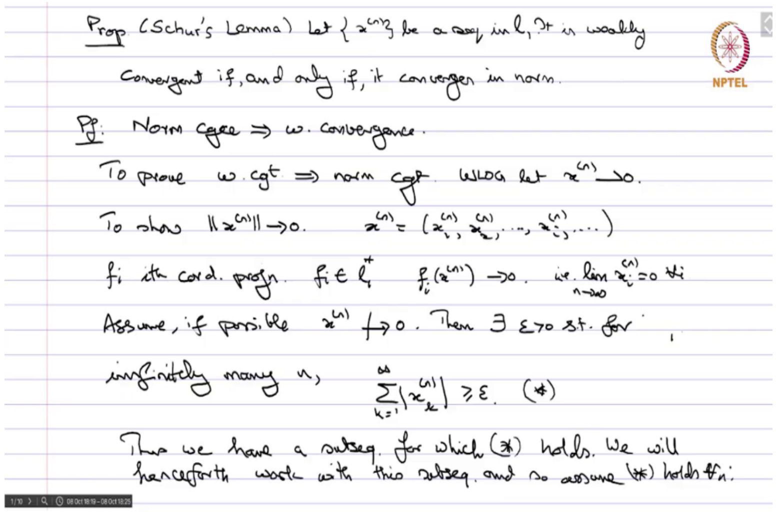
Functional Analysis
Professor. S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No. 28
Weak Topology – Part 3

(Refer Slide Time: 00:16)



Now we will prove a result in l_1 and this is a wonderful source of constructing concrete examples. So, the following proposition is called Schur's Lemma.

Proposition (Schur's Lemma): Let $\{x_n\}$ a sequence in l_1. It is weakly convergent if and only if it converges in norm.

So, this is a remarkable result. So, weakly convergent means convergent in the weak topology. And then it says there is no difference between the convergence sequences in the two topologies. So, proof.

Proof: So, we know that norm convergence implies weak convergence. So, we have, so we have to prove. So, to prove weakly convergent implies norm convergent. So, let us take without loss of generality, let $x^{(n)} \rightharpoonup 0$ because if it converges to anything else we can consider the difference $x^{(n)} - x$ that will converge weakly to 0. So, it is enough to, so to show, $\|x^{(n)}\| \to 0$.

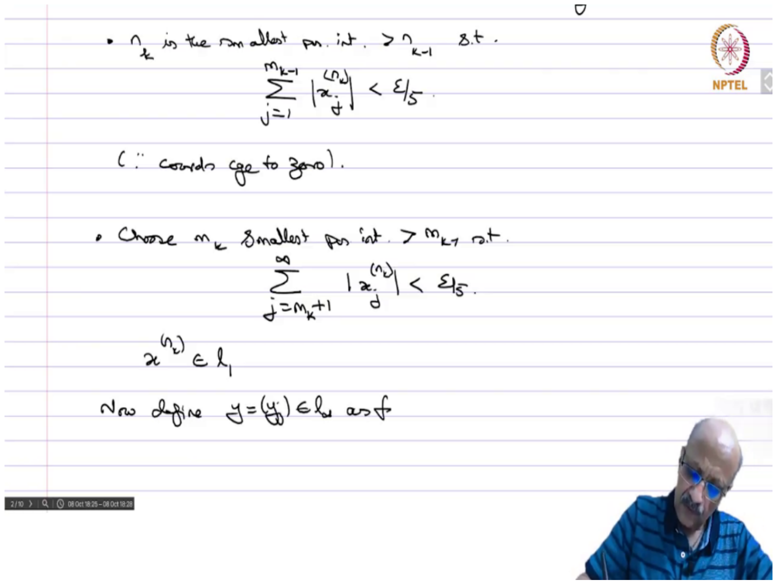
So, let us write $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \cdot, \cdot, \cdot, x_i^{(n)}, \cdot, \cdot, \cdot)$. So, f_i is the i'th coordinate prediction. And that is the continuous linear functional and therefore, you have, so $f_i \in l_1^{\star}$ and therefore, you have that $f_i\left(x^{(n)}\right) \to 0$. That means, that is $x_i^{(n)} = 0$ for all i. So now, assume, if possible, you have that $\{x^{(n)}\}$ does not converge to 0 in norm. Then there exists an $\epsilon > 0$ such that for infinitely many n we have

