Functional Analysis
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Lecture No. 31
Reflexive Spaces

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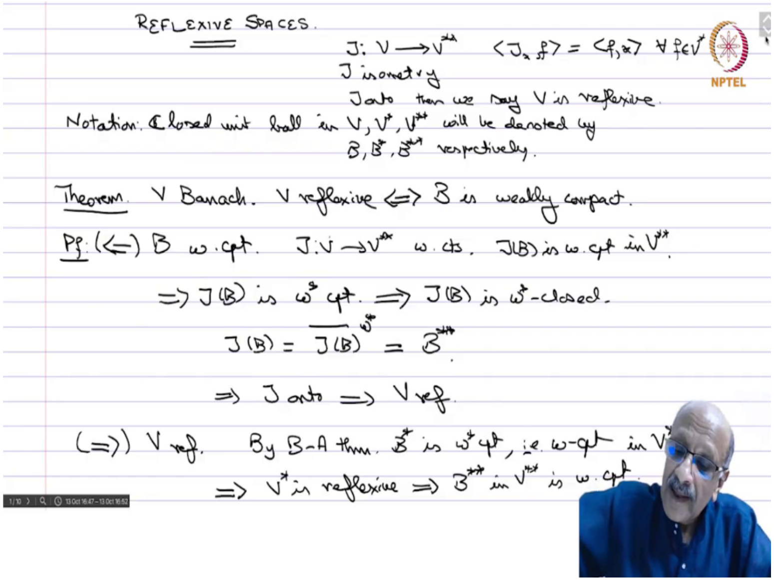

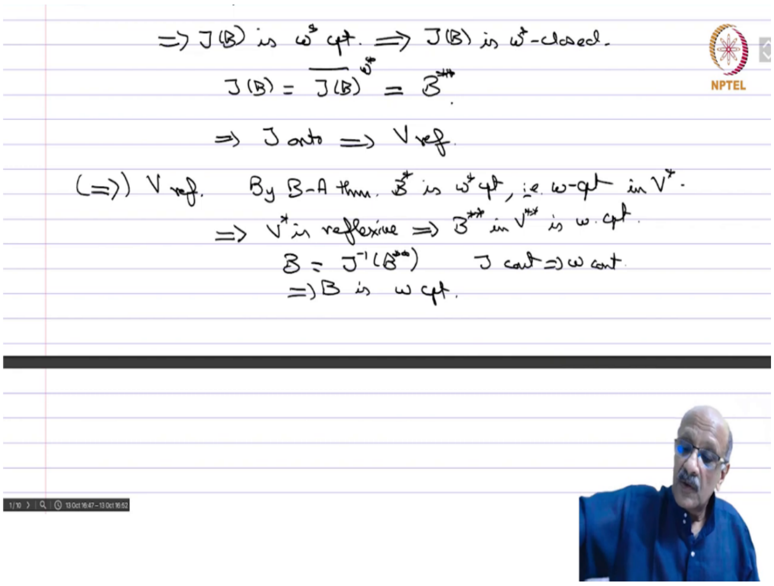

We will now look at applications of weak and weak star topologies in functional analysis. So, we will start with Reflexive Spaces. So recall, V is norm linear space then you have a canonical embedding $J: V \to V^{**}$. So, $\langle J_x, f\rangle = \langle f, x\rangle$ for all $f \in V^{*}$. This is the canonical embedding, this an isometry, so J is isometry. And if J onto, then we say V is reflexive.

So, now we are going to characterise reflexive spaces using the weak topology. So notation,

Notation: So unit, closed unit ball in V, $V^{\star}$, $V^{\star\star}$ will be denoted by B, $B^{\star}$, $B^{\star\star}$ respectively. So, the first theorem is a very nice and important theorem.

Theorem: So, V Banach. V reflexive if and only if B is weakly compact.

That means, compact in the weak topology. So, you know that B can never be non-compact and if you are in $V^{\star}$ then you know $B^{\star}$ is $W^{\star}$ compact but for reflexive spaces the unit ball, closed unit ball is W compact and that characterises reflexive spaces.

