Functional Analysis
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Lecture No. 32
Separable Spaces - Part 1

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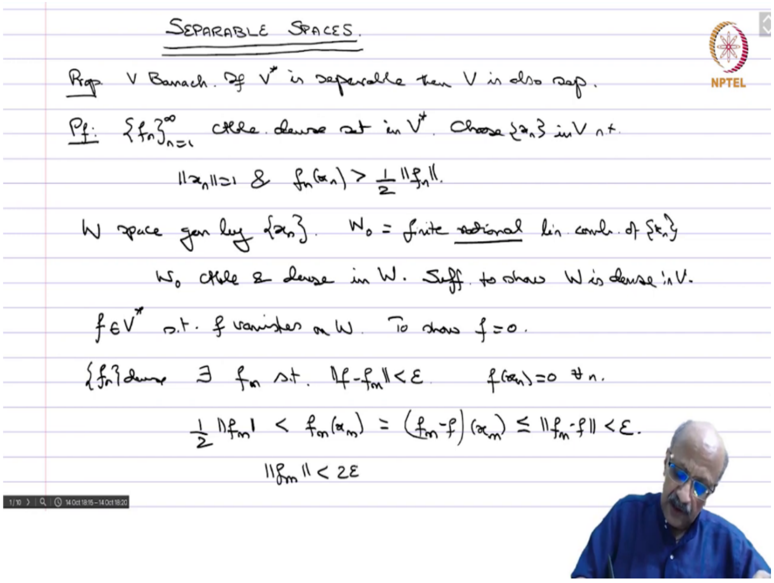

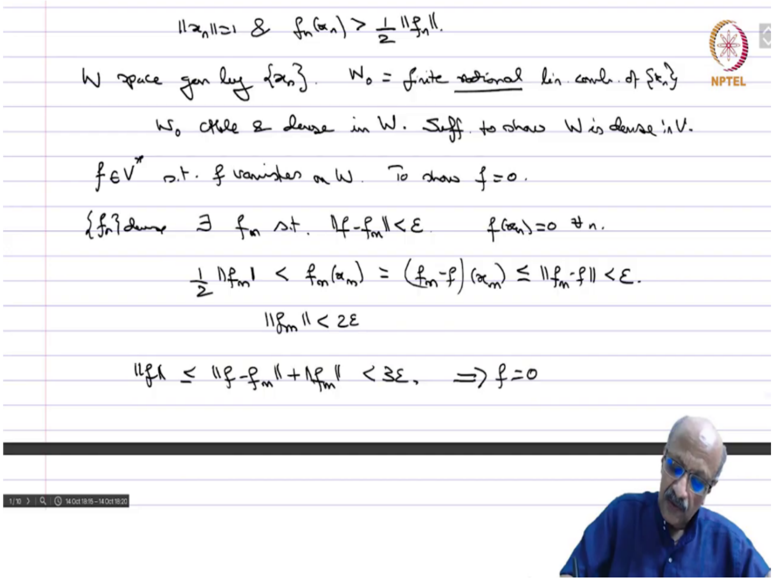

We will now talk about Separable Spaces. A topological space is separable if it has a countable dense set. So, we have the following proposition.

Proposition: So, V Banach. If $V^{\star}$ is separable then V is also separable.

Proof: So let $\{f_n\}_{n=1}^{\infty}$ be a countable dense set in $V^{\star}$. So, you choose $x_n \in V$ such that you have norm $\|x_n\| = 1$ and $f_n(x_n) \geq \frac{1}{2}\|f_n\|$. This is possible because $\|f_n\|$ is nothing but the $sup\{f_n(x): \|x\| = 1\}$. So, definitely you can find an x_n with $\|x_n\| = 1$ such that it is bigger than this.

So, assume we will assume for simplicity that the base field is R, otherwise you argue only with the real and, imaginary parts, whatever we do now. So now let us assume, so W be the space generated by the $\{x_n\}$. So, these are the set of all finite linear combinations of $\{x_n\}$. So, then W_0 is finite rational linear combinations of the $\{x_n\}$'s.

So, phi, W is the set of all it is a space generated by $\{x_n\}$ that means these are all finite linear combinations. Now, we are taking finite linear but with rational coefficients. So, then W_0 is dense because you are taking finite rationals accountable and you are taking finite linear combinations of $\{x_n\}$ which is also a countable set.