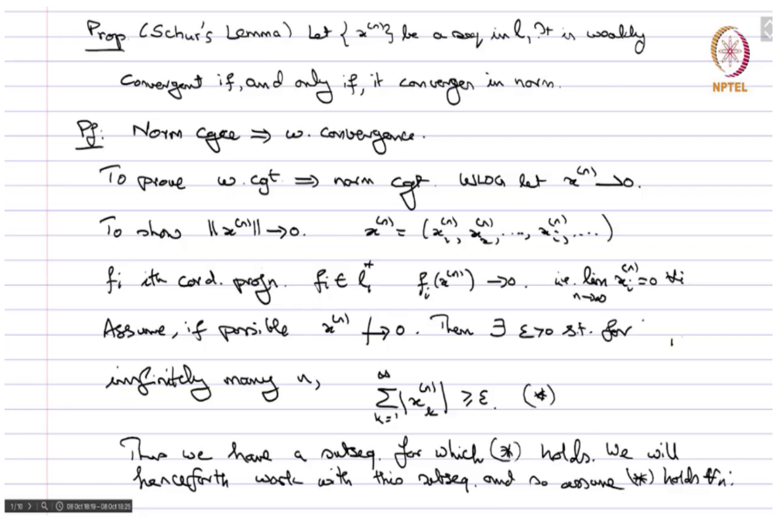
$$\sum_{k=1}^{\infty} \left|x_k^{(n)}\right| \geq \epsilon. \qquad (\star)$$

So, it has to be bounded away from a neighbourhood. So, you take a ball of radius ϵ near 0 the sequence does not converge to 0, means there are infinitely many outside. Thus, we have sub sequence for which ($\star$) holds. So, we will henceforth work with the subsequence.

And so, assume ($\star$) holds for all n, because otherwise I have to put another index for the n and there are too many indices that will be confusing. So, we just work with the sub sequence for which this is true and then we will say. So, given any sequence, so if we can prove that the sub sequence goes to 0 in norm, then automatically every subsequence will have a further subsequence which goes to 0 and therefore, the entire sequence will have to go to 0. So, this is the standard method in topology and therefore, without loss of generality, we can do this.

(Refer Slide Time: 06:08)

