if you look at these points the midpoint is also on the boundary so it would not satisfy this condition on the l_1 and l_∞ cases. But, in the l_2 case you we can see that it definitely satisfies. So, this means, so this is what we mean by uniformly bulging in all directions.

If you have two points on the boundary which are a certain distance apart then irrespective of the position of the two points the midpoint must be uniformly away from the boundary at a certain distance. So, so example

Example: So l_1^N and l_∞^N that means R^N with the $||\cdot||_1$ or $||\cdot||_\infty$ are not uniformly convex.

In fact they are not even strictly convex which I introduced a little earlier.

Example: So, l_2^N is uniformly convex. So, let us take $x, y\in l_2^N$ this is the R^N with $||\cdot||_2$ then this straightforward calculation so $||\frac{x+y}{2}||_2^2 + ||\frac{x-y}{2}||_2^2 = \frac{1}{2}\left(||x||_2^2 + ||y||_2^2\right)$. This is called the parallelogram identity or the Apollonius's theorem from plane geometry. So, if you have so this is x and y then this will be x plus if you take the midpoint here then this will be $\frac{x+y}{2}$, this will be $\frac{x-y}{2}$ in length and therefore this is the standard theorem in plane geometry known as Apollonius theorem or this is also called the parallelogram law. Because, you can complete the parallelogram and say that half the diagonal square on half the diagonal is equal to one half of the sum of the squares on the other two sides. So, that is this theorem so this is straightforward checking you can do it.

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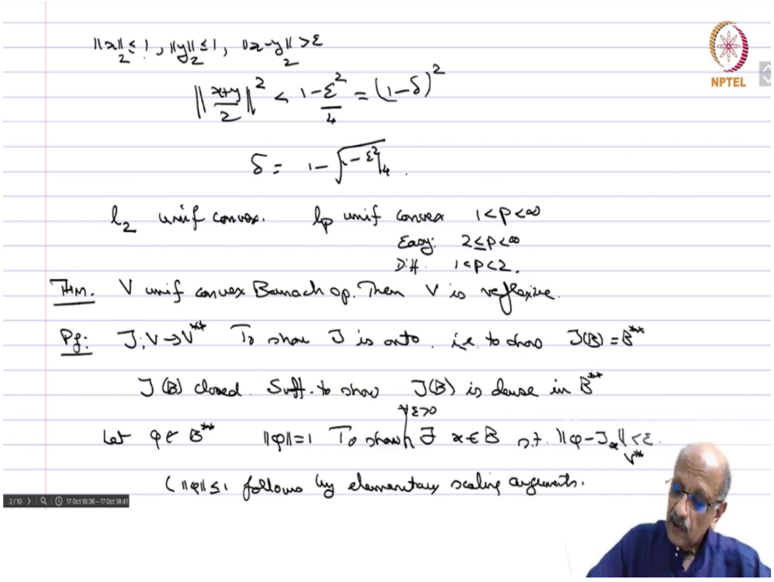

So, then if $\|x\|_2 \leq 1$, $\|y\|_2 \leq 1$, $\|x - y\|_2 > \epsilon$. Then from this identity it follows that $\|\frac{x+y}{2}\|_2^2 < 1 - \frac{\epsilon^2}{4} = (1 - \delta)^2$. And where δ is now given in a straightforward fashion by $1 - \sqrt{1 - \frac{\epsilon^2}{4}}$. So, once you have this, you have uniform convexity.

So, we can also show in fact l_2 for the same reason. In l_2 also you have the same identity so you can check it. So, l_2 is uniformly convex l_p is uniformly convex if $1 < p < \infty$. So, if in fact it is easy we will see this later if $2 < p < \infty$ and then difficult if $1 < p < 2$ so we will see this in a when the time comes when we do l_p spaces. So, you can see that these spaces are all uniformly convex. Now, the important theorem why are we concerned about uniformly convex spaces, so theorem this is the major theorem here:

Theorem: V uniformly convex Banach space. Then V is reflexive. So, you see the geometry of the norm tells you when the space is reflexive.