And therefore, this is W_0 is countable and dense in W. So, sufficient to show W is dense in V. Because, every element in V can be approximated by W which is element in W which in turn can be approximated by something in W_0. So, W_0 will be dense in V and W_0 is countable.

So, this will prove that you have a countable dense set and therefore it is separable. So, let us assume, so the Hahn Banach method. So, let us assume $f \in V^{\star}$ such that f f vanishes on W. So to show, $f = 0$. So, $\{f_n\}$'s are dense therefore there exists an f_m such that $\|f - f_m\| < \epsilon$. And now, $f(x_n) = 0$ for all n. Because, f vanishes on W, W is generated by the $\{x_n\}$'s and therefore in particular $f(x_n) = 0$ for all n. And therefore, $\frac{1}{2}\|f_m\| < f_m(x_m) = (f_m - f)(x_m)$. And that is less than equal to $\|f_m - f\|\|x_m\|$, $\|x_m\| < 1$ and $\|f_m - f\| < \epsilon$. Therefore, $\|f_m\| < 2\epsilon$. And therefore,

$\|f\| \leq \|f - f_m\| + \|f_m\| < 3\epsilon$. ϵ is arbitrary, so this implies that f has to be 0. So, this proves that if V^* is separable then V is separable.

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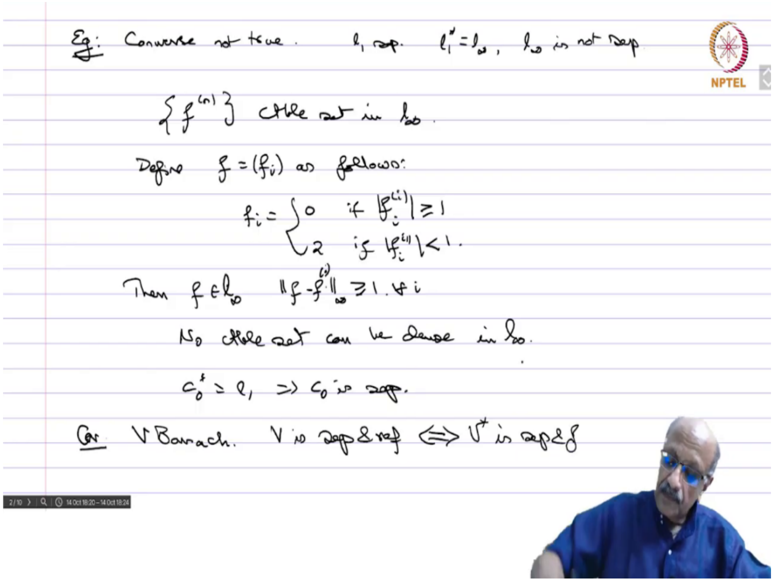

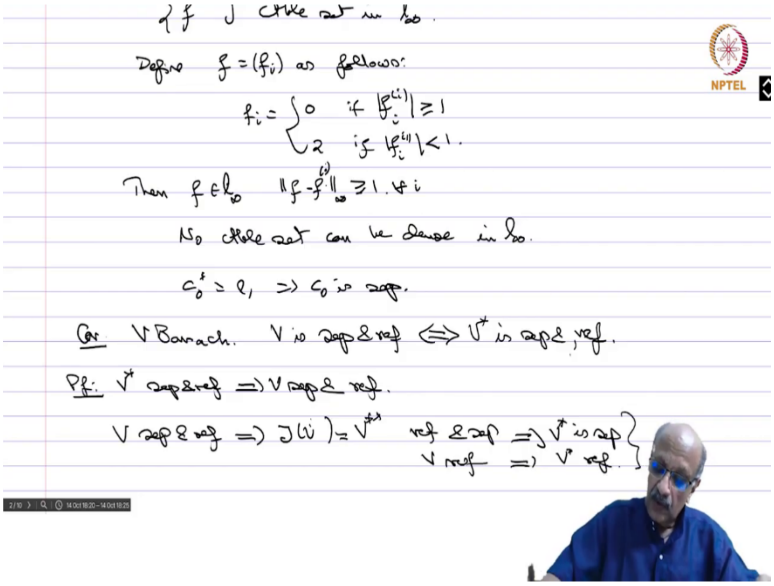

So, now let us, so example, converse not true.