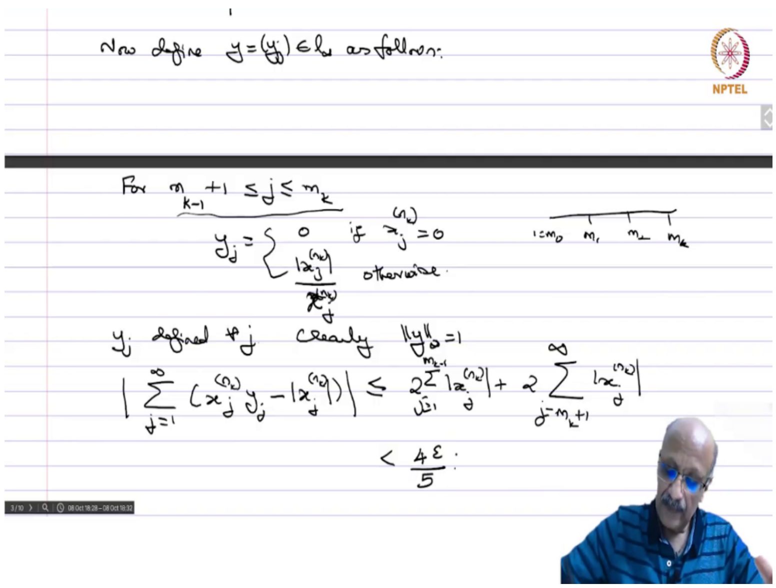


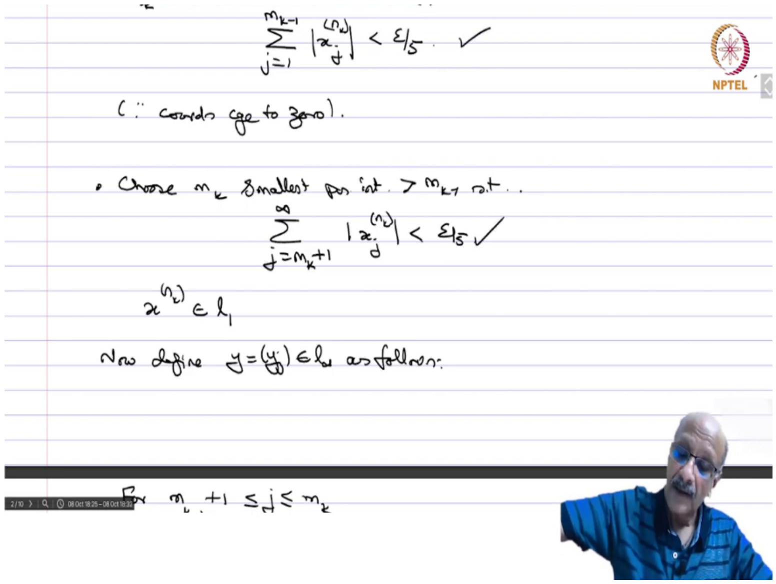


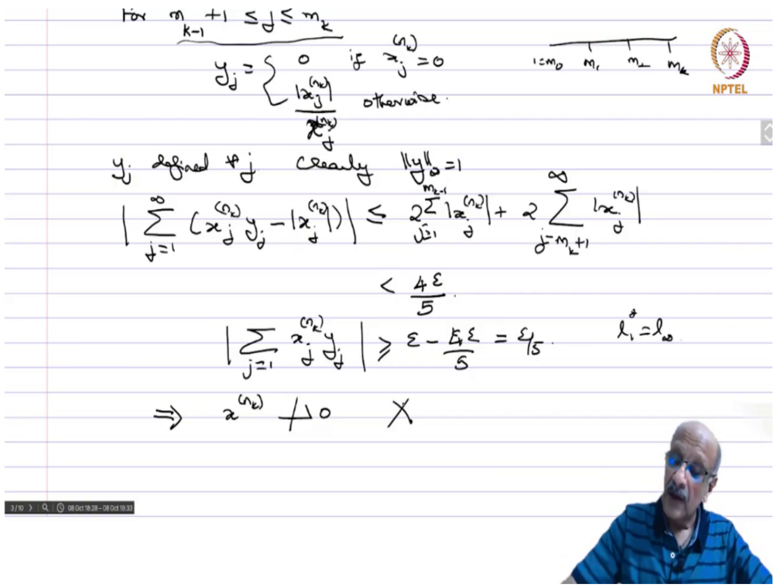
So now, we set $n_0 = m_0 = 1$, for $k \geq 1$ define n_k and m_k inductively as follows. So, the first one, you take n_k is the smallest positive integer strictly bigger than n_{k-1} such that $\sum_{j=1}^{m_{k-1}} \left| x_j^{(n_k)} \right| < \frac{\epsilon}{5}$. Why can you do this? Because we know every coordinate sequence converges to 0 and therefore, here you have only a finite number of such sequences, coordinate sequences, namely $j = 1$ to m_{k-1}. So, m_{k-1} is already determined and therefore, you have a finite number of them.