Proof: So, $J: V \to V^{**}$ canonical embedding and to show J is onto. So, that is to show $J(B) = B^{**}$. Where according to our notation B is the closed unit ball in V, B^{**} is the closed unit ball in V^{**}. Now, J is an isometry so and so $J(B)$ is closed so sufficient to show $J(B)$ is dense in B^{**}. Then $J(B)$ will be, $J(B)$ is closed so $J(B) = \overline{J(B)}, \overline{J(B)}$ is B^{**}. So $J(B) = B^{**}$. So, this is what we want to show, we want to show that given any continuous linear functional on the dual space namely a member of B^{**} which is in V^{**} you can always approximate it by something from $J(B)$ as much as close this.

So, let $\phi \in B^{**}$ and we assume that $||\phi|| = 1$. Now, if you prove it for $||\phi|| = 1$ then so to show for every $\epsilon > 0$ there exists an $x \in B$ such that $||\phi - J_x|| < \epsilon$. So, this is what, this is in V^{**} of course.

This is what we want to show, so this is what we mean by dense. Now, I have said for $||\phi|| = 1$ but then once you show it for $||\phi|| = 1$ then by scaling argument you can easily show it for any other ϕ with $||\phi|| \leq 1$. So, $||\phi|| = 1$ follows by elementary scaling arguments. Namely, the theorem will be true for $\frac{\phi}{||\phi||}$ and then you have to do it from there you can easily deduce it for any ϕ.

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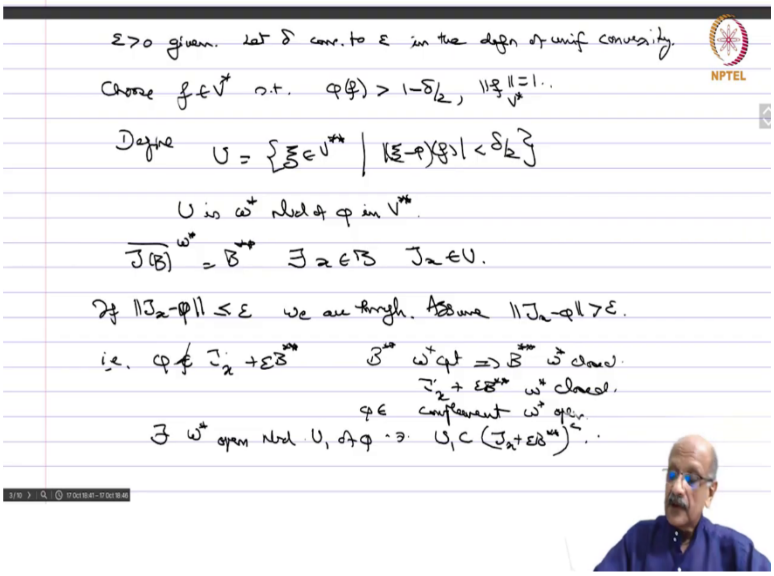

So, now we are going to assume that. So, ϵ given, so let δ correspond to ϵ in the definition of uniform convexity. So, you choose that particular δ. Now, choose $f \in V^*$ such that $\phi(f)$ so ϕ is a functional on V^*, $||\phi|| = 1$ and therefore we can choose $\phi(f) > 1 - \frac{\delta}{2}$ and $||f||_{V^*} = 1$. So, $||f||_{V^*} = 1$ because what is $||\phi||$, it is a supremum of $\phi(f)$ or $|\phi(f)|$ if you like but then $\phi(f)$, $\phi(-f)$ together give you $|\phi(f)|$. So, supremum of the $\phi(f)$ over $||f||_{V^*} = 1$ will be equal to 1. So, you can always find an f because that is bigger than $1 - \frac{\delta}{2}$.

So, now you define $U = \left\{\xi \in V^{**} : |(\xi - \phi)f| < \frac{\delta}{2}\right\}$. So, f is a single point in the pre space so V^{**} we are working on so this is U is W^* neighbourhood of ϕ in V^{**}. So, W^* open set and $J(B)$ but what do you know about $J(B)$. $\overline{J(B)}$ in W^* is equal to B^{**}. This we have already seen several and used in previous occasions also. Therefore, there exists an $x \in B$ such that $J_x \in U$. Because, every neighbourhood must intersect $J(B)$ and therefore in the W^* sense and therefore there exists an x such that this holds.