Example: You take l_1, l_1 is separable because you take all finite sequences that are 0 after a finite stage that is dense in l_1 because of the norm. And we have seen this already. And then if you take finite rational sequences, they are the countable set which will be dense in the whole space.

So, l_1 is separable, in fact all l_p other than l_∞ is separable. So l_1 separable, $l_1^\star = l_\infty$, but l_∞ is not separable. In fact, when trying to prove $l_\infty^\star$ is not l_1 we even showed, the came across this fact that finite sequences are not dense in l_∞. But l_∞ is not separable. Let us take any $\{f^{(n)}\}$, countable set in l_∞. Now, define $f = \{f^{(i)}\}$ in the following fashion. So, $f^{(i)} = 0$ if $\left|f^{(i)}\right| \geq 1$ and $f^{(i)} = 2$ if $\left|f^{(i)}\right| < 1$. Then obviously $f \in l_\infty$ and what is norm $\|f - f^{(i)}\|_\infty$. Well, at the i 'th coordinate the distance is certainly bigger than 1 and therefore this has to be bigger than equal to 1 for all i. So, this shows that no countable set can be dense in l_∞. So, l_∞ is not dense. But $C_0^\star$ is l_1, so $C_0^\star = l_1$, so this implies that C_0 is separable. But this you can check directly you can produce a countable dense set, in fact again finite sequences with rational coordinates this will be automatically a countable dense set.

Corollary: V Banach. So, V is separable and reflexive if and only if $V^\star$ is separable and reflexive.

Proof: So $V^\star$ separable and reflexive implies that V is separable and reflexive because if $V^\star$ is separable, V is separable, if $V^\star$ is reflexive, V is reflexive, we know all this. So, now we assume that V is separable and reflexive. So, the point is l_1 is not reflexive that is why this we have a counter example here, this on the proof if you like that l_1 is not reflexive if V separable and reflexive, then this implies $J(V) = V^{\star\star}$ which is isometrically $J(V)$ which is $V^{\star\star}$ is automatically reflexive and separable. And therefore, this implies that $V^\star$ is separable. And of course, V reflexive implies $V^\star$ is reflexive. And therefore, V reflexive $\Rightarrow$ $V^\star$ reflexive. Therefore, you have all these things, so we have that it is.

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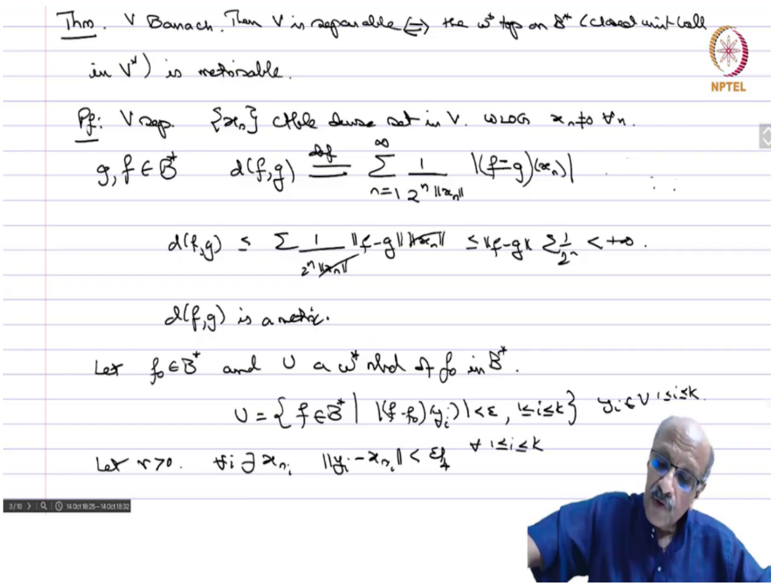

