Functional Analysis
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Lecture No. 30
Weak* Topology - Part 2

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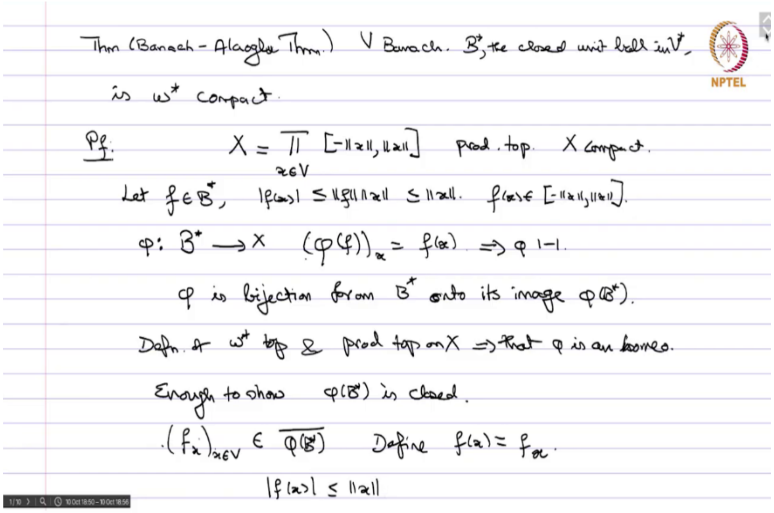

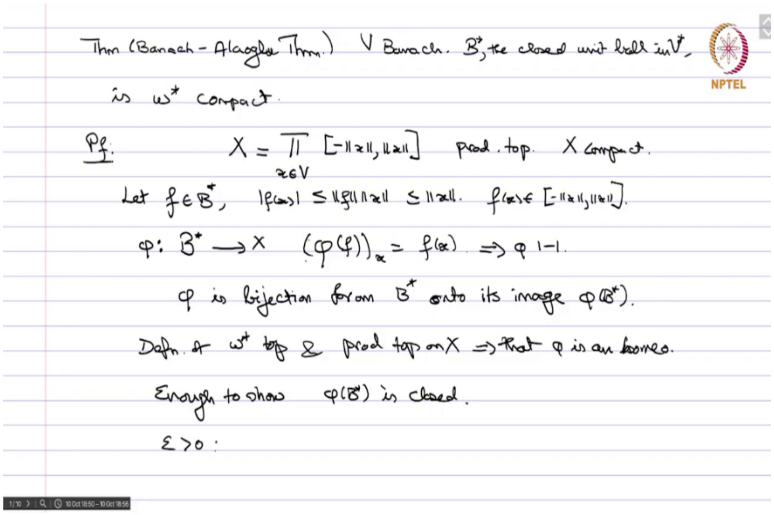

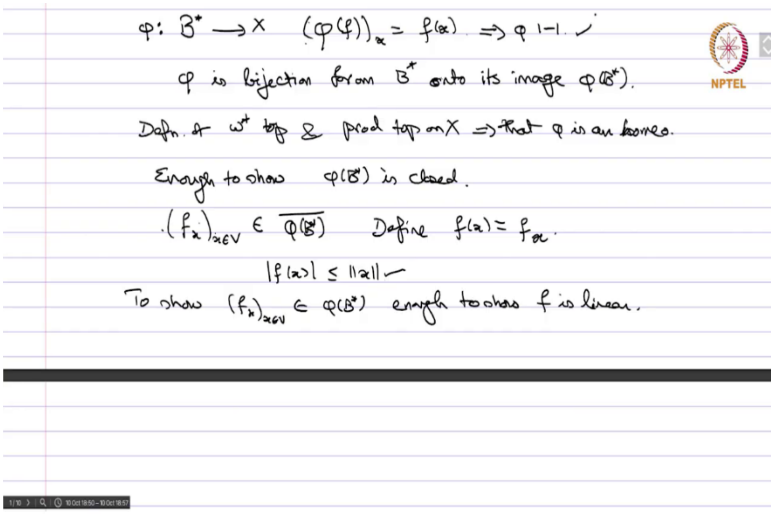

So, we will now prove this important theorem, it is called the Banach-Alaoglu Theorem.