So, you can always find a sufficiently large m_k such that this is true for all $j = 1$ to m_{k-1} and consequently. Since, coordinate converge to 0. Now, having chosen n_k choose m_k smallest, smallest positive integer strictly bigger than m_{k-1} such that $\sum_{j=m_k+1}^{\infty} \left|x_j^{(n_k)}\right| < \frac{\epsilon}{5}$. Why is this possible? We have that $x^{(n_k)} \in l_1$, therefore this is the tailor for convergent series, so you can always make it as small as you like. So, now define, $y = (y_j) \in l^{\infty}$ as follows:

(Refer Slide Time: 09:24)





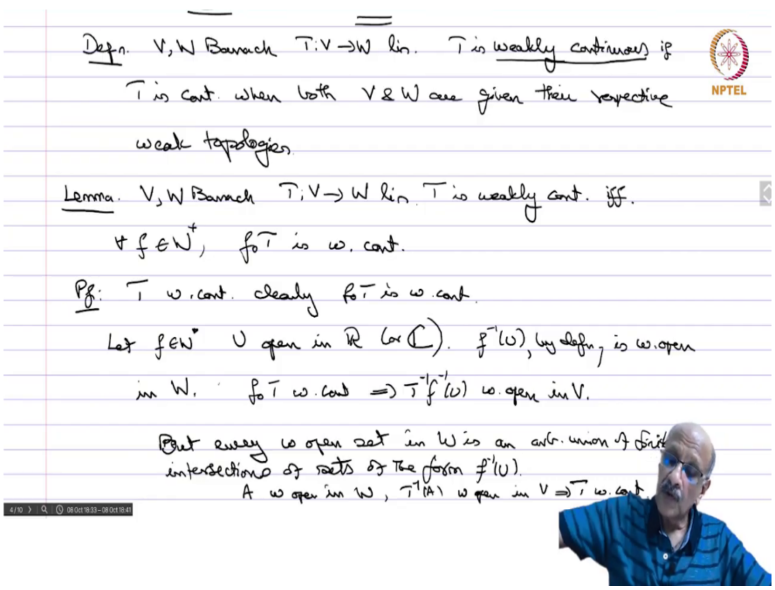


So, for $m_{k-1} + 1 \le j \le m_k$ you will define $y_j = 0$ if $x_j^{(n_k)} = 0$, $y_j = \frac{\left|x_j^{(n_k)}\right|}{x_j^{(n_k)}}$ otherwise. So, $\{m_k\}$ is a subsequence which goes to infinity. So, m_k is strictly bigger than m_{k-1}, and therefore, it is a subsequence which goes to infinity and starting from 1 of course, and therefore, every j will fall in some such interval. So, every 1 falls in some such interval, and therefore, y_j is defined, so y_j defined for all j.

So, you are taking 1 to infinity the integers, so you break them up as $1 = m_0$, then you have m_1, m_2, m_k, and so on. So, in each interval you define y_j, if j falls in any of these intervals, you define them and therefore, you have defined it for all j. Then clearly, $||y||_\infty = 1$, because supremum is just 1. So, now you look at $\left|\sum_{j=1}^{\infty}(x_j^{(n_k)}y_j - \left|x_j^{(n_k)}\right|)\right|$. So, I am going to take the modulus inside. So, I get $\left|x_j^{(n_k)}\right|$, $\left|y_j\right|(\leq 1)$ plus $\left|x_j^{(n_k)}\right|$. So that you will meet $2\left|x_j^{(n_k)}\right|$, but I am going to split this sum $j = 1$ to ∞ as $2\sum_{j=1}^{m_{k-1}}\left|x_j^{(n_k)}\right| + 2\sum\left|x_j^{(n_k)}\right|$. Now, from m_{k-1} to m_k, $x_j y_j = 0$, $x_j y_j$ is $x_j^{(n_k)}y_j - \left|x_j^{(n_k)}\right|$ that gets 0. So, $\sum\left|x_j^{(n_k)}\right|$ will only come from $j = m_k + 1$ to ∞, so in between in the, in this, in this particular interval $m_{k-1} + 1 \leq j \leq m_k$, $x_j y_j = x_j^{(n_k)}y_j - \left|x_j^{(n_k)}\right|$ and therefore, it is killed and therefore, it is. Now by our choice each one of these is less than $\frac{\epsilon}{5}$, so this is less than $\frac{4\epsilon}{5}$. So, the two choices we have made here and here that makes it less than $\frac{4\epsilon}{5}$. Consequently, by the triangle inequality what do you get, $\left|\sum_{j=1}^{\infty}x_j^{(n_k)}y_j\right| \geq \epsilon - \frac{4\epsilon}{5} = \frac{\epsilon}{5}$.

