Functional Analysis
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Lecture No. 29
Weak* Topology – Part 1

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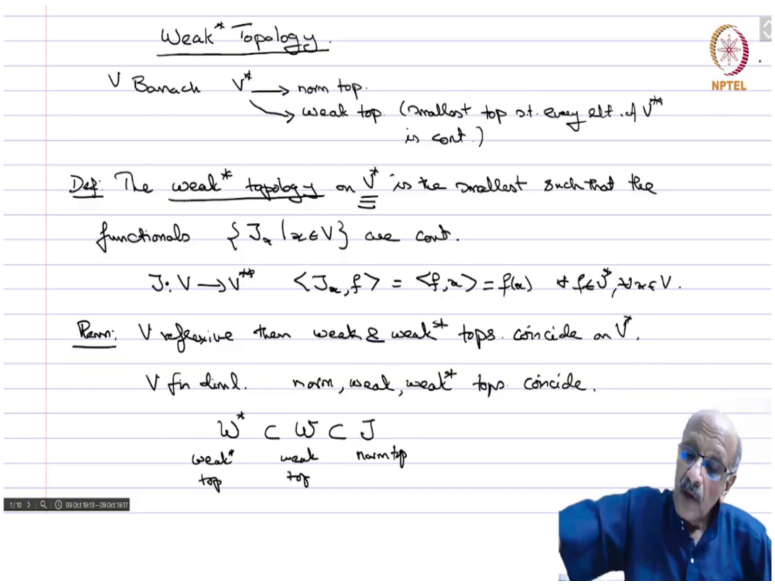

We will now discuss the weak star topology. So, V is a Banach space and V^* is the dual space. So, V^* you already have the norm topology. V^* is a Banach space we also have the weak topology, what is the weak topology, smallest topology such that every element of V^{**} is continuous. So, this would be the weak topology on V^*.

Now, we are going to define the weak star topology.

Definition: The weak star topology on V^*, so this is always defined only on the dual space, is the smallest topology, smallest in the sense of smallest number of open sets such that the functionals $\{J_x : x \in V\}$ are continuous.

Recall J is the canonical embedding from V into V^{**}. J_x acting on any f is nothing but f acting on x which in the old notation is $f(x)$, of course, this is for every $f \in V^*$, for every $x \in V$. So, this is the, so V^{**} all elements of V^{**} continuous, it is a weak topology, but we are only now asking for a smaller number of functions, namely the functions of the form J_x to be continuous and therefore, this gives you a smaller topology.

So, immediate remark,

Remark: V reflexive means J is onto, then there is nothing new. Weak and weak star topologies coincide on $V^\star$. So, this makes sense only in the non reflexive case. So, if V is finite dimensional, then it is reflexive and we already know that weak and norm topologies are the same. So, norm weak, weak star topologies coincide. So, that is nothing absolutely new in those cases. So, so, if you have weak star, is the weak star topology, so $W^\star$ is contained in the weak topology W, W is contained in the norm topology J. J is norm topology, W is the weak topology, and $W^\star$ is the weak star topology. So,

$$W^\star \subset W \subset J$$

is the inclusion between them.

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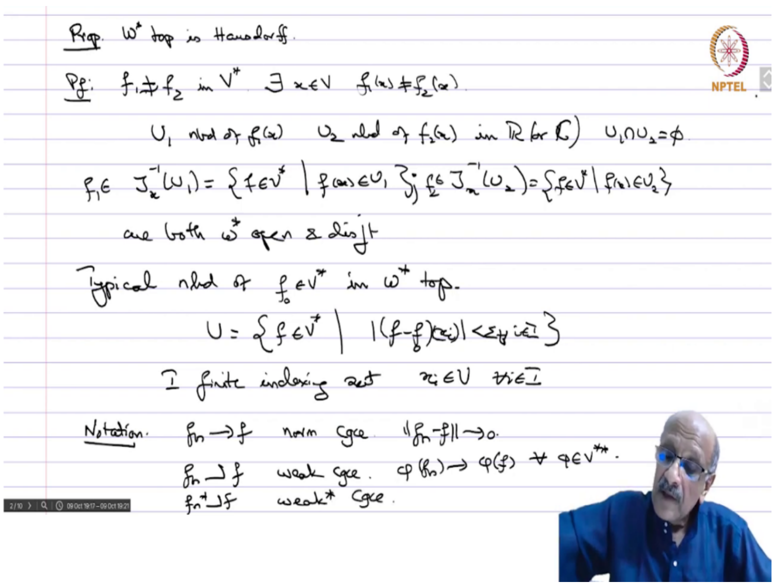

And then, so first immediate proposition, any self respecting topological space better be Hausdorff.