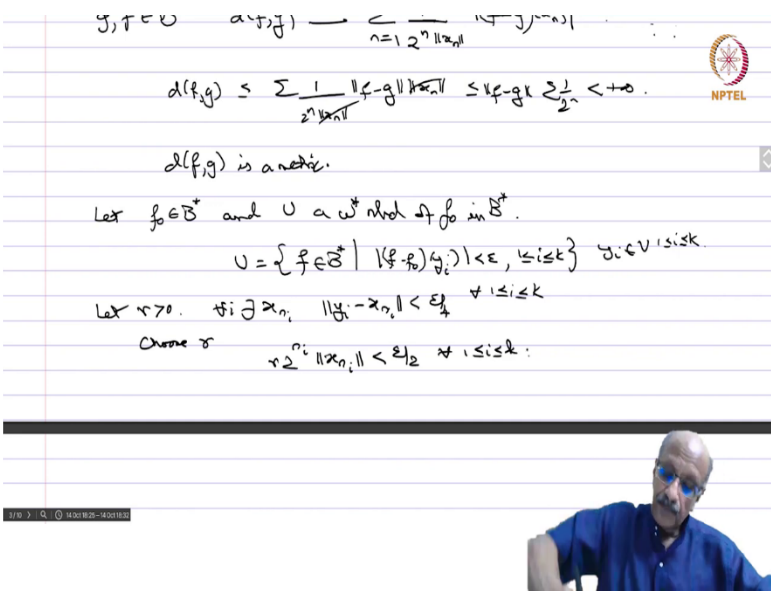

So now we have a very important theorem.

Theorem: V Banach, then V is separable if and only if the weak star topology on $B^{\star}$ (this is the closed unit ball in $V^{\star}$) is metrizable.

That means, you take if V separable if and only if what is the condition you take $B^{\star}$ the closed unit ball in $V^{\star}$ and you put the weak star topology restricted to $B^{\star}$ then it is equivalent to a metric topology. There exists a metric whose topology is the same as the weak star topology restricted to $B^\star$, so it is essentially a metric space.

Proof: So we are going to first assume that V is separable. Then you can have a countable dense set, so $\{x_n\}$ countable dense set. So, without loss of generality we can assume $x_n \neq 0$ for all n. Why is this so? So, if you take, if suppose 0 is in that countable dense set take any neighbourhood of 0. Then, if you take any ball which does not contain the origin because of density there will be a x_n here. So, every neighbourhood of 0 will also contain a neighbour, a point from the, from the will contain the non-zero element of the x_n's. And therefore, the non-zero, and therefore given any open set you even if you have 0 if it intersects 0 then it will also have to intersect a non-zero element. So, it is enough to assume that you have that all the elements are non-zero. So, now you define, so $g, f \in B^\star$ you define $d(f, g)$:

$$d(f,g) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n \|x_n\|} \left|(f - g)x_n\right|$$

that is why I wanted all the x_n's to be non-zero. So, why is this well defined, because if you take $d(f,g) \leq \sum_{n=1}^{\infty} \frac{1}{2^n \|x_n\|} \|f - g\| \|x_n\|$ and therefore that cancels and $\sum_{n=1}^{\infty} \frac{1}{2^n \|x_n\|} \|f - g\| \|x_n\| \leq \|f - g\| \sum_{n=1}^{\infty} \frac{1}{2^n}$, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is 1 and therefore is a convergent geometric series. So, $\|f - g\| \sum_{n=1}^{\infty} \frac{1}{2^n}$ is finite, so $d(f, g)$ is well defined for all f and g. And it is easy to check that it is non negative and that is clear. If it is 0 then f and g agree on $\{x_n\}$ which is a dense set, therefore they have to agree everywhere. So, $f = g$ and conversely also triangle inequality is trivial, it just comes on the triangle inequality of the modulus. And therefore, we have that this is symmetric. So, $d(f, g)$ is a metric. So, now we want to show that this metric topology is equivalent to the weak star topology in the thing. So, let us take a weak star neighbourhood. So, so let $f_0 \in B^\star$ and U a $W^\star$ neighbourhood of f_0 in $B^\star$. What does it mean, so U will be of the form $U = \{f \in B^\star : \left|\left(f - f_0\right)y_i\right| < \epsilon,\ 1 \leq i \leq k\}$ and then you have $y_i \in V$ for $1 \leq i \leq k$. So, this is how a W^* neighbourhood will look like. So, so let $r > 0$, then by the density for every i there exists x_{n_i} such that $\|y - x_{n_i}\| < \frac{\epsilon}{4}$. So, ϵ is given here. So, then choose r, so this is for each $1 \leq i \leq k$. Now, choose r such that $r2^{n_i}\|x_{n_i}\| < \frac{\epsilon}{2}$ for all $1 \leq i \leq k$.