Proof: So let us proof, first assume that B is W compact. So, B is W compact, then $J: V \rightarrow V^{\star\star}$ is an isometry. Therefore it is continuous, therefore we have seen that it is also W continuous. And therefore, $J(B)$ is W compact in $V^{\star\star}$, the continuous image of a compact set is compact, that is all. So then, if it is compact in W, then it will be compact in any smaller topology. So, this implies $J(B)$ is $W^{\star}$ compact, $W^{\star}$ is Hausdorff compact in a Hausdorff space are close, so $J(B)$ is $W^{\star}$ closed. Therefore, $J(B) = \overline{J(B)}$ in $W^{\star}$, but we saw this result in the previous session that $\overline{J(B)}$ is nothing but $B^{\star\star}$. So, $J(B) = B^{\star\star}$, so J is onto the unit ball to the unit ball and therefore, it is onto from, so this implies J is onto. Because if it is onto on the unit ball it will be onto on the entire space, so this implies V reflexive.

So, now let us do the opposite thing. So, V is reflexive, that means the W and $W^{\star}$ on $V^{\star}$ coincide. Therefore, by Banach-Alaoglu Theorem $B^{\star}$ is $W^{\star}$ compact, that is W compact in $V^{\star}$, in $V^{\star}$. So, if $B^{\star}$ is W compact the previous, by previous arguments this shows that $V^{\star}$ is reflexive by the first part of this theorem. If $V^{\star}$ this reflexive, then this implies that $B^{\star\star}$ is W compact because again in $V^{\star\star}$, the W and $W^{\star}$ coincide, therefore $B^{\star\star}$ by the Banach-Alaoglu is $W^{\star}$ compact and therefore, it is W compact also. And then, we have that $B = J^{-1}(B^{\star\star})$ because V is reflexive and therefore, J and J^{-1} are isometry, so they are continuous, therefore W continuous and consequently implies B is W compact. So, J is continuous implies W continuous. So, B, so this proves the theorem completely. So now, there are lots of nice corollaries to this result.

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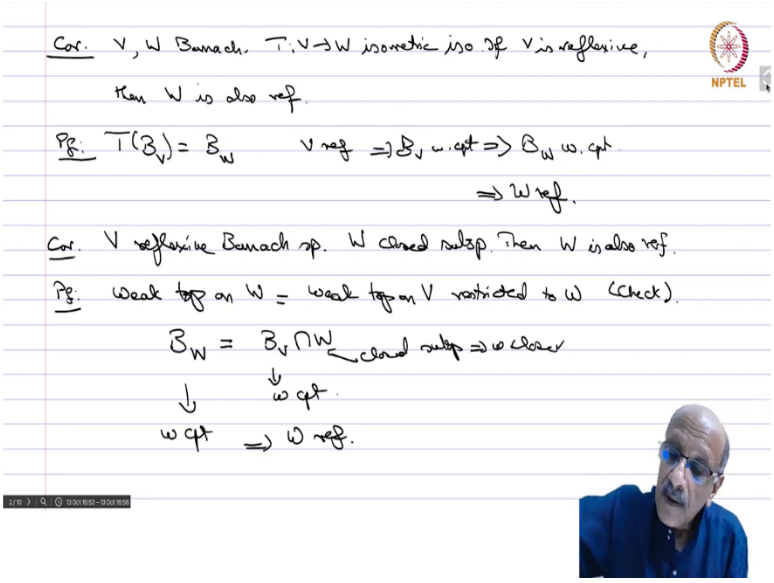

Corollary: So V and W Banach and $T: V \to W$ isometric isomorphism, that means T is an isomorphism, T and T^{-1} are continuous and it is a one-one onto map and T is isometric and therefore, it preserves the norms. Then, so if V is reflexive, then W is also reflexive.

Proof: If you take B_V the closed unit ball in V and B_W the closed unit ball in W, then $T\left(B_V\right) = B_W$. Now, V reflexive implies B_V is W compact and T is continuous and therefore, it is weakly continuous, so B_W is W compact implies W reflexive.

Next corollary,

Corollary: V reflexive Banach space and W closed subspace, then W is also reflexive.

Proof: So what is the weak topology on W is nothing but the weak topology on V restricted to W. So check, we will do this in the exercises but it, I would recommend that you check it yourself just a simple consequence of the Hahn-Banach theorem because every continuous linear functional can be extended to the whole space. So, you can check the W neighbourhoods are precisely the W neighbourhood in $V \cap W$. So, that is all that you have to show. And therefore, if you have, so we have B what is the closed unit ball in W this is nothing but the closed unit ball in $V \cap W$. So, W is closed and it is a subspace, so it is a closed subspace implies weakly closed. And B_V is weakly compact, so B_W is also weakly compact implies W reflexive.

