Functional Analysis
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Lecture No. 33
Separable Spaces – Part 2

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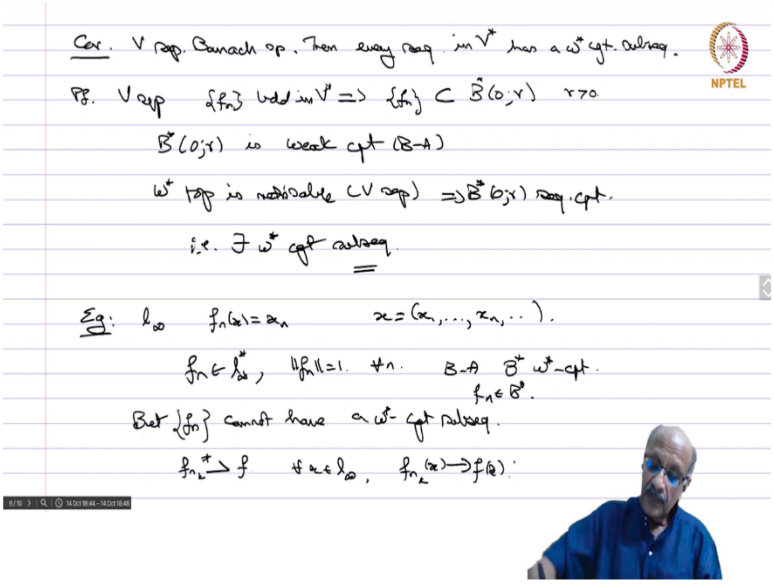

Corollary: V separable Banach space then every sequence in $V^{\star}$ has a $W^{\star}$ convergent subsequence.

Proof: So, this is almost immediate because if V separable then and so $\{f_n\}$ bounded in $V^{\star}$ implies this $\{f_n\}$ is contained in some ball $B^{\star}(0;r), r > 0$. Some, some. But, then so $B^{\star}(0;r)$ which means the ball center origin radius r in $V^{\star}$. So, $B^{\star}(0;r)$ is $W^{\star}$ compact (this is Banach Alaoglu) and weak star topology is metrizable. Since, V separable and in a metric space compactness is same as sequential compactness therefore $B^{\star}(0;r)$ is sequentially compact. That is there exists a weak star convergent subsequence. So, this is just.

So, in a compact space in compact metric space we know that compactness and sequential compactness are equivalent. But, in general topological spaces again one of the arguments where you need nets filters etc. is that a sequence may in a compact topological space may not necessarily have a convergent subsequence.

So, examples are difficult to usually write and here is a nice example which we have.

Example: So, let us take the space l_∞ which is not separable which you know. So, now you define $f_n(x) = x_n$ so this is n'th coordinate projection so $x = (x_n, \cdot, \cdot, \cdot, x_n, \cdot, \cdot, \cdot)$. So, $f_n \in l_\infty^\star$ and $\|f_n\| = 1$ for all n. So, this is a bounded sequence in $l_\infty^\star$ and by the Banach-Alaoglu $B^\star$ is $W^\star$ compact and $f_n \in B^\star$. So, but $\{f_n\}$ cannot have a weak star convergent subsequence. Why? Suppose, it had let us say f_{n_k} converges $W^\star$ to some f then for every $x \in l_\infty$ we have $f_{n_k}(x) \rightarrow f(x)$. That means.

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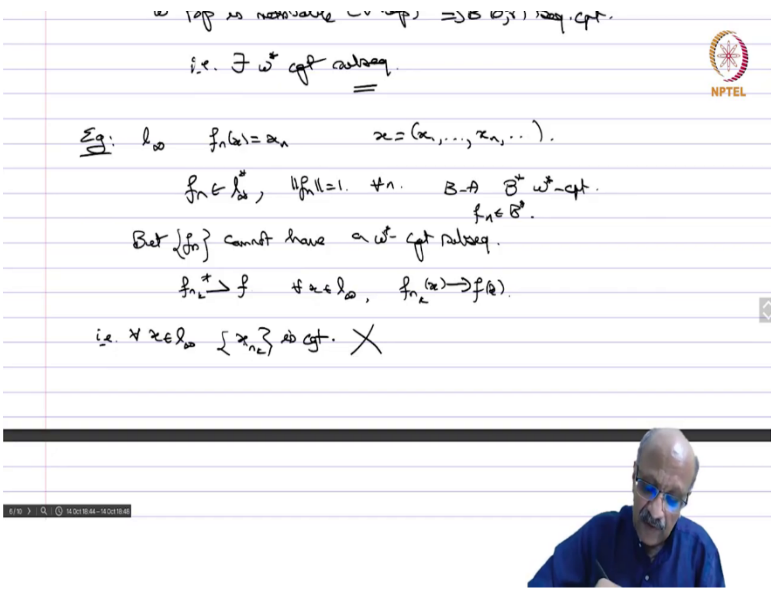

That is for every $x \in l_\infty$, $\{x_{n_k}\}$ is convergent. And that is observed. Because, you have l_∞ is just bounded sequences and you cannot say there is a fixed subsequence which will always converge. So, that you cannot say. This n_k does not depend on x. So, if you have bounded sequence you have a convergent subsequence but that would depend on the sequence namely

x itself. But, here I am saying n_k is independent of x and therefore that is not true and that is observed and therefore it does not have a convergent subsequence.