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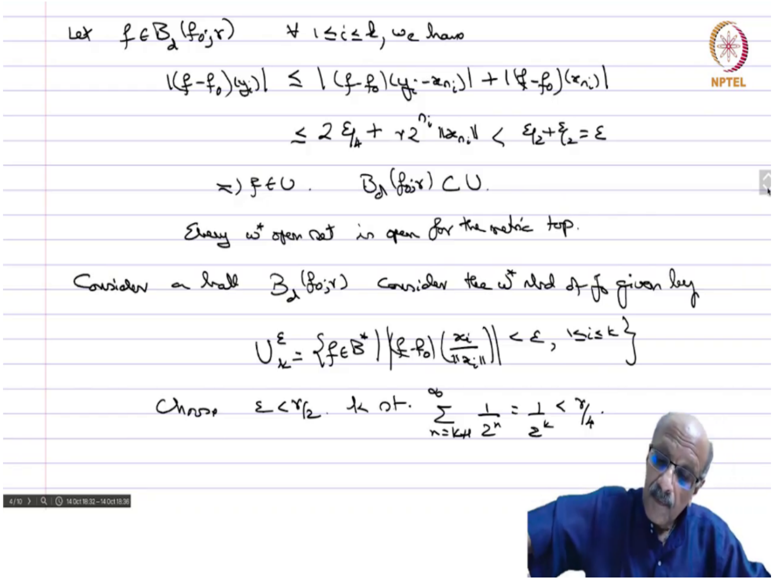