Theorem (Banach-Alaoglu Theorem): So, V Banach, $B^\star$ the closed unit ball in $V^\star$ is $W^\star$ compact.

Proof: So a closed unit ball in infinite dimensional space is not non-compact and we are saying that it is weak star compact. So, first we define the product space $X = \prod_{x \in V} [-\ \|x\|,\ \|x\|]$, for every x you have this interval and then you have product space with the product topology. So, this is each interval is a compact interval and therefore, product of compact spaces by the Tychonoff's theorem X is compact, so X is compact. Now, let $f \in B^\star$ then $|f(x)| \leq \|f\|\|x\|$, $\|f\| \leq 1$, so $\|f\|\|x\| \leq \|x\|$. Therefore, $f(x) \in [-\ \|x\|,\ \|x\|]$. So, given any, so from $B^\star$ to X, I define a map ϕ, so we have $\phi(f)$ whose x'th coordinate is nothing but $f(x)$. So, since if two functionals agree at all points then they have to be the same functional, so this implies that ϕ is one-one. So, ϕ is a bijection from $B^\star$ onto its image $\phi(B^\star)$. Now, you look at the topologies, what is the product topology, product topology typical open set looks like open in a finite number of the x's and the rest of it is the full thing. So, that is the definition of the product topology. So, you have $-\ \epsilon$ to ϵ in a finite number of once and then, you have the rest of it can be open. And, so if you look at the $W^\star$, what is that, it is set of all f such that $\left(f - f_0\right)\left(x_i\right) < \epsilon$ for a finite number of i's, so this is exactly the topology there. So, the definition of $W^\star$ and product topology on X imply that ϕ is an isomorphism or homeomorphism. Therefore we just have to show, so enough to show $\phi(B^{\star})$ is closed because a close subspace of a compact set is compact. So, if $\phi(B^{\star})$ is closed, then $\phi(B^{\star})$ will be compact. And since, ϕ is a homeomorphism you have that $B^{\star}$ will also be compact.

So, we just realized that the product topology, which means a finite number of sets are open for a finite number of X's and that is exactly how your weak topologies are also defined, so that is the observation which we make. So, now let us take any $\epsilon > 0$, before that, so we have to show that this is closed. So, let us take $(f_x)_{x\in V}$, so this is in the product space which belongs to $\overline{\phi(B^{\star})}$. Define $f(x) = f_x$, then what do you know $|f(x)| \leq \|x\|$. So, to show $(f_x)_{x\in V} \in \phi(B^{\star})$ enough to show f is linear. If it is linear, this condition already tells you it is continuous and therefore, its image of a continuous linear operator, continuous linear functional under this mapping ϕ, which we have defined here. So, we just have to show that f is linear. So, how do we do that?

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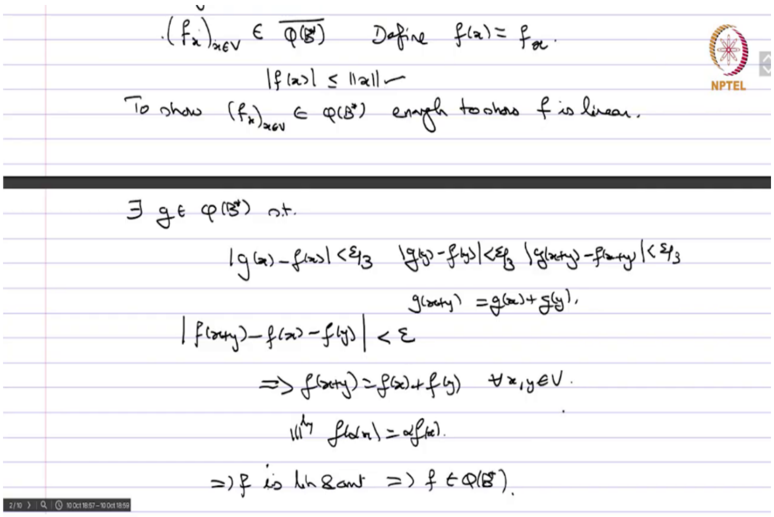