Proposition: $W^{\star}$ is Hausdorff.

Proof: So what should you do, you should find two $W^{\star}$ open sets if so, let $f_1 \neq f_2$. So, what does it mean, that means there exists a $x \in V$ such that $f_1(x) \neq f_2(x)$, this is what we mean when two of the linear functionals are not the same. So, let us take U_1 neighbourhood of $f_1(x)$, U_2 neighbourhood of $f_2(x)$ in R or C if you are dealing with complex spaces, then $U_1 \cap U_2 = \phi$ because the real line or complex plane is itself Hausdorff. So, you now you take $J_x^{-1}(U_1) = \{f \in V^{\star} : f(x) \in U_1\}$ and $J_x^{-1}(U_2) = \{f \in V^{\star} : f(x) \in U_2\}$ are both $W^{\star}$ open by a definition because it is J_x^{-1} something and disjoint. And you have $f_1 \in J_x^{-1}(U_1)$ and $f_2 \in J_x^{-1}(U_2)$. So, you have neighbourhood, $W^{\star}$ neighbourhoods of this. So, we can now also define how is the, how is a typical neighbourhood. So, typical neighbourhood of $f_0 \in V^{\star}$ in $W^{\star}$. So, U must be set of all $f \in V^{\star}$ such that, so just as we had weak neighbourhood was $f - f_0 < \epsilon$. So, you have J_x minus neighbourhood of f_0. So, $U = \{f \in V^{\star} : \left|(f - f_0)(x_i)\right| < \epsilon, \ \forall\, i \in I\}$ where I a finite indexing set and $x_i \in V$ for all $i \in I$. So, you take a finite number of points. So, you take $J_{x_1}, J_{x_2}, J_{x_n}$ and inverse image of that should be less than ϵ. So, so $J_{x_i}(f - f_0)$ which is $(f - f_0)(x_i)$ should be less than ϵ. So, this defines a neighborhood. So, then notation again,

Notation: So we say $f_n \to f$, this is norm convergence. Therefore, $\|f_n - f\| \to 0$. Then $f_n \rightharpoonup f$ this is called the usual weak convergence. What does this mean, $\phi(f_n) \to \phi(f)$ for every $\phi \in V^{\star\star}$. And then we have $f_n \ \star \to f$, this is weak star convergence.

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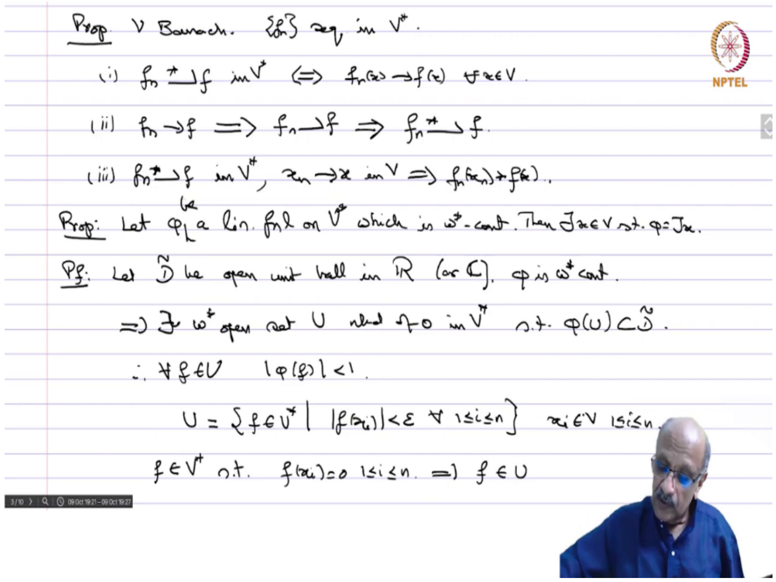