So this, what does this mean? I have found $\left(y_j\right)$ which is in $l_1^{\star}$ is l^∞ remember that. So, $(y_j) = y \in l^\infty$ and I found $x^{(n_k)}$ and therefore, $x^{(n_k)}$ fails to converge to 0 weakly because it is bounded, this thing is away, but that is a contradiction because you know that $x^{(n)}$ converges to 0 weakly, So, any sub sequence must also converge to the same limit weakly. So, this proves the thing and consequently you have established this thing.

(Refer Slide Time: 14:08)



So, what have we shown, so this is, as I said, has lots of applications and source of counter examples. So, the first immediate remark is the following.

Remark: So, l^1 is an infinite dimensional space. Hence, norm and weak topologies are distinct. So, weak topology is strictly contained in the normal topology. So, there are normal open sets like the unit ball open unit ball which is not weakly open and therefore, these topologies are distinct.

So weak topology is strictly contained in the norm topology. So, there are norm open sets like the unit ball, open unit ball which is not weakly open, and therefore these topologies are distinct. However, convergent sequences in both topologies are the same. So, when studying topology you would have seen that sequences are insufficient, and therefore people made a lot of noise. That is why you study filters and nets and these things.

So, in metric spaces, if two metric spaces, if two metric spaces, two topologies have the same convergent sequences then the two metric topologies are one and the same. However, in general topological spaces this is not true and here is an example you have the norm topology which is a metric topology and the weak topology which is different from that they have the same convergence sequences, but the topology is not the same.

So, you cannot just work with sequences. So, this justification for studying nets and filters and so, on. So, this is the, we will see several locations to use Schur's lemma to construct counter examples or show examples for various things. So now, we have a definition

Definition: V, W Banach and $T: V \rightarrow W$ linear. So, we say T is weakly continuous, if T is continuous when both V and W are given their respective weak topologies.

So, you put the weak topology here in V, the weak topology in W and then if T is a continuous mapping there then you say that it is weakly continuous. So now, we want to characterize this, so the lemma.

Lemma: So, V, W Banach T: $V \rightarrow W$ linear, then T is weak weakly continuous if and only if for every $f \in W^{\star}$, $f \circ T$ is weakly continuous. So, $f \circ T$ is a linear map from V into the scalar field R or C whatever you have, let us say R.

And therefore, now weak topology in R, of course, there is only one topology, namely the usual topology which is both the weak and strong. For V if you put the weak topology then f composed with T into R, which is a linear functional must be continuous with the weak topology in V. So, then you say that characterizes the thing.