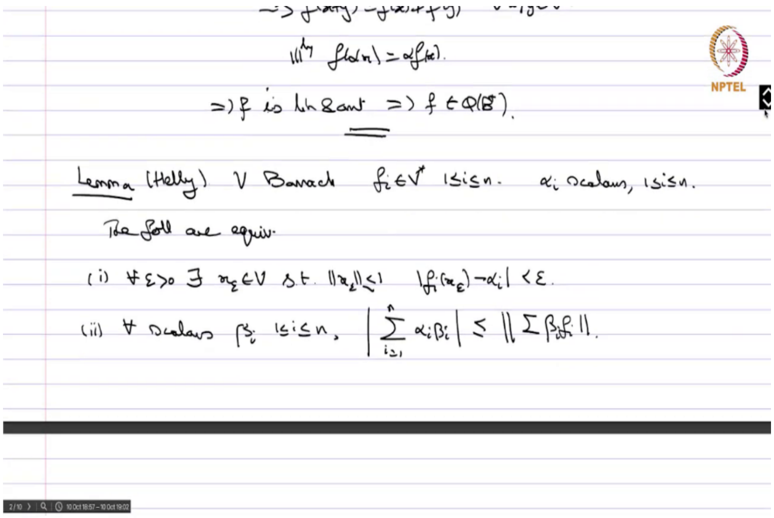

Illy $f(\alpha x) = \alpha f(x)$.

$\Rightarrow f$ is linear $\Rightarrow f \in \Phi(B^*)$.

Lemma (Helly) V Banach $f_i \in V^*$ $1 \le i \le n$. α_i scalars, $1 \le i \le n$.

The foll are equiv.

(i) $\forall \varepsilon > 0 \; \exists x_\varepsilon \in V$ s.t. $\|x_\varepsilon\| \le 1$ $|f_i(x_\varepsilon) - \alpha_i| < \varepsilon$.

(ii) $\forall$ scalars β_i $1 \le i \le n$, $\left|\sum_{i=1}^{n} \alpha_i \beta_i\right| \le \|\sum \beta_i f_i\|$.

So, it is in the closure, therefore there exists a $g \in \phi(B^{\star})$ such that g lies in any neighbourhood of f. So, such that, so I am going to take a neighbourhood namely consisting of three points x, y and $x + y$. So, so we take $|g(x) - f(x)| < \frac{\epsilon}{3}$, $|g(y) - f(y)| < \frac{\epsilon}{3}$ and the third point I am going to take $|g(x + y) - f(x + y)| < \frac{\epsilon}{3}$. So, I have taken three points x, y and $x + y$ and take an $\frac{\epsilon}{3}$ neighbourhoods around f of those points and then that gives me a weak star neighbourhood and that weak star neighbourhood must intersect $\phi(B^{\star})$ because, the given function is in the closure and therefore, I can find such g. Then what do you know about $|f(x + y) - f(x) - f(y)|$. So, you add and subtract the g's, since g's and $\phi(B^{\star})$ we know that $g(x + y) = g(x) + g(y)$ because $g \in \phi(B^{\star})$. So therefore, $|f(x + y) - f(x) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$ this is less than ϵ and this is true for all ϵ, so this implies that $f(x + y) = f(x) + f(y)$ for all x, y in V. Similarly, we can prove $f(\alpha x) = \alpha f(x)$, same way you produce a g and then that is very small by $|\alpha|$ etc. and then you can do it. So this implies that, f is linear, linear than continuous and therefore, implies $f \in \phi(B^{\star})$. Therefore, $\phi(B^{\star})$ is closed, hence it is compact ϕ is an homeomorphism $f(B^{\star})$ is also compact. So, this proves the Banach-Alaoglu theorem.

So, this is the advantage I said, by going to the weak star topology the open sets become fatter and they are fewer and therefore, the chances of a set being compact are better. And in this case we see that the unit, closed unit ball which is would not be compact in the norm topology is now compact.

So, now we have a lemma due to Helly.