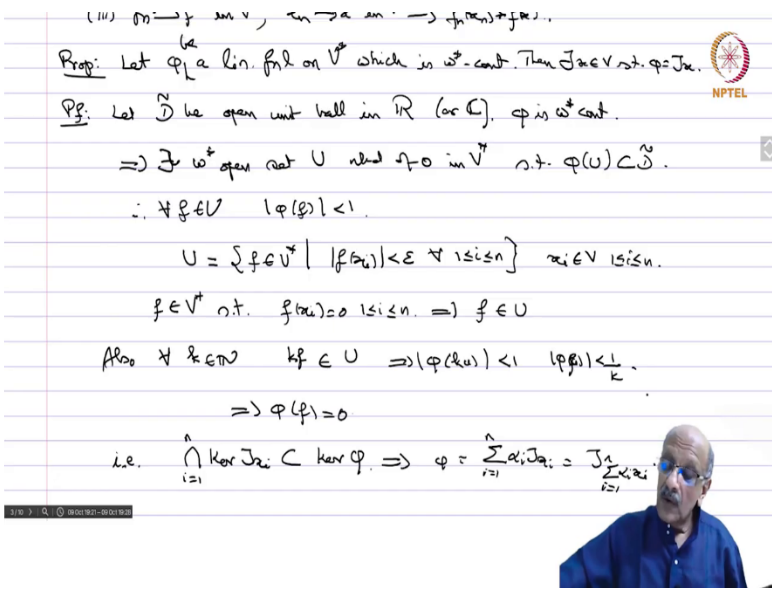

So, we have the immediately following proposition: you can do it yourself because it is exactly like we did the previous proposition. So, proposition

Proposition: V Banach, $\{f_n\}$ sequence in $V^\star$.

1. $f_n \ \star \to f$ in $V^\star$ if and only if $f_n(x) \to f(x)$ for every $x \in V$. One way is obvious, return you just look at the point based neighbourhood system and you will get this.

2. $f_n \rightharpoonup f$. Obviously, we have already seen $f_n \rightharpoonup f$, that means for all functionals it converges, in particular it converges for all j axis and therefore, $f_n \rightharpoonup f$ implies that $f_n \star \rightarrow f$.
3. $f_n \star \rightarrow f$ in $V^\star$ and $x_n \rightarrow x$ in V, this implies $f_n(x_n) \rightarrow f(x)$.

Again, you take the difference, and then you, you just have to, you just take the difference and add and subtract the usual term like we did last time and you will get it. Again, if you have it is convergent in $W^\star$ it has to be bounded because $f_n(x) \rightarrow f(x)$ every x, by the uniform boundedness principle $||f_n|| \leq C$. So, we have that automatically. So, we do not have anything new. So now, proposition

Proposition: Let ϕ be linear functional on $V^\star$ which is, let ϕ be a linear function which is $W^\star$ continuous, it means continuous with respect to the $W^\star$. Then, there exists $x \in V$ such that $\phi = J_x$

.

So, these are precisely the functionals which will be continuous with respect to the W, no other linear functionals which can be continuous. So this is,

Proof: So let $\tilde{D}$ be the open unit ball in R or C depending on what is your base field. It does not matter we are going to (())(12:13) So, so ϕ is $W^\star$ continuous implies there exists $W^\star$ open set U neighbourhood of $0 \in V^\star$ because $\phi(0) = 0$, so it comes to, it belongs to $\tilde{D}$.

So, if you take $\phi^{-1}(\tilde{D})$ it should be an open set, it should contain a neighbourhood of 0 such that $\phi(U) \subset \tilde{D}$. Therefore, for every $f \in U$ we have $|\phi(f)| < 1$. Now what is U going to look like? It is a neighbourhood of the origin. So, $U = \{ f \in V^\star : \left|f\left(x_i\right)\right| < \epsilon, \forall\, 1 \leq i \leq n\}$. So, in some finite collection of points you have taken and this is a neighbourhood. So, assume $f \in V^\star$ such that $f(x) = 0$, then this implies that $f \in U$ because $f\left(x_i\right) = 0$ is strictly less than ϵ, so it belongs to U. And then also for every $k \in N$, you have $kf \in U$, because $(kf)\left(x_i\right) = kf\left(x_i\right)$ and that is also belongs to U. And this implies that $|\phi(kf)| < 1$ or $|\phi(f)| < \frac{1}{k}$. And this implies, that $\phi(f) = 0$. So, if $f(x) = 0$ we have shown that $\phi(f) = 0$, that is $\cap_{i=1}^{n} Ker\left(J_{x_i}\right) \subset Ker(\phi)$. And if you remember, we have done this exercise already in the Hahn-Banach theorem Chapter. This implies that $\phi = \sum_{i=1}^{n} \alpha_i J_{x_i} = J_{\sum_{i=1}^{n} \alpha_i x_i}$. And that is exactly what we claim. So, x the point x is $\sum_{i=1}^{n} \alpha_i x_i$ and that proves this proposition. So, the only functions which are continuous with respect to the $W^{\star}$ are the J_x functions nothing else.

