Functional Analysis
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IMSc
Examples, contd.

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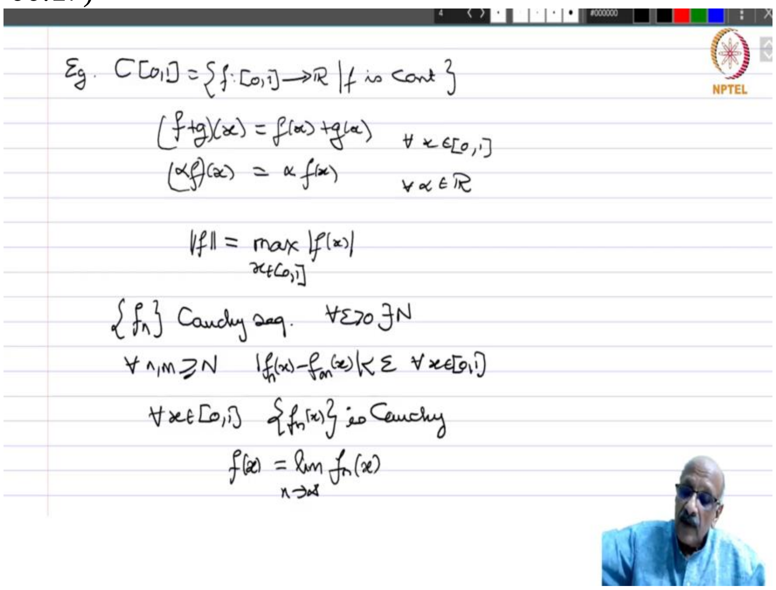

Our next example is the function space $C[0,1] := \{f: [0,1] \mapsto \mathbb{R} |\ f\ is\ continuous\}$. This becomes a vector space with the point wise addition and scalar multiplication, which are defined as $(f + g)(x) = f(x) + g(x); \forall\ x \in [0,1]$ and $(\alpha f)(x) = \alpha f(x); \forall\ x \in [0,1],\ \alpha \in \mathbb{R}$. Now, every continuous function on a compact interval is bounded and attains its maxima and minima, and therefore we can define the following.

$$||f|| = \max_{x \in [0,1]} |f(x)|$$

It is a very simple exercise for you to check that this defines a norm, and therefore I will leave it to you to do it; even the triangle inequality is also very trivial (just as we did it in the l_∞ case). We want to show that this defines a Banach space; so we take a Cauchy sequence (f_n) in $C[0,1]$. Then, for every $\epsilon > 0$, there exists N such that for all $n, m \geq N$, we have

$$|f_n(x) - f_m(x)| < \epsilon, \forall\ x \in [0,1].$$

Therefore, $\forall\ x \in [0,1], (f_n(x))$ is Cauchy (since the norm is defined by the maximum, this happens for every x) and therefore it converges.

So, let us define $f(x) = \lim_{n \to \infty} f_n(x)$ i.e., we now have a function from $[0,1]$ to $\mathbb{R}$; we have a candidate for the limit of the Cauchy sequence. And therefore we ask the usual question whether this candidate is eligible.

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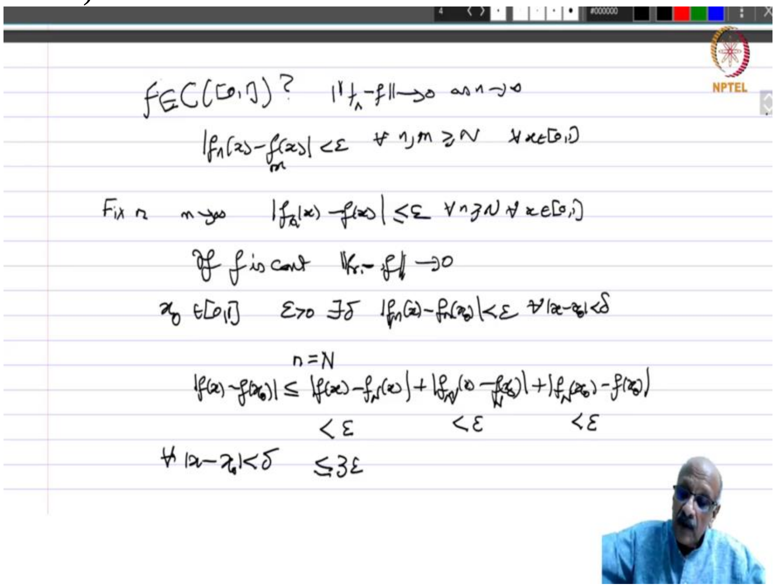

So, we want to know if $f \in C[0,1]$, and also, whether we have that $||f_n - f|| \to 0$ as $n \to \infty$. If we answer again these two questions affirmativly, then the Cauchy sequence will have a limit; and therefore the space will become a Banach space.