Lemma (Helly): So, V Banach and $f_i \in V^\star$, $1 \leq i \leq n$. And α_i scalars again $1 \leq i \leq n$. The following are equivalent,

1. For every $\epsilon > 0$ there exists an $x_\epsilon \in V$ such that $\|x_\epsilon\| \leq 1$ and $\left|f_i(x_\epsilon) - \alpha_i\right| < \epsilon$. That means, I can approximately I can find x_ϵ such that the $f_i(x_\epsilon)$ approximate the α_i as closely as I wish, so that is the.
2. The second condition which is equivalent to this is that, for all scalars β_i, $1 \leq i \leq n$, you have $\left|\sum_{i=1}^{n} \alpha_i \beta_i\right| \leq \|\sum_{i=1}^{n} \beta_i f_i\|$. So, this inequality guarantees the existence of an x_ϵ which approximates at each f_i, α_i as closely as we wish.

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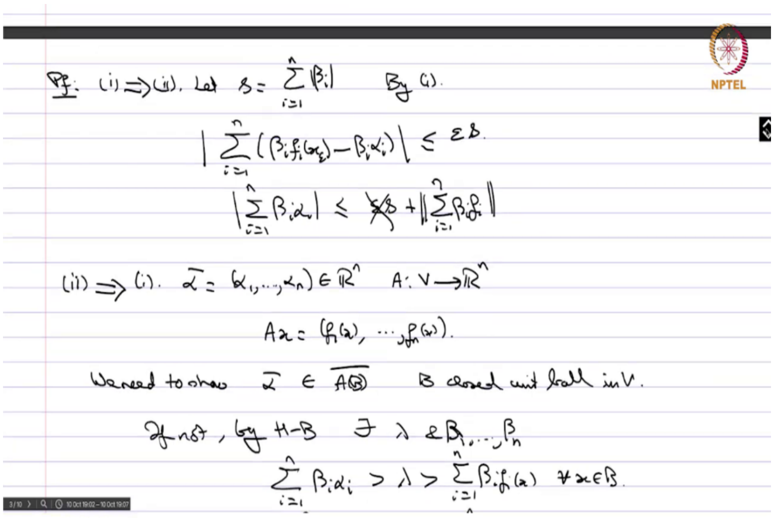

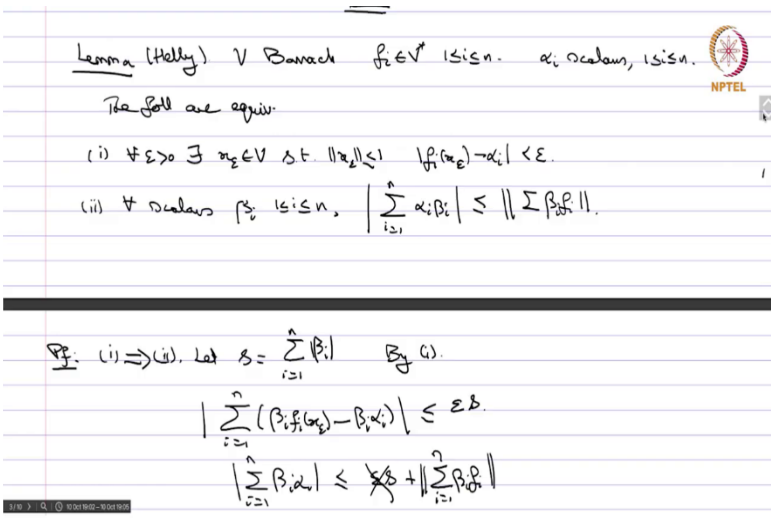

Lemma (Helly) V Banach $f_i \in V^*$ $1 \le i \le n$. α_i scalars, $1 \le i \le n$.

The foll are equiv.

(i) $\forall \varepsilon > 0 \ \exists x_\varepsilon \in V$ s.t. $\|x_\varepsilon\| \le 1$ $|f_i(x_\varepsilon) - \alpha_i| < \varepsilon$.

(ii) $\forall$ scalars β_i $1 \le i \le n$, $\left|\sum_{i=1}^{n} \alpha_i \beta_i\right| \le \left\|\sum \beta_i f_i\right\|$.