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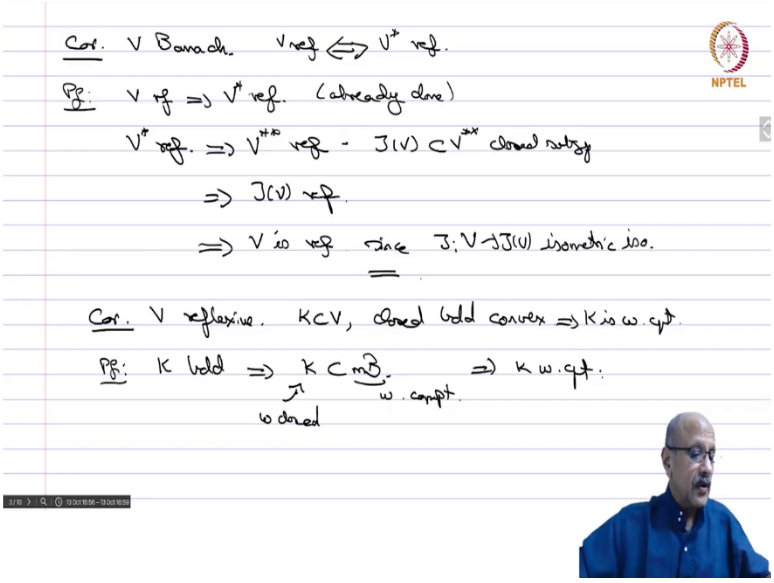

Next corollary,

Corollary: V Banach. Then V reflexive if and only if V^* reflexive.

Proof: So V reflexive implies V^* reflexive already done. Why? we have seen this namely if you V is reflexive then B^* is weakly compact by the Banach-Alaoglu Theorem because the W and W^* are the same. And therefore, if it is W compact we already showed that V is, the space is reflexive therefore, V^* is reflexive. So, V reflexive implies V^* reflexive, we have already shown. So, now let us assume V^* reflexive. So, this implies that V^{**} is reflexive by this argument and $J(V) \subset V^{**}$ is a closed subspace. So, by the previous corollary we have $J(V)$ is reflexive and this implies V is reflexive, since $J: V \to J(V)$ is isometric, isomorphism. So that proves, so V is reflexive if and only if V^* is reflexive.

So, for instance we know that l_1 is not reflexive because l_∞^* is not l_1 but then, so l_1 is not reflexive, so l_∞ cannot be reflexive either. So next corollary,

Corollary: V reflexive and $K \subset V$ closed bounded and convex implies K is W compact. So, it is not just the ball which is W compact every closed bounded convex set is automatically W compact.

Proof: So K is bounded, so this implies that K is contained in some m times the unit ball B.

So, mB this is weakly compact because it is just scaling the unit ball and there since V is reflexive. And V is a closed convex set, so it is W closed and therefore, a closed subset of a compact set is compact in the Hausdorff space therefore, K is W compact, so that proves that result.

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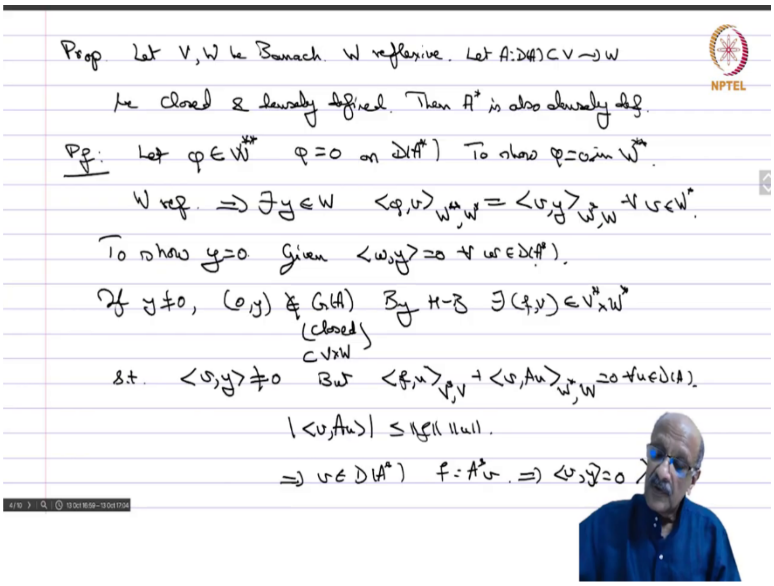

Proposition: Let V, W be Banach and W reflexive. Let A, which is unbounded operator defined in $D(A)$ taking values in W be closed and densely defined. Then, so is densely defined so the adjoint is defined, so then $A^\star$ is also densely defined. So, we made this remark already $A^\star$ need not be in general densely defined but $A^\star$ is always closed.