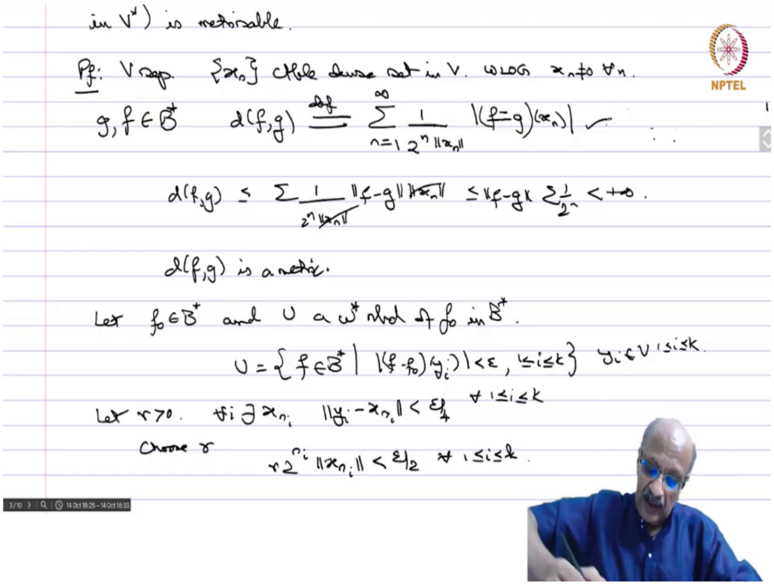

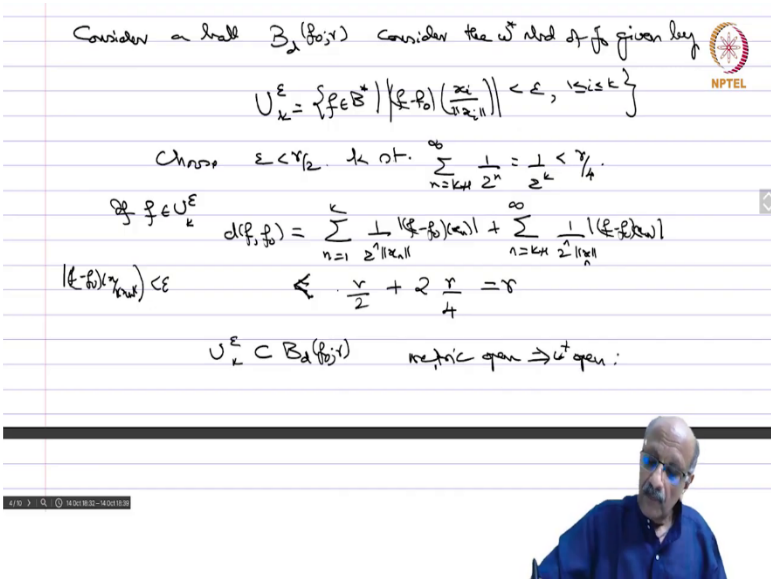

Now, let $f \in B_d(f_0; r)$ that is the ball with respect to the metric centre f_0 and radius r. Then for all $1 \leq i \leq k$, we have $\left|(f - f_0)y_i\right| \leq \left|(f - f_0)\left(y_i - x_{n_i}\right)\right| + \left|(f - f_0)\left(x_{n_i}\right)\right|$. Now, the first one is less than or equal to $f - f_0$ in norm each of them is in $B^\star$ so the norm is less than 1, so $\|f - f_0\| \leq 2$, $\|y - x_{n_i}\|$ we have chosen to be less than $\frac{\epsilon}{4}$, plus $\left|(f - f_0)\left(x_{n_i}\right)\right|$. Now, what is that, that is connected to the definition. So, if you have, f of is one term in the infinite series $\sum_{n=1}^{\infty} \frac{1}{2^n \|x_n\|}\left|(f - g)x_n\right|$, so $\left|(f - f_0)\left(x_{n_i}\right)\right|$ will be less than equal to $d(f, g)$ which is less than $r2^{n_i}\|x_{n_i}\|$. So, from the definition of the thing $\left|(f - f_0)\left(x_{n_i}\right)\right| \leq r2^{n_i}\|x_{n_i}\|$. And that we have chosen to be less than $\frac{\epsilon}{2}$. So, $\left|(f - f_0)\left(y_i - x_{n_i}\right)\right| + \left|(f - f_0)\left(x_{n_i}\right)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. And this implies that $f \in U$. Therefore, $B_d\left(f_0; r\right) \subset U$. So, what have we proved that every weak star open set contains a ball in the metric space. Therefore, every weak star open set is open for the metric topology.

So, now for the converse, so now we have to take any ball and show that it is also weak star. So, let us consider a ball $B_d(f_0; r)$. And consider the $W^\star$ neighbourhood of f_0 given by

$U_k^\epsilon = \left\{ f \in B^\star : \left|\left(f - f_0\right)\right|\left(\frac{x_i}{\|x_i\|}\right) < \epsilon, 1 \leq i \leq k \right\}$. So, this x_i's are from the countable dense set which we already have. Now, I am going to choose a $\epsilon < \frac{r}{2}$ and k such that $\sum_{n=k+1}^{\infty} \frac{1}{2^n}$ which is $\frac{1}{2^k}$ and that I want to be less than $\frac{r}{4}$. So, if $f \in U_k^\epsilon$ where ϵ and k have been chosen in this fashion what is $d(f, f_0)$, $d\left(f, f_0\right) = \sum_{n=1}^{k} \frac{1}{2^n \|x_n\|}\left|(f - f_0)x_n\right| + \sum_{n=k+1}^{\infty} \frac{1}{2^n \|x_n\|}\left|(f - f_0)x_n\right|$. So, the first one you have norm, this we already saw $\frac{\left|(f - f_0)(x_n)\right|}{\|x_n\|} < \epsilon$. So, $\sum_{n=1}^{k} \frac{1}{2^n \|x_n\|}\left|(f - f_0)x_n\right| \leq \frac{r}{2} \sum_{n=1}^{k} \frac{1}{2^n}$ which I will take instead of k I will take all the way to infinity whatever may be k and therefore this is less than $\frac{r}{2}$. So, let me explain that so $\frac{\left|(f - f_0)(x_n)\right|}{\|x_n\|} < \epsilon$ and therefore that will come out, so you will just get $\sum_{n=1}^{k} \frac{1}{2^n}$, I can take less than 1 to ∞. And consequently, I will get $\sum_{n=1}^{k} \frac{1}{2^n \|x_n\|}\left|(f - f_0)x_n\right|$ is less than just ϵ and ϵ is less than $\frac{r}{2}$, plus the remaining thing $\frac{\left|(f - f_0)(x_n)\right|}{\|x_n\|} \leq \|f - f_0\| \leq 2$ plus $\sum_{n=k+1}^{\infty} \frac{1}{2^n}$ that is $\frac{r}{4}$. So, that is $2\frac{r}{4}$. So $\frac{r}{2} + \frac{r}{2}$ which is r, so $d\left(f, f_0\right) < r$. And therefore, you have $U_k^\epsilon \subset B_d(f_0; r)$. So, every metric open implies $W^\star$ open. So, this proves that the metric and weak star topologies are the same if the space is separable.