First of all we have that $|f_n(x) - f_m(x)| < \epsilon, \forall n, m \geq N, \forall\, x \in [0,1]$. So, you fix n and let m tend to infinity to have $|f_n(x) - f(x)| < \epsilon, \forall n \geq N, \forall\, x \in [0,1]$. So, we have that f_n converges uniformly to f, and the convergence in this norm is essentially uniform convergence. So, if we prove that f is continuous, then we are done. Now, we already know from real analysis that the uniform limit of continuous functions has to be continuous. I am just going to reprove that fact; Let us take any $x_0 \in [0,1]$. Now, given any $\epsilon > 0$, there exists a $\delta > 0$ such that you have $|f_n(x) - f_m(x)| < \epsilon, \forall\, |x - x_0| < \delta$. In particular, I am going to do it for $n = N$; I am going to choose this δ. So, now

$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < 3\epsilon, \forall\, |x - x_0| < \delta$. Therefore, we have shown that f is a continuous function; so, every Cauchy sequence converges and therefore $C[0,1]$ is a Banach space. So, this is an example of a function space, we will see many more function spaces later; so we will leave it at this.

So, before concluding, let me do one final example if you like. So, this is how to produce new Banach spaces from old (i.e., new normed linear spaces from old). This is standard in mathematics we do it all the time; whenever we have a structure, there are at least two different ways of producing new objects with the same structure. One is known as the subspace; if you have a group then you have a subgroup, if you have a topological space, you have a subspace, which inherits the same structure from above.

So, same way if you have a normed linear space and you have any vector subspace of V; you put the same norm there, you get a subspace. The other structure is the quotient space.

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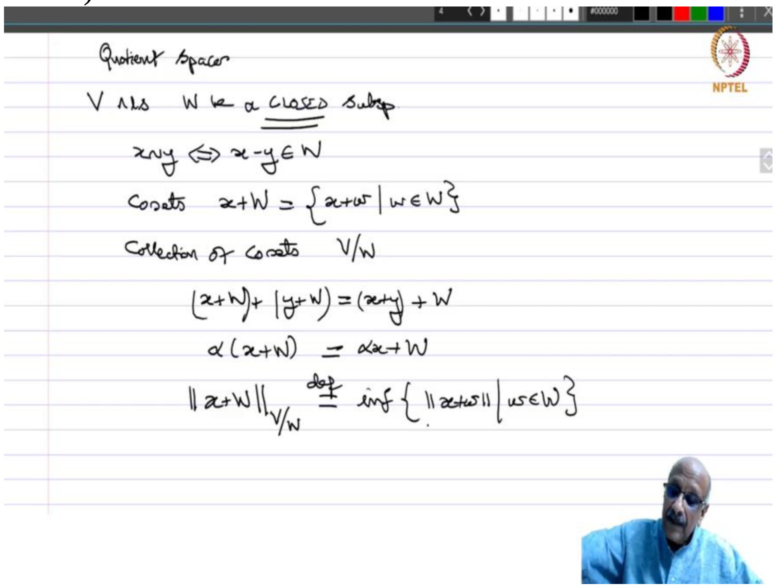

We will give you the example or we will discuss quotient spaces. Ler V be a normed linear space and V be a closed subspace (this is important and we will see in a moment). Then we define the equivalence relation $x \sim y \iff x - y \in W$. This defines an equivalence relation and then it partitions V into equivalence classes. The equivalence classes are are called the cosets and denoted by $x + W \coloneqq \{x + w : w \in W$. The set of all such cosets i.e., the collection of cosets, we call the quotient space $\frac{V}{W}$. We can give this vector space structure namely, if you have two cosets $x + W$ and $y + W$, then we can define the addition as $(x + W) + (y + W) = (x + y) + W$. x and the scalar multiplication as $\alpha(x + W) = \alpha x + W$. One can check that these are well defined, because if $x \sim x'$, $y \sim y'$; since W is a subspace, $x + y \sim x' + y'$. Similarly, if $x \sim x'$, then $\alpha x \sim \alpha x'$; and therefore, these are well defined. So, the cosets can be defined, the operation does not depend on the representative you are taking for the coset, and therefore you can do it.

Now we are going to define a norm on $\frac{V}{W}$. We define $||x + W||_{\frac{V}{W}} \coloneqq \inf \{ ||x + w|| : w \in W \}$.

(You take all elements of the coset, take their norms and take the minimum of that).

Now have to check that this is a norm.