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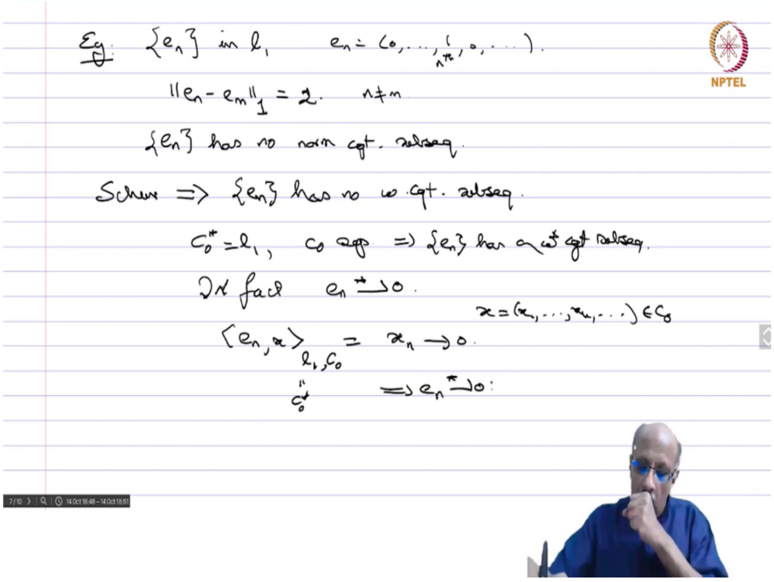

So, another, another example so why did this fail because, l_∞ is not separable. Therefore, you cannot apply the previous theorem.

Example: So, now let us take the sequence $\{e_n\}$ in l_1. If you take the sequence $\{e_n\}$ in l_1, $e_n = (0, \cdot, \cdot, \cdot, 1, \cdot, \cdot, \cdot)$. So, $\{e_n\} \in l_1$. And then so $\|e_n - e_m\| = 2$. Because, if you take $e_n - e_m$ you have in one place you have 1 you have another place you have -1 and everywhere else you have 0. If $n \neq m$. So, you will have 2 and therefore $\{e_n\}$ has no norm convergent subsequence. But Schur Lemma says weakly convergent and non-convergent subsequences are the same. So, by sure this implies $\{e_n\}$ has no weakly convergent subsequence either. But, $C_0^\star = l_1$, C_0 is separable. Implies $\{e_n\}$ has a $W^\star$ convergent subsequence.

But in fact, $\{e_n\}$ itself $W^\star$ converges to 0. Why? Because, if you take $\langle e_n, x\rangle_{l_1, C_0}$ so l_1 is $C_0^\star$. What is $\langle e_n, x\rangle_{l_1, C_0}$? $\langle e_n, x\rangle_{l_1, C_0}$ is nothing but x_n, $x = \left(x_n, \cdot, \cdot, \cdot, x_n, \cdot, \cdot, \cdot\right) \in C_0$. And then you know $x_n \to 0$. So this implies that $\{e_n\}$ weak star converges to 0. So, we conclude with another important theorem.

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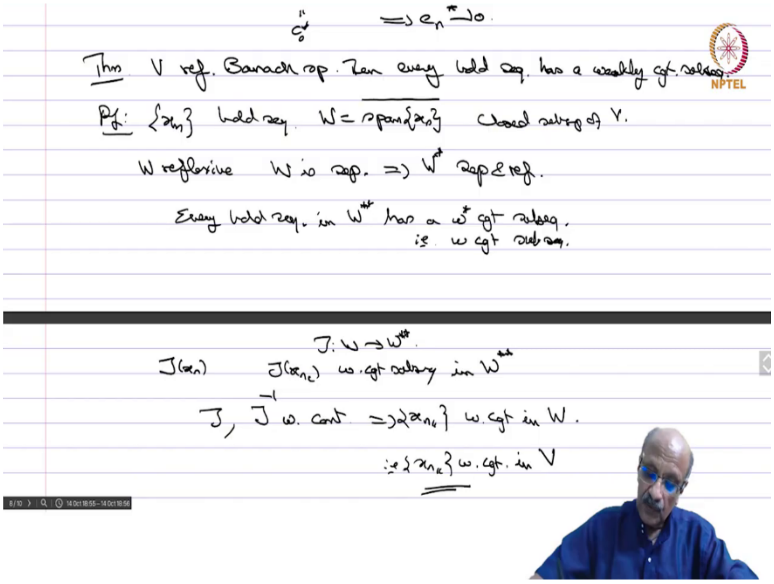

And this is a cornerstone of many existence proofs in the calculus of variations and so on.