So, if $||J_x - \phi|| \le \epsilon$ we are through. We said strictly less than but less than or equal to does not matter so we are through. So, so assume this is not true so $||J_x - \phi|| > \epsilon$. That means what that is $\phi \notin J_x + \epsilon B^{**}$. B^{**} is a unit ball ϵB^{**} is the closed set of all vectors whose norm is less than or equal to ϵ. You are translating this ball by J_x, ϕ does not belong to this means what $||\phi - J_x||$ must be bigger than strictly bigger than ϵ and that is the condition which we have put in here. So, this now B^{**} is W^* compact because of the Banach Alaoglu theorem. And therefore, it is by since the W^* is Hausdorff implies B^{**} is W^* closed. So, $J_x + \epsilon B^{**}$ is just a translation of a closed set. So, this also W^* closed. So, complement is W^* open and ϕ belongs to this complement because we are assuming ϕ is not in that thing. Therefore there exists a W^* open neighbourhood U_1 of V such that U_1 also is contained in $(J_x + \epsilon B^{**})^c$.

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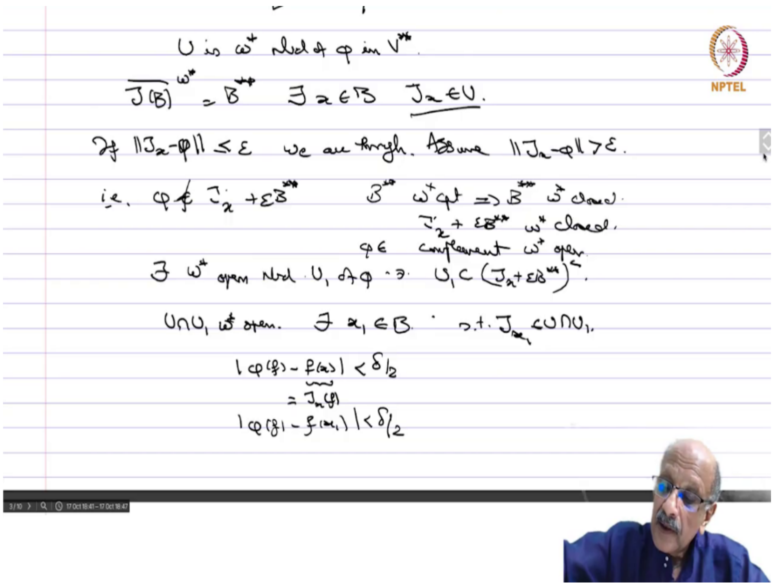

Then again $U \cap U_1$ is W^* open and so by the W^* density of $J(B)$ in B^{**} there exists $x_1 \in B$ such that $J(x_1) \in U \cap U_1$. So, what do you mean by this? So, $|\phi(f) - f(x)| < \frac{\delta}{2}$. Because, $f(x)$ is nothing but $J_x(f)$. So, $\phi(f) - J_x(f), J_x \in U$ and therefore you have $|\phi(f) - f(x)| < \frac{\delta}{2}$. J_{x_1} is also in U so $|\phi(f) - f(x_1)| < \frac{\delta}{2}$.

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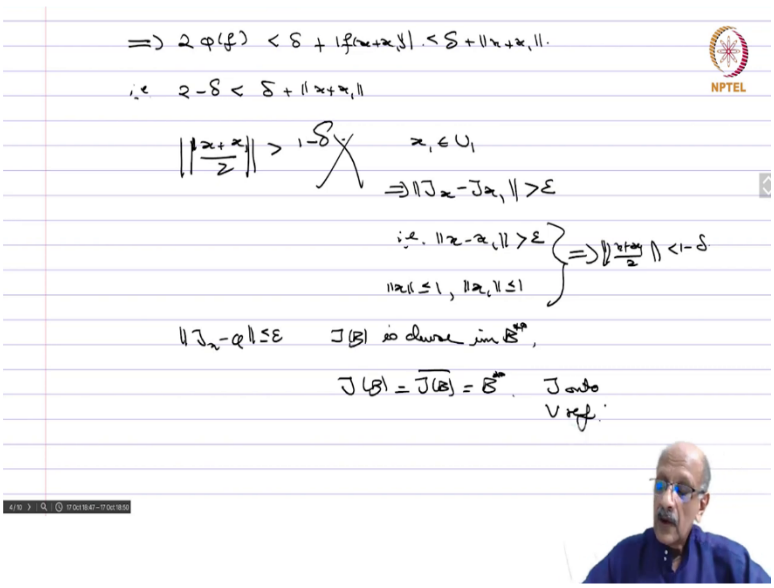