Pf: (i) $\Rightarrow$ (ii). Let $s = \sum_{i=1}^{n} |\beta_i|$ By (i).

$$\left|\sum_{i=1}^{n} (\beta_i f_i(x_\varepsilon) - \beta_i \alpha_i)\right| \le \varepsilon s.$$

$$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| \le \varepsilon s + \left\|\sum_{i=1}^{n} \beta_i f_i\right\|$$

Proof: So let us show that $1 \Rightarrow 2$. So, you let $s = \sum_{i=1}^{n} \left|\beta_i\right|$, then by 1 what do you have, $\left|\sum_{i=1}^{n} (\beta_i f_i\left(x_\epsilon\right) - \beta_i \alpha_i)\right|$ this will be less than equal to $\left|f_i\left(x_\epsilon\right) - \alpha_i\right| < \epsilon$ and therefore that comes out, so you get $\left|\sum_{i=1}^{n} (\beta_i f_i\left(x_\epsilon\right) - \beta_i \alpha_i)\right| < \epsilon s$.

Therefore, you have $\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| \le \epsilon s + \sum_{i=1}^{n} \left|\beta_i f_i\left(x_\epsilon\right)\right|$ but that is less than or equal to $\|\sum_{i=1}^{n} \beta_i f_i\|$. Because $\beta_i f_i\left(x_\epsilon\right)$ in modulus is less than, that is $\sum_{i=1}^{n} \beta_i f_i\left(x_\epsilon\right)$ and that is less than $\|\sum_{i=1}^{n} \beta_i f_i\|$. So, this implies 2, so, and this is true for allϵ so we can let ϵ go to 0 and then you get the second condition.

Now, we want to show that $2 \Rightarrow 1$. So, you post $\overline{\alpha} = \left(\alpha_1, \cdot, \cdot, \cdot, \alpha_n\right) \in R^n$ and you, or C^n if you like, we will do it with R^n. A you define from V to R^n in the usual way we are going to define $A(x) = (f_1(x), \cdot, \cdot, \cdot, f_n(x))$. So, what do we need to show, we need to show, if you look at the condition one what is it saying, you should be able to find $x_\epsilon \le 1$ such that $f_i\left(x_\epsilon\right)$ is less than as close to α_i as you like.

That means we have to show that $\bar{\alpha} \in \overline{A(B)}$ where B is the closed unit ball. So, if not you have $\overline{A(B)}$, now that is a convex set, closed convex set and you have $\bar{\alpha}$ which is a single point which is a compact convex set. Therefore by the Hahn-Banach, we can find there exists λ and $\beta_1, \cdot, \cdot, \cdot, \beta_n$ these scalars come from the linear functional on R^n. So, this linear we have done this kind of argument before such that $\sum_{i=1}^{n} \beta_i \alpha_i$. So, this is the action of this linear functional on $\bar{\alpha}$, so that is there is strictly bigger than λ, you can strictly separate and that is bigger than $\sum_{i=1}^{n} \beta_i f_i(x)$ for every $x \in B$. And therefore, from this you get that $\| \sum_{i=1}^{n} \beta_i f_i \|$ because that is the supremum of $\sum_{i=1}^{n} \beta_i f_i(x)$ for all $x \in B$ that will be give you the norm that is less than or equal to λ which is strictly less than $\sum_{i=1}^{n} \beta_i \alpha_i$ and that is a contradiction of statement number 2. Because, we have assumed 2 which goes the other way around. So, that is, so that proves this proposition or lemma.