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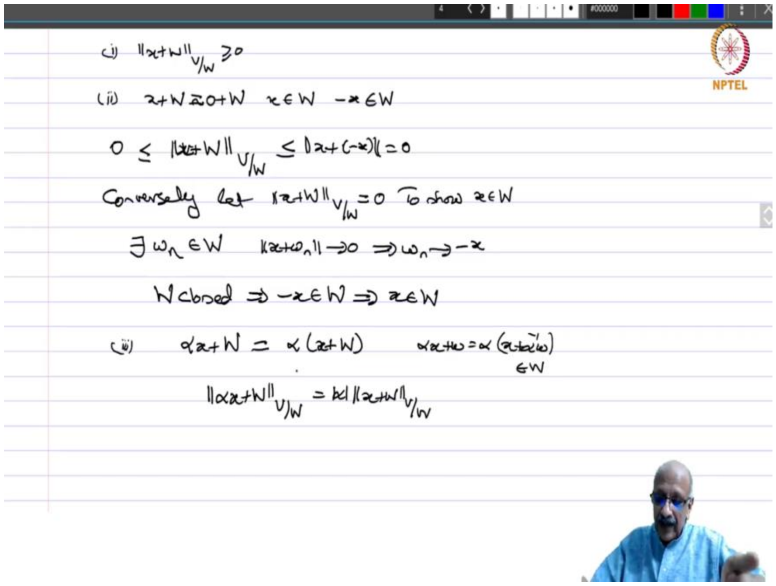

First of all, $||x+W||_{\frac{V}{W}} \geq 0$, because you are taking the infimum of non-negative numbers and therefore it is non-negative. Let us see the second one. If $x+W=0+W$, then $x,-x \in W$ (because W is a subspace). Therefore, $0 \leq ||x+W||_{\frac{V}{W}} \leq ||x+(-x)|| = 0$.

So, if $x+W$ is the zero element, then $||x+W||_{\frac{V}{W}} = 0$. Conversely, let $||x+W||_{\frac{V}{W}} = 0$. Then we have to show that $x+W=0$, or in other words $x \in W$. What does the definition says? Since $||x+W||_{\frac{V}{W}} = 0$, there exists $w_n \in W$ such that $||x+w_n|| \to 0$ (by definition). This means that $w_n \to -x$. Now W is closed. This implies that $-x \in W$ (this is where the hypothesis closed is important) and this implies that $x \in W$, and therefore we are true.

Now, we want to show the the third property. Take $\alpha(x+W)$ for some $\alpha \in F$. If $\alpha = 0$, then there is nothing to do. So let us assume $\alpha \neq 0$; and **therefore, $\boldsymbol{\alpha(x+W)=\alpha x+W}$ this can be written as alpha times x plus W again because, if you take alpha x plus w; this can be written as alpha times x plus alpha inverse of w, which is again an element of W.**

And therefore from this it immediately follows that the norm of alpha x plus W in V by W is nothing but mod alpha times norm of x plus W; V by W because every element here can be written as alpha times something and you are taking the infimum of the norms.

Now we are going to show that triangle inequality;

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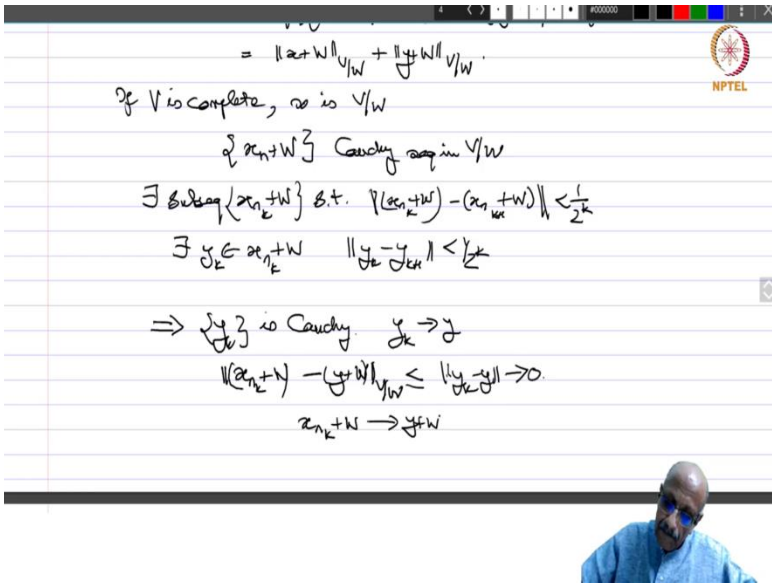
$$\|x+y+W\|_{V/W} = \inf\{\|x+y+w\| \mid w \in W\}$$
$$= \inf\{\|x+y+w+w'\| \mid w, w' \in W\}$$
$$\leq \inf\{\|x+w\| + \|y+w'\| \mid w, w' \in W\}$$
$$= \inf\{\|x+w\| \mid w \in W\} + \inf\{\|y+w'\| \mid w' \in W\}$$
$$= \|x+W\|_{V/W} + \|y+W\|_{V/W}.$$