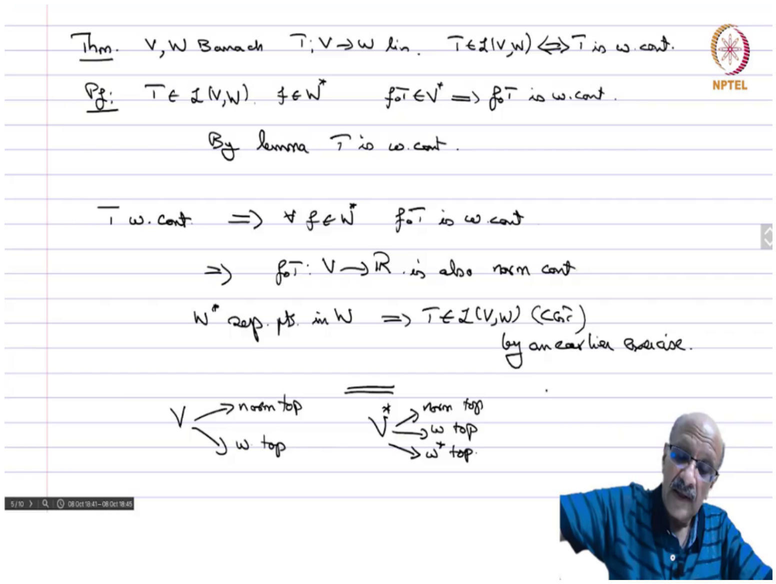
Proof: So assume T is weakly continuous then clearly $f \circ T$ is weakly continuous because what is the weak topology in W it is the best, smallest topology such that each $f \in W^{\star}$ is continuous.

So, every linear, continuously functional is automatically weakly continuous and therefore, if you compose two weakly continuous mapping you get weakly continuous. So, this is, this part is trivial. So, now let $f \in W^{\star}$, and U open in R, of course or C if you like, if you are working with complex spaces.

Then, $f^{-1}(U)$ by definition is weakly open in W. And we are also given that $f \circ T$ weakly continuous implies $T^{-1} f^{-1}(U)$ is weakly open in V. So, T^{-1} that is this, but every weakly open set in W is nothing but is an arbitrary union of finite intersections of sets of the form $f^{-1}(U)$. And therefore, you have T^{-1} of all these sets. So, that means, if, if you have V, not V, so if, so U weakly open not U again, so let me, I need a better symbol. So, if A weakly open in W then

$T^{-1}(A)$ weakly open in V. So, this implies T is weakly continuous. So, this proves the particular lemma.

(Refer Slide Time: 21:44)



So now, we have the theorem, that actually we do not have to make so much of a fuss because of the following theorem.

Theorem: So, V, W Banach, $T: V \to W$ linear then $T \in L(V, W)$ that means it is a continuous linear transformation with the norm topology on both sides, if and only if T is weakly continuous.

So, we made this definition with and made some fuss, but actually usual continuity and weak continuity are one and the same for linear maps. So, that is a very nice thing. So, proof.

Proof: So, let us take $T \in L(V, W)$ then if $f \in W^{\star}$ then $f \circ T \in V^{\star}$ and this implies that $f \circ T$ is weakly continuous because every element of $V^{\star}$ is automatically weakly continuous. And therefore, by lemma T is weakly continuous. Now, let us assume the converse T is weakly continuous and we want to show that T is one-one. So, then what does this mean, T is weakly continuous in place for every $f \in W^{\star}$, we have $f \circ T$ is weakly continuous. But $f \circ T$ is weakly continuous means what, $f \circ T: V \to R$ or C. And that means the inverse image of every open set is weakly open, but weakly open topology is smaller than the norm topology and therefore, f composed with T^{-1} of every open set is also norm open. Therefore, implies $f \circ T: V \to R$ sis also norm continuous. So, if it is continuous with respect to a poorer topology it is going to be certainly continuous with respect to the richer topology because there are more open sets there. And therefore, this is true then we saw this exercise, you know that $W^{\star}$ separates points and in W and we use this and closed graph theorem we did an exercise in the last chapter. And therefore, this implies that $T \in L(V, W)$ by the closed graph theorem, by an earlier exercise. We precisely showed that a linear map is continuous if $f \circ T$ is continuous for all $f \in W^{\star}$. So, this is exactly what we have. So then that proves this.

So, we now have norm Banach space V. So, you have two topologies here. We have the norm topology and we have the weak topology. So now if you have $V^{\star}$ which is also a Banach space you will have the norm topology you have the weak topology and then we are going to define one more which is called the weak star topology and study its properties. So, we will see that next time.