So, A is densely define you can define $A^\star$ and $A^\star$ is closed this is what we know but if A is closed then densely defined than $A^\star$ is densely defined and you know $A^\star$ is closed, so $A^\star$ is also closed and densely defined.

Proof: So what do you want to show. So, let ϕ we want to show that $D(A^{\star})$ is dense, so we are going to use the Hahn-Banach theorem. Let $\phi \in W^{\star\star}$ such that $\phi = 0$ on $D(A^{\star})$. To show $\phi = 0$ everywhere in $W^{\star\star}$. So, something which vanishes on a set, vanishes everywhere and therefore, we want to show that it is. But W is reflexive. So, this implies that there exists a $y \in W$ such that $\langle \phi, v\rangle_{W^{\star\star}, W^{\star}} = \langle v, y\rangle_{W^{\star}, W}$ for all $v \in W^{\star}$. This is the meaning of reflexive, so that means every functional occurs as an evaluation. So, to show $y = 0$ because ϕ is essentially determined by y, so we want to show that $y = 0$. So, what is given, we are given that $\langle w, y\rangle = 0$ for all $w \in D(A^{\star})$. So, if not, if $y \neq 0$, then $(0, y) \notin G(A)$. $0 \in D(A)$ and if y has to be in graph of A, y has to be $A(0)$ that is 0 which is not true, so $(0, y) \notin G(A)$. So, $y \in W$, so it is not in $G(A)$. So therefore, by Hahn-Banach there exists (f, v) and $G(A)$ is closed, remember that, that is given to you because A is a closed operator that means $G(A)$ is closed. So, there exists $(f, v) \in V^{\star} \times W^{\star}$, so this is closed and this is contained in $V \times W$, so the dual space is $V^{\star} \times W^{\star}$ which does not vanish on $(0, y)$. So that means, such that $\langle v, y\rangle \neq 0$. But $\langle f, u\rangle_{V^{\star}, V} + \langle v, Au\rangle_{W^{\star}, W} = 0$ for all $u \in D(A)$. Then, what is this $|\langle v, Au\rangle| \leq \|f\|\|u\|$ and this implies that $v \in D(A^{\star})$ and $f = A^{\star}v$. But, if $v \in D(A^{\star})$ this implies $\langle v, y\rangle$ has to be 0 that is a given condition which is a contradiction, we are given the $\langle w, y\rangle = 0$ for all $w \in D(A^{\star})$ and therefore, this is a contradiction. So, this proves this theorem. So, this is a contradiction so, y has to be 0 that means $D(A^{\star})$ is dense.

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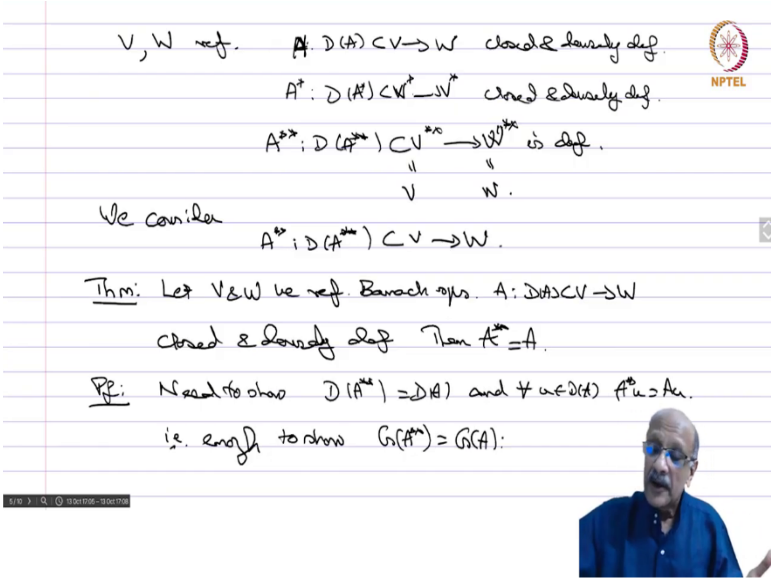