Theorem: So, V reflexive Banach space. Then every bounded sequence has a weakly convergent subsequence.

So, as I told you, weak topology is not metrizable topology and so on. Therefore, you cannot take, you know, bounded sets are weakly compact in reflexive spaces. But, you cannot take for granted that therefore sequence will have a weakly convergent subsequence just as we saw an example of arbitrary if you do not have metric spaces you cannot take this. But, nevertheless in a reflexive Banach space every bounded sequence has a weakly convergence subsequence. Namely, the weakly compact sets have this property in a reflexive space.

Proof: So, so let $\{x_n\}$ be a bounded sequence and you let $W = \overline{span\{x_n\}}$ that means finite linear combinations of $\{x_n\}$ and then take the closure. So, this is a closed subspace of V. Then W is reflexive because closed subspace of reflexive space is reflexive. And W is separable is it separable. Because, you have this $W = \overline{span\{x_n\}}$. So, every element in W can be approximated as closely as you like by a finite linear combination of the $\{x_n\}$. Which in turn can be approximated as closely as you like by finite rational linear combinations of the $\{x_n\}$. And the finite rational linear combinations of $\{x_n\}$ is a countable set. So, every element in W can be approximated as close as you like by a member of a countable set or this that means W has a countable dense set. Therefore, W is separable and reflexive. So, this implies that $W^{\star}$ is also separable and reflexive. So, that means every bounded sequence in $W^{\star\star}$ which is, because $W^{\star}$ is separable every bounded sequence $W^{\star\star}$ has a $W^{\star}$ convergent subsequence. But, we are in reflexive spaces so that is weakly convergent. Because, $W^{\star}$ is reflexive therefore the weak and weak star topologies on $W^{\star\star}$ are the same.

So, in particular if you take $J\left(x_n\right)$, $J: W \to W^{\star\star}$ the canonical embedding. Then $\{J\left(x_n\right)\}$ is a bounded sequence. And therefore, it will have a weakly convergent subsequence. So, $\{J\left(x_{n_k}\right)\}$ weakly convergent subsequence in $W^{\star\star}$. But, J is W continuous. J, $J^{-1}$are all W continuous. And this implies that $\{x_{n_k}\}$ weakly convergent in W. But, then the topology that is $\{x_{n_k}\}$ weakly convergent in V. Because, this is nothing but the weak topology in W is nothing but the inheritance from the weak topology in V. Therefore, it will automatically converge. Therefore, you have that this is true.

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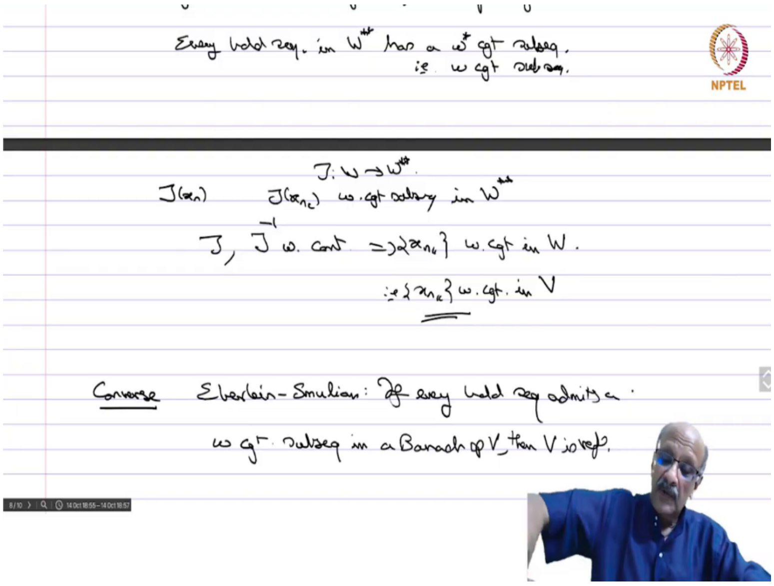

Now, the converse of this theorem is a very deep theorem. This is due to Eberlain-Smulian.

Converse: If every bounded sequence admits a weakly convergent subsequence in a Banach space V then V is reflexive.

This Eberlain-Smulian theorem is a converse, it is a very deep theorem that is fairly difficult to prove. So, we will stop here.