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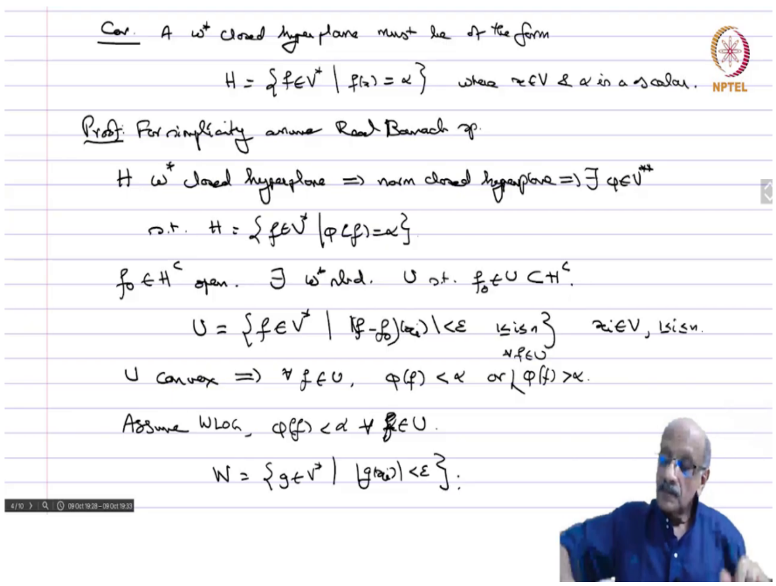

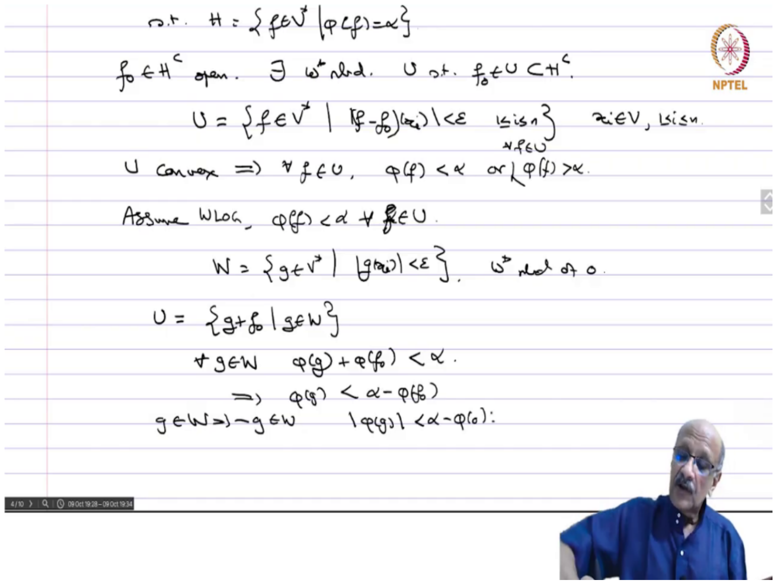

So corollary,

Corollary: A $W^{\star}$ closed hyperplane must be of the form $H = \{f \in V^{\star} : f(x) = \alpha\}$ and α is a scalar.

So, we are going to prove it.

Proof: For simplicity, assume real Banach space. So, H is $W^{\star}$ closed hyperplane, $W^{\star}$ is contained in W which is contained in J. So, implies J closed hyperplane and implies there exists a $\phi \in V^{\star\star}$ such that $H = \{f \in V^{\star} : \phi(f) = \alpha\}$. Because we know these are the only closed hyperplanes which in, in the J. So, α is some real number. So, now let us assume $f_0 \in H^c$ which is now open. So, there exists, so, so there exists a $W^{\star}$ neighborhood U such that $f_0 \in U$ and that is completely contained in H^c. So, what does U look like, U is going to look like set of all $f \in V^{\star}$ such that $\left|(f - f_0)\left(x_i\right)\right| < \epsilon$, $1 \leq i \leq n$ and $x_i \in V$, and ϵ is positive. So, U is convex and therefore, if you remember a proposition where we studied when doing the Hahn-Banach theorem. In fact, there I remarked that everything was dependent on the ball being convex and we would prove the same theorems of Hahn-Banach if you had locally convex spaces, namely spaces with convex neighborhoods and that is exactly what we are having here. And therefore, U is convex.