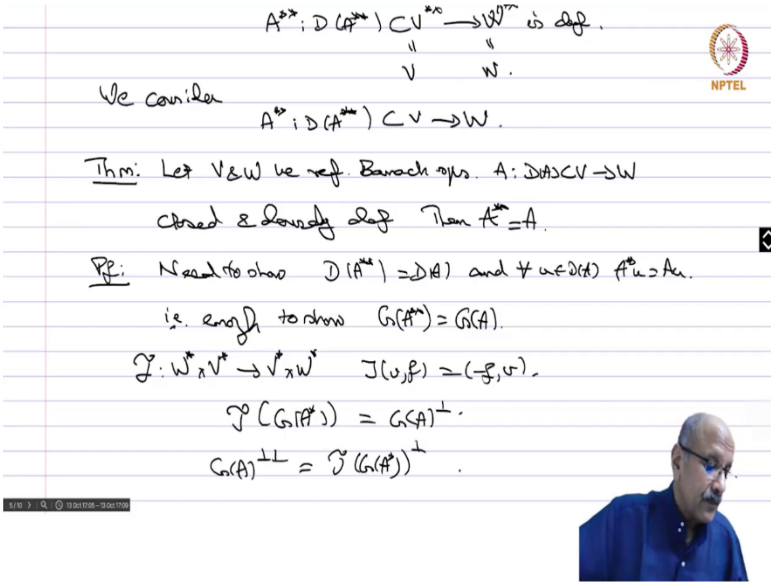

So, now let us assume that V and W are both reflexive and $A: D(A) \subset V \to W$ is closed and densely defined. Then, we just saw that $A^{\star}$ which is defined on $D(A^{\star}) \subset W^{\star}$ going into $V^{\star}$. And this is defined because A is densely defined, $A^{\star}$ is also closed and densely defined this is by the previous proposition.

So, $A^{\star\star}: D\left(A^{\star\star}\right) \subset V^{\star\star} \to W^{\star\star}$ is defined because $A^{\star}$ is densely defined $A^{\star\star}$ is defined. But we are going to take V reflexive, so $V^{\star\star}$ is can be identified with V and $W^{\star\star}$ can be identified with W. So, we consider $A^{\star\star}: D\left(A^{\star\star}\right) \subset V \to W$. So, we can consider it like this and because we can identify being both all spaces being reflexive V^{**} can be identified with V and W^{**} can be identified with W.

So, then the theorem is a nice important theorem.

Theorem: So, let V and W be reflexive Banach spaces and $A: D(A) \subset V \to W$ closed and densely defined. Then $A^{**} = A$. So, these two mappings are the same. So, when you say two unbounded operators are the same, you have to show that the domains are the same and you must also show that the action on each member of the domain is the same.

Proof: Need to show $D(A^{**}) = D(A)$ and for every $u \in D(A)$ we have to show $A^{**}u = Au$ that is enough to show $G(A^{**}) = G(A)$. Because what is $G(A)$? it is a domain all the first coordinates are in the domain and the second coordinate is the action and these two are the same that means the mappings are one and the same.

So recall, you have $I : W^{*} \times V^{*} \to V^{*} \times W^{*}$, $I(v, f) = (-f, v)$. And then, we saw that $I\big(G\big(A^{*}\big)\big)$ this was already shown is nothing but $G(A)^{\perp}$. So, $G(A)^{\perp\perp}$ will be $I\big(G\big(A^{*}\big)\big)^{\perp}$. So what does this mean?