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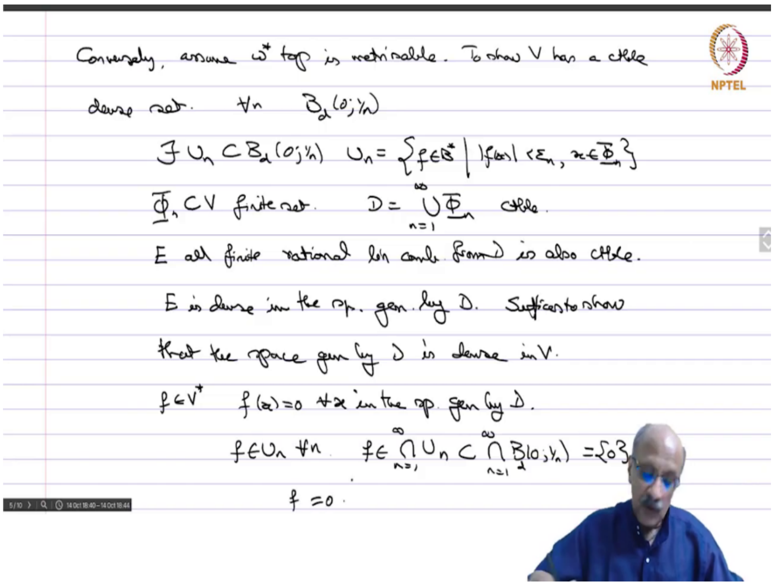

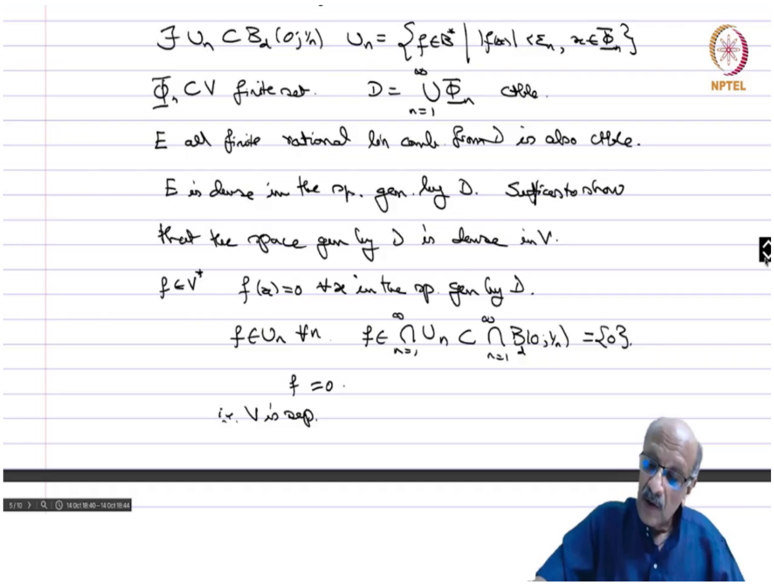

So, now conversely assume weak star topology is metrizable. Then to show V has a countable dense set. So, for each n consider the ball $B_d(0; \frac{1}{n})$. So, this is a ball in the metric space and that is the same as the weak star open thing. Then this ball contains a $W^\star$ open neighbourhood. So, there exists $U_n \subset B_d(0; \frac{1}{n})$ and U_n is a weak star neighbourhood of the origin, so this is set of all $f \in B^\star$ such that $|f(x)| < \epsilon_n$ and for all x in some Φ_n. So, $\Phi_n \subset V$ is a finite set. So, now you take $D = \cup_{n=1}^{\infty} \Phi_n$. So, D is countable. So, E all finite rational linear combinations from D is also countable. And E is dense in the space generated by D that is all linear combinations of elements from D that is a subspace E is dense in it. So, suffices to show, D is, so the space that the space generated by D is dense in V. So, you take $f\in V^{\star}, f(x) = 0$ for all x in the space generated by D. Then if $f(x) = 0$ for all x in particular it is 0 for all $x\in \Phi_n$ and therefore you have $f\in U_n$ for all n. Therefore, $f\in\bigcap_{n=1}^{\infty}\Phi_n$ which is contained in the $\bigcap_{n=1}^{\infty}B_d(0;\frac{1}{n})$ and that is nothing but $\{0\}$. And therefore, $f = 0$. So, this proves by therefore, the space that is, so that is V is separable because E is a countable set and that is dense in D in the space generated by D which in turn is dense in the whole space.