So, add these two from this you get that $2\phi(f) < \delta + \left|f(x + x_1)\right| < \delta + \|x + x_1\|$. Because, $\|f\|$. So, you have this. Now, what is $\phi(f)$ we have chosen it in such a way that is $2 - \delta < \delta + \|x + x_1\|$. Because, we have chosen $\phi(f)$ in that fashion $\phi(f) > 1 - \frac{\delta}{2}$. Therefore, $\|\frac{x+x_1}{2}\| > 1 - \delta$. But, $x_1 \in U_1$ that we should we have not used that fact and that implies $\|J_x - J_{x_1}\| > \epsilon$ that is $\|x - x_1\| > \epsilon$. And we also know that $\|x\| \leq 1$, $\|x_1\| \leq 1$ because both of them are elements of B. So, these two and uniform convexity implies that $\|\frac{x+x_1}{2}\| < 1 - \delta$ and that is a contradiction. So, this is a contradiction.

And therefore we have we have that norm of $\|J_x - \phi\|$ has to be less than equal to ϵ and therefore $J(B)$ is dense in B^{**}. That is $J(B)$ which is closed so $J(B) = \overline{J(B)}$ which is equal to B^{**}. So, J is onto. So, B is reflexive. So, this proves the theorem completely.

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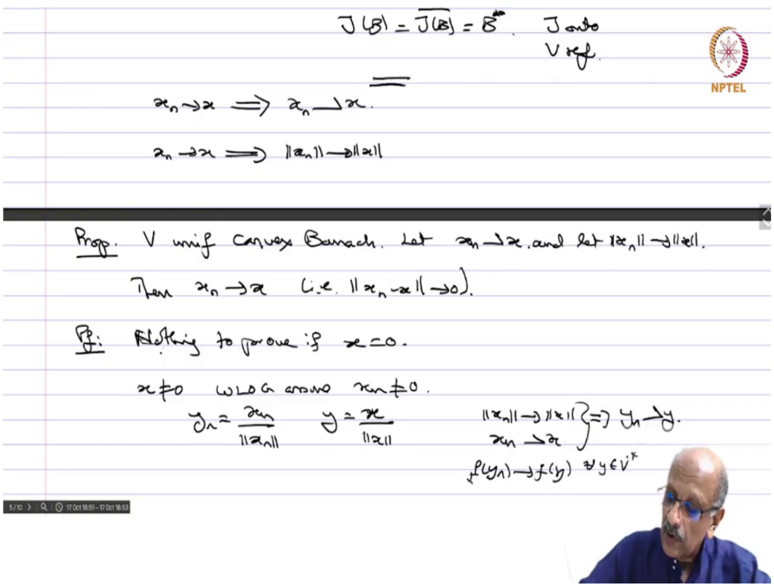

So, now let us that suppose $\{x_n\}$ converges to x in norm then this implies that $\{x_n\}$ converges to x weakly this we have seen. And we have also seen that the converse is not true. We have examples like $\{e_n\} \in l_2$ that goes to 0 weakly but, it cannot converge strongly. So, $\{x_n\}$ converges to x in norm implies by continuity of the norm $||x_n|| \to ||x||$.

And obviously this converse is also not true you can have any set of vectors $||x_n|| \to ||x||$ but that does not mean that $\{x_n\}$ should go to x. But, if you combine these two relationships on the right hand side, if these two things happen, namely if you have weak convergence and the convergence of the norm then it means convergence in the norm. So, provided you are in a uniformly convex space. So, proposition

Proposition: So V uniformly convex Banach space. And let $x_n \rightharpoonup x$ and let $||x_n|| \to ||x||$. Then $x_n \to x$ that is $||x_n - x|| \to 0$. Usually, this will be difficult to prove proving weak convergence is will be relatively easier because you have to show that $f(x_n) \to f(x)$ for every linear function.

And also showing that the norm converges will also be relatively easier and together these two in a uniformly convex Banach space tells you that you have norm convergence so this is a very useful result.