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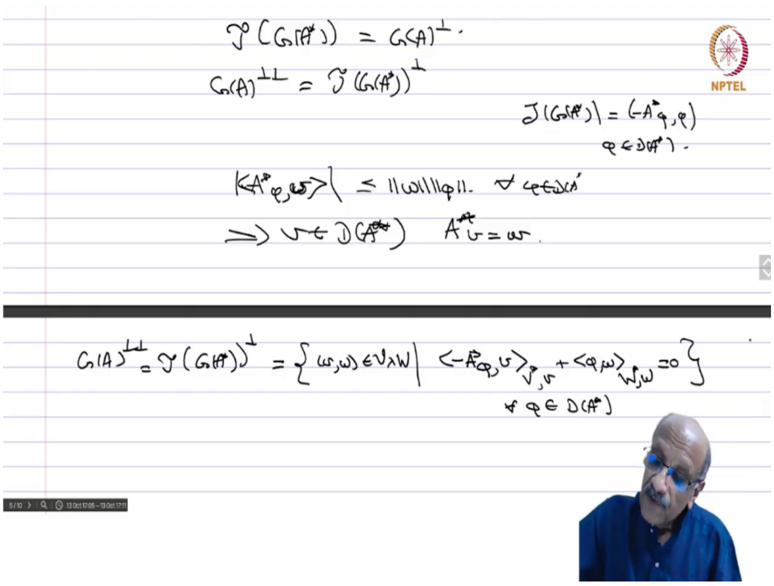

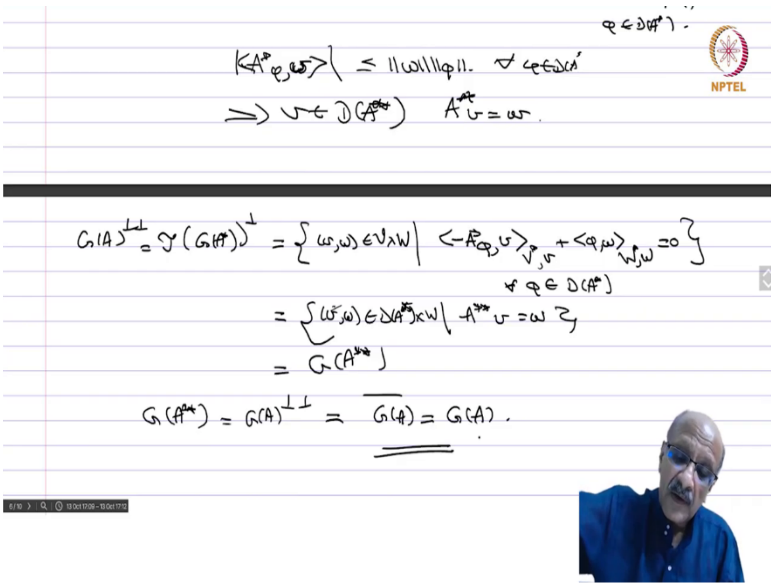

So, what is $G(A)^{\perp\perp}$? this is $I\left(G\left(A^{\star}\right)\right)^{\perp}$. So, this should consist of all elements $(v, w)\in V\times W$, which kill every element of $I\left(G\left(A^{\star}\right)\right)$. Now, what is $I\left(G\left(A^{\star}\right)\right)$ looks like, so $I\left(G\left(A^{\star}\right)\right)$ will be looking like $\{\left(-\ A^{\star}\phi, \phi\right): \phi\in D\left(A^{\star}\right)\}$. So, that is $I\left(G\left(A^{\star}\right)\right)$, so we want (v, w) to kill all of them. So, we want $I\left(G\left(A^{\star}\right)\right)^{\perp} = \{(v, w)\in V\times W\colon \langle -\ A^{\star}\phi, v\rangle_{V^{\star}, V} + \langle\phi, w\rangle_{W^{\star}, W} = 0\}$ and this should be true for all $\phi\in D(A^{\star})$. Now, this relationship tells you that $\left|\langle -\ A^{\star}\phi, v\rangle_{V^{\star}, V}\right| \leq \|w\|\|\phi\|$ and therefore, this tells you that $v\in D(A^{\star\star})$ and $A^{\star\star}v = w$. And therefore, so keeping that in mind $\{(v, w)\in V\times W\colon \langle -\ A^{\star}\phi, v\rangle_{V^{\star}, V} + \langle\phi, w\rangle_{W^{\star}, W} = 0\}$ is nothing but, $\{(v, w)\in V\times W\colon A^{\star\star}v = w\}$. And that is precisely $G(A^{\star\star})$ therefore, $G(A^{\star\star}) = G(A)^{\perp\perp}$, but you know double perp in the base space is nothing but the closure. So, $\overline{G(A)}$ but A is a closed operator and therefore, this equal to $G(A)$ and that completes the proof of the theorem.

So, that is, so, we will next look at separable spaces in the next session.