Therefore, for every $f\in U$ you have $\phi(f) < \alpha$ or $\phi(f) > \alpha$. So, one of these two only will happen, always only one of them. So, for every $f\in U$, $\phi(f) > \alpha$. So, let us assume. So, without loss of generality $\phi(f) < \alpha$ for every, $f\in U$. Now, let us look at $W = \{ g\in V^{\star}: \left|g\left(x_i\right)\right| < \epsilon\}$, this is the neighbourhood of the origin $\left|g\left(x_i\right)\right| < \epsilon$.

So, this is a $W^{\star}$ neighbourhood of the origin. Now, then what is U, $U = \{g + f_0: g\in W\}$. So, this is exactly you put $f + f_0$ then $g + f_0 - f_0$ is g that is less than ϵ. So, every U can be written as, every element of U is nothing but $g + f_0$. So, you get therefore that if for every $g\in W$ you have $\phi(g) + \phi\left(f_0\right)$ that belongs to U, $g + f_0$ is in U, so $\phi(g) + \phi\left(f_0\right) < \alpha$. Therefore, $\phi(g) < \alpha - \phi\left(f_0\right)$. But then g is symmetric namely $g\in W$ implies $-g\in W$ and therefore, if you apply the same thing we get $|\phi(g)| < \alpha - \phi\left(f_0\right)$.

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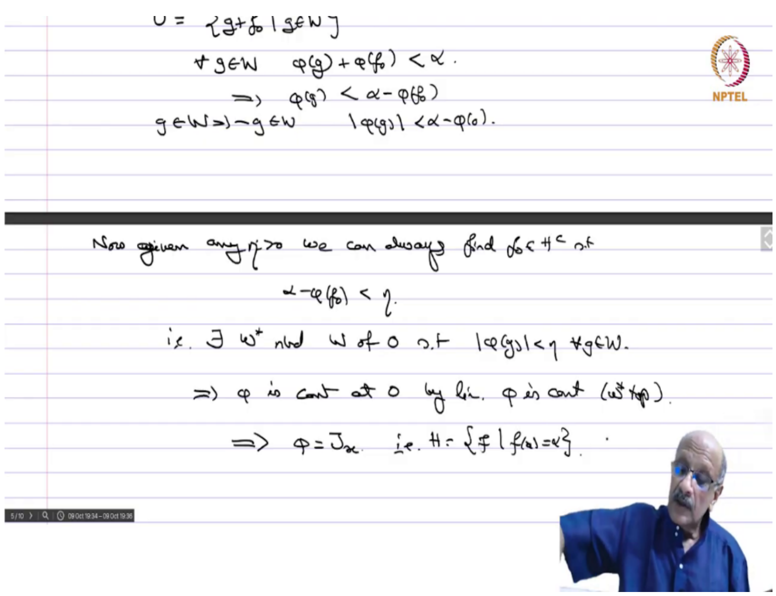

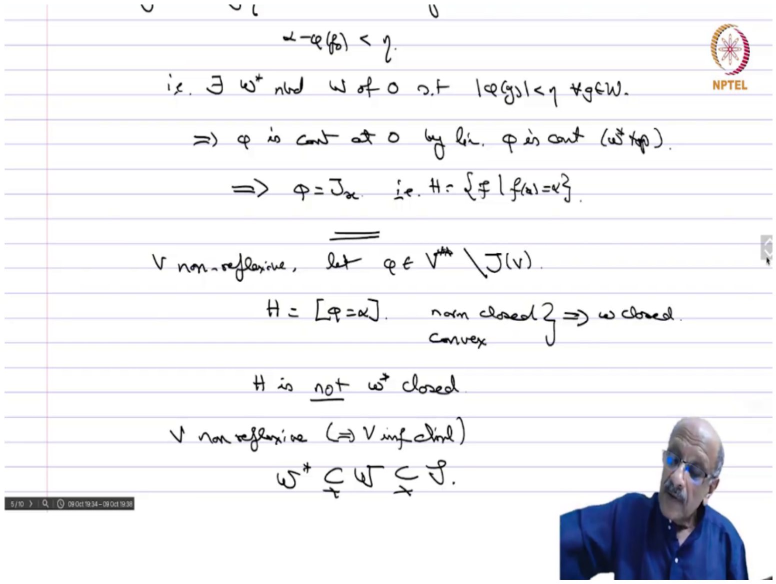

Now, we can given any ϵ or η if you like, maybe I used ϵ somewhere. So, given any $\eta > 0$ we can always find $f_0 \in H^c$ such that $\alpha - \phi(f_0) < \eta$. Of course, you can choose because you draw take a point in H and then you take any point f and then you take H everything is $f(x)$, $\phi(f) = \alpha$. Take a f_0 which is $\phi(f_0)$ which is strictly less than α, draw a line between these two and then as you move along the line by the intermediate value theorem you can get any number which is less than η. So, this is possible.