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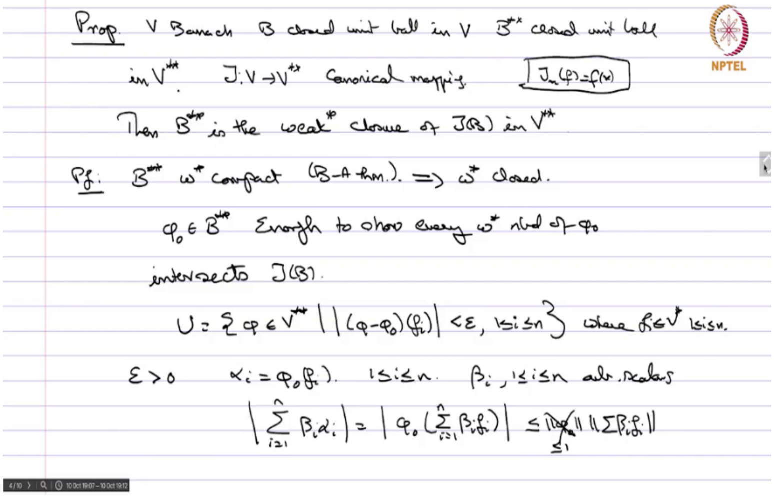

Lemma (Helly) V Banach $f_i \in V^*$ $1 \le i \le n$. α_i scalars, $1 \le i \le n$.

The foll are equiv.

(i) $\forall \varepsilon > 0\ \exists x_\varepsilon \in V$ s.t. $\|x_\varepsilon\| \le 1$ $|f_i(x_\varepsilon) - \alpha_i| < \varepsilon$.

(ii) $\forall$ scalars β_i $1 \le i \le n$, $\left|\sum_{i=1}^{n} \alpha_i \beta_i\right| \le \left\|\sum \beta_i f_i\right\|$.

Pf: (i) $\Rightarrow$ (ii). Let $\delta = \sum_{i=1}^{n} |\beta_i|$ By (i).

$$\left|\sum_{i=1}^{n} (\beta_i f_i(x_\varepsilon) - \beta_i \alpha_i)\right| \le \varepsilon\delta.$$

$$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| \le \cancel{\varepsilon\delta} + \left\|\sum_{i=1}^{n} \beta_i f_i\right\|$$

intersects $J(B)$.

$$U = \{\varphi \in V^{**} \mid |(\varphi - \varphi_0)(f_i)| < \varepsilon,\ 1 \le i \le n\} \text{ where } f_i \in V^* \ 1 \le i \le n$$

$\varepsilon > 0$ $\alpha_i = \varphi_0(f_i)$. $1 \le i \le n$. β_i, $1 \le i \le n$ arb. scalars

$$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| = \left|\varphi_0\left(\sum_{i=1}^{n} \beta_i f_i\right)\right| \le \underbrace{\|\varphi_0\|}_{\le 1} \left\|\sum \beta_i f_i\right\|$$

By Helly, $\exists x_\varepsilon \in V$ $\|x_\varepsilon\| \le 1$ $|f_i(x_\varepsilon) - \alpha_i| < \varepsilon$

i.e. $|J_{x_\varepsilon}(f_i) - \varphi(f_i)| < \varepsilon$ $\forall 1 \le i \le n$.

i.e. $J_{x_\varepsilon} \in U$. $J_{x_\varepsilon} \in J(B)$.

So, now we have a proposition it is interesting,

Proposition: So V Banach B is the closed unit ball in V and B^{**} closed unit ball in V^{**}. $J: V \to V^{**}$ canonical mapping. That means what, $J_x(f) = f(x)$ this is how we studied reflexivity and so on, so, this is the canonical map. Then, B^{**} is the W^* closure of $J(B)$ in V^{**}.

Proof: So, B^{**} is the closed unit ball in V^{**}, V^{**} is already a dual space and therefore, B^{**} is W^* compact by the Banach-Alaoglu Theorem. Therefore, it is W^* closed. Now, you take $\phi_0 \in B^{**}$. And so, we have to show that every W^* neighbourhood of ϕ_0 will intersect $J(B)$