If V is complete, so is V/W

$\{x_n + W\}$ Cauchy seq in V/W

$\exists$ subseq $\{x_{n_k} + W\}$ s.t. $\|(x_{n_k}+W) - (x_{n_{k+1}}+W)\| < \frac{1}{2^k}$

$\exists\ y_k \in x_{n_k} + W \quad \|y_k - y_{k+1}\| < 1/2^k$



So we have to look at $||x+y+W||_{\frac{V}{W}} = \inf\{||x+y+w|| : w \in W\} = \inf\{||x+y+w+w'|| : w, w' \in W\} \le \inf\{||x+w|| + ||y+w'|| : w, w' \in W\} \le \inf\{||x+w||\} + \inf||y+w'|| : w, w' \in W\} = ||x+W||_{\frac{V}{W}} + ||y+W||_{\frac{V}{W}}$. Therefore we have shown that the triangle inequality is satisfied.

So, $\frac{V}{W}$ where W is a closed subspace makes a normed linear space.

If V is complete so is $\frac{V}{W}$. So, if the original space is Banach, then the quotient space also is Banach; so the quotient space inherits the Banach or completeness property. We shall rpove this. We have to show that every Cauchy sequence converges; so let us take a Cauchy sequence Cauchy sequence $(x_n + W)$ in $\frac{V}{W}$. Now, given any Cauchy sequence, you can always find a sub-sequence, such that consecutive elements differ by whatever quantity you decide.

So, there exists subsequence $(x_{n_k} + W)$, such that $\left|\left|x_{n_k} + W - x_{n_{k+1}} + W\right|\right|_{\frac{V}{W}} < \frac{1}{2^k}$. This means there exists $y_k \in x_{n_k} + W$ such that $||y_k - y_{k+1}|| < \frac{1}{2^k}$. This implies that (y_k) is Cauchy (whenever, you have a sequence, where consecutive terms differed by something; and that something forms a summable series, then the original sequence is Cauchy).

Since V is complete, (y_k) will converge to y. Now we have $\left|\left|x_{n_k} + W - y + W\right|\right|_{\frac{V}{W}} \le ||y_k - y|| \to 0$. Therefore $x_{n_k} + W \to 0$. Therefore, we have produced a subsequence which is convergent.

Again you have a Cauchy sequence property, if you have subsequence converges in a Cauchy sequence. Then the original sequence also converges to the same limit.

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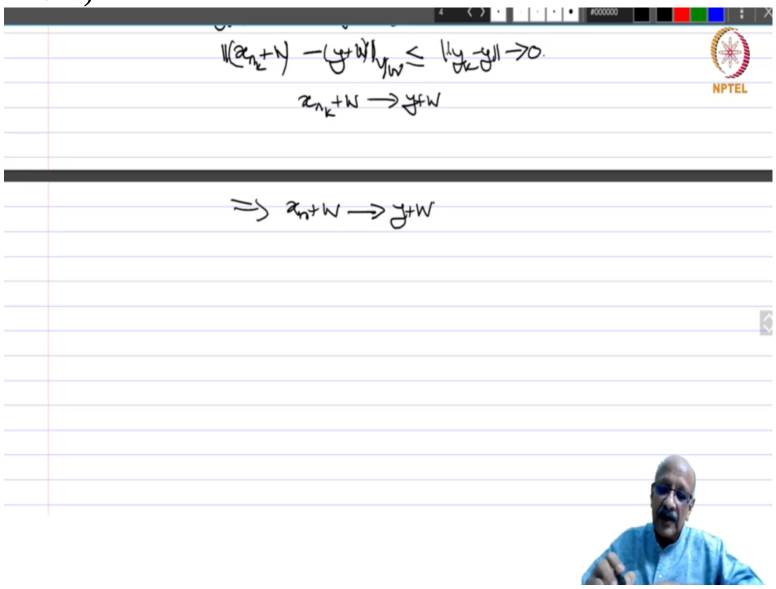

Therefore, so this implies that $x_n + W \rightarrow y + W$. We have used three properties of Cauchy sequences throughout this proof and you should check them all. First, in a Cauchy sequence you can find a subsequence whose consecutive terms differ by whatever quantity you decide. Now, if a sequence has consecutive terms differing by a general term of a convergence series; then the sequence is Cauchy. And thirdly, if you have a Cauchy sequence and you have a convergent subsequence; the sequence converges to the same limit.

So, using these three, we have shown that if V is complete, then $\frac{V}{W}$ is also complete. And therefore the it becomes a Banach space; so with these examples we will wind up. And then we will next next topic for discussion will be continuous linear transformations. Thank you.