And therefore, you have that, that there exists W^* neighborhood W of 0 such that $|\phi(g)| < \eta$ for all $g \in W$. This implies ϕ is continuous at 0 and by linearity ϕ is continuous everywhere, this is in W^*. So, ϕ is a continuous linear functional in the W^* and by the previous proposition $\phi = J_x$ that is $H = \{f : f(x) = \alpha\}$. So, that proves the theorem completely.

So, we are in a finite dimensional space we already observed that all the topologies coincide J, W, and W^*. In infinite dimensional spaces we saw the W is strictly less than the smaller than the J. And if the space is reflexive, of course W and W^* coincide. So, now if you take V non reflexive, let $\phi \in V^{**} \setminus J(V)$.

Then you look at the hyperplane $H = [\phi = \alpha]$. So, this is we know J closed because any closed hyperplane is precisely of this form if you take $\phi \in V^{**}$. And this is convex. And therefore, this implies this is W closed. We proved it yesterday if you have a J closed convex set it is automatically W closed. But H is not W^* closed because it is not of the form ϕ is not of the form J_x.

Therefore, it is not W^* closed that is what we have just proved now. So, if you have a non reflexive space the weak so, we non reflexive, that means automatically implies V infinite dimensional then you have W^* is strictly contained in W which is strictly contained in J or the strong topology. So, you have these things.

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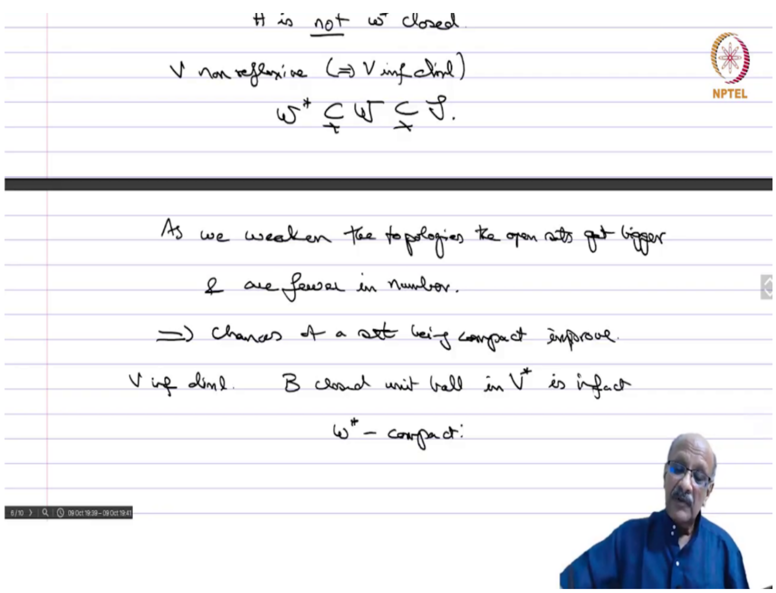

So, now you must be asking this question, what is this madness of weakening our topologies. We had a perfectly good norm topology on all our spaces, which is matrix pay topology and therefore, very easy to work with. Then we threw away some sets and produced a weak topology. And now, we come to a dual space. We throw away some more open sets and then have, so we are impoverishing our topologies continuously, repeatedly.

What is the purpose of this if you noticed the weak, W open sets are pretty big. Yesterday we saw that D bounded open set cannot be W open, because it contains full defined subspaces in it. So, the W open sets are big. And so, as we weaken the topologies the open sets get bigger and are fewer in number. So, this means chances of a set being compact improve. So, what is compactness, every open cover has a finite subcover. So, you have fewer open sets. So already the open sets, open covers come down and then the open sets get bigger.

So, hopefully the finite number of them will be covered. So, the chances of this. And in fact, if V is finite dimensional again and then B closed unit ball in $V^{\star}$ is in fact $W^{\star}$ compact, it is compact in the $W^{\star}$. It is not compact in the J we know, it is not compact in the W either if the topologies are different, but if the, if it is compact in the $W^{\star}$.

So, you see by impoverishing our topologies we increase the chances of getting compact sets. And compact sets means convergence, if you have sequences, nets and so on you will have convergent subsequences and so on. And therefore, that is important. So, this is a very landmark, big landmark theorem in the weak topology chapter, which we will next prove.