because that is what we mean by showing that it is dense. Because then that will make B^{**} to be in $\overline{J(B)}$ but B^{**} is already weak star closed and therefore, it is in fact (())(19:27). So, enough to show every W^{*} neighbourhood of ϕ_0 intersects $J(B)$. So, let us take U which is a W^{*} neighbourhood. So what is this, this set of all $\phi \in V^{**}$ such that $(\phi - \phi_0)(f_i)$, so you have to go to the space which gave you whose dual is V^{**}, V^{**} is the dual of V^{*} and therefore, we have to take $\left|\left(\phi - \phi_0\right)\left(f_i\right)\right| < \epsilon$, $1 \leq i \leq n$ where $f_i \in B^{*}$ for $1 \leq i \leq n$. So, this is the standard neighbourhood of this point. So, let $\epsilon > 0$ and let us take, that is given of course and then let us take $\alpha_i = \phi_0\left(f_i\right)$. Then, if β_i, $1 \leq i \leq n$ arbitrary scalars then what do you can you say at $\left|\sum_{i=1}^{n} \beta_i \alpha_i\right|$. This is equal to modulus. What is α_i, $\phi_0\left(f_i\right)$, so $\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| = \left|\phi_0(\sum_{i=1}^{n} \beta_i f_i)\right|$. And $\phi_0 \in B^{**}$, so its norm is less than equal to 1 so $\left|\phi_0(\sum_{i=1}^{n} \beta_i f_i)\right|$ is less than equal $\|\phi_0\|$ (which is less than equal to 1) $\times \|\sum_{i=1}^{n} \beta_i f_i\|$. So, $\|\phi_0\| \leq 1$, I forget this, so I have $\left|\phi_0(\sum_{i=1}^{n} \beta_i f_i)\right| \leq \|\sum_{i=1}^{n} \beta_i f_i\|$ which is the condition 2. So, this is equivalent to saying there exists a x_ϵ such that $\left|f_i\left(x_\epsilon\right) - \alpha_i\right| < 1$. Now, what is, so there exists. So by Helly, there exists $x_\epsilon \in V$, $\|x_\epsilon\| \leq 1$ and $\left|f_i\left(x_\epsilon\right) - \alpha_i\right| < \epsilon$. That means what, that is $\left|J_{x_\epsilon}\left(f_i\right) - \phi(f_i)\right| < \epsilon$ for all $1 \leq i \leq n$. And that is $J_{x_\epsilon} \in U$. And it also J_{x_ϵ} is where it belongs to $\phi(B)$. So, $J_{x_\epsilon} \in \phi(B) \cap U$ and that proves the result. So, what is the meaning of this result?

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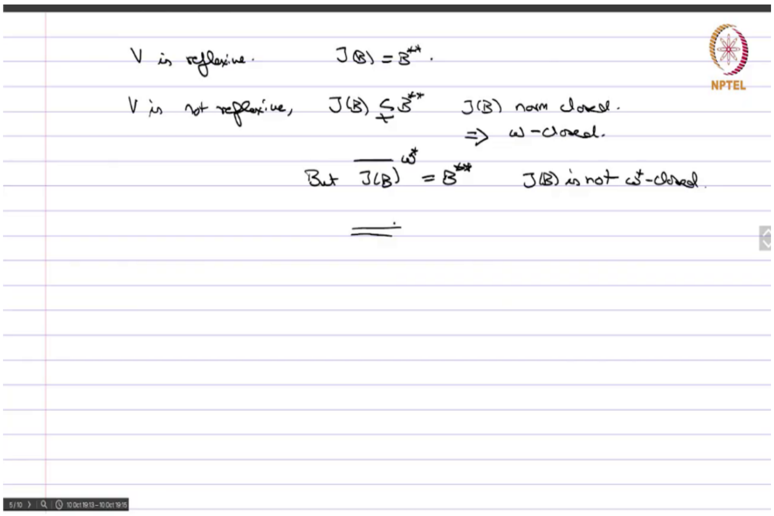

So, assume V is reflexive, then of course $J(B) = B^{\star\star}$. If V is not reflexive J is an isometry, so $J(B)$ is strictly contained in $B^{\star\star}$ and $J(B)$ is J closed. So, in fact it is therefore, since it is J closed and convex, so it is also W closed. But $\overline{J(B)}$ in the $W^{\star}$ topology is $B^{\star\star}$ and therefore, $J(B)$ is not $W^{\star}$ closed. So, we have this another example of a set which is J closed, W closed, but it is not $W^{\star}$ because we are again in a non-reflexive space that means automatically in infinite dimensions. So, this is about the weak star topology. So, in the next lectures we will look at the applications of the weak and weak star topology to the theory of Banach spaces.