Proof: So nothing to prove if $x = 0$ because $||x_n|| \to 0$ that is exactly saying now $x_n \to 0$ in norm. So, nothing to prove if $x = 0$. So, if $x \neq 0$ so without loss of generality assume $x_n \neq 0$ it is true for n sufficiently large and therefore, we can assume this for all of them. So, now you define $y_n = \frac{x_n}{||x_n||}$ and $y = \frac{x}{||x||}$. Now, $||x_n|| \to ||x||$ and $x_n \rightharpoonup x$. Together these two imply that $y_n \rightharpoonup y$. This you can check so all you have to do is take any linear functional and you have to show that $f(y_n) \to f(y)$ for every $y \in V^{*}$. This is what we have to show and that is easy to show because you are given these two things you can check it yourself.

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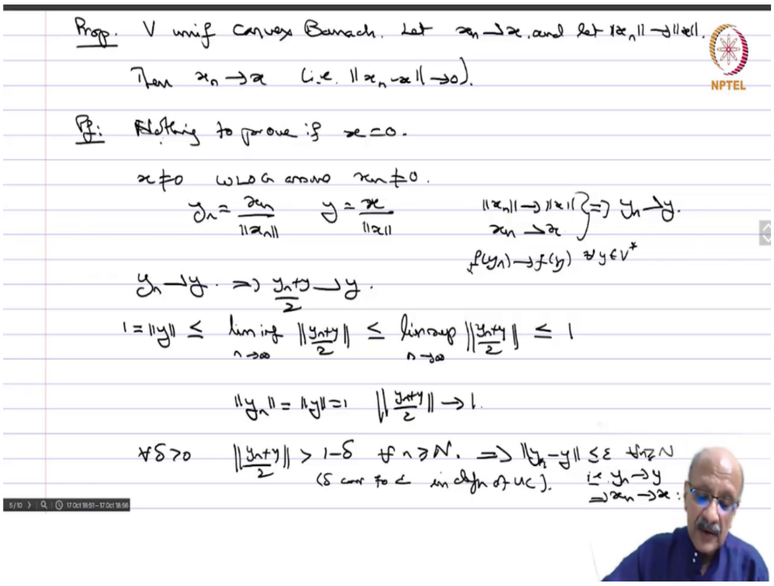

So, we have that $y_n \rightharpoonup y$ and now we have to show that $y_n \to y$. So, $y_n \rightharpoonup y$ therefore $1 \leq \|y\|$ is less than or equal to lim inf. So, $y_n \rightharpoonup y$ implies that $\frac{y_n + y}{2} \rightharpoonup y$. Because, $y_n \rightharpoonup y$, $\frac{y_n + y}{2}$ also goes to y weakly. Therefore, $\|y\| \leq \|\frac{y_n + y}{2}\|$. Because, we know if something goes weakly then the limit and for norm of the limit is less than the lim inf. Now, $\|\frac{y_n + y}{2}\| \leq \|\frac{y_n + y}{2}\|$. But, $\|y_n\|$ and $\|y\|$ are all 1. And therefore, by the triangle inequality $\|\frac{y_n + y}{2}\|$ is always less than equal to 1 and therefore $\|\frac{y_n + y}{2}\| \leq 1$. Therefore, we have that all equality is there throughout and therefore you have that $\|y_n\| = \|y\| = 1$ and $\|\frac{y_n + y}{2}\| \to 1$. That means for any $\delta > 0$ we have $\|\frac{y_n + y}{2}\| > 1 - \delta$ for all $n \geq N$. But, this means that $\|y_n - y\| \leq \epsilon$ for all $n \geq N$. Because, if it was strictly bigger than ϵ that will contradict the uniform convexity. You have $\|y_n\| = \|y\| = 1$, $\|y_n - y\| > \epsilon$ then you will have norm $\|\frac{y_n + y}{2}\|$ has to be less than $1 - \delta$ and that is not true. So, δ corresponds to ϵ in the definition of uniform convex. Now, you take δ corresponding to ϵ in uniform convexity. Then $\|\frac{y_n + y}{2}\| > 1 - \delta$ for all $n \geq N$ implies that $\|y_n - y\| \leq \epsilon$ that is $y_n \to y$ and this implies that $x_n \to x$. And that completes your proof of this theorem. So, we will next look at applications of these results to the calculus of variations.