Functional Analysis
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Lecture No. 27
Weak Topology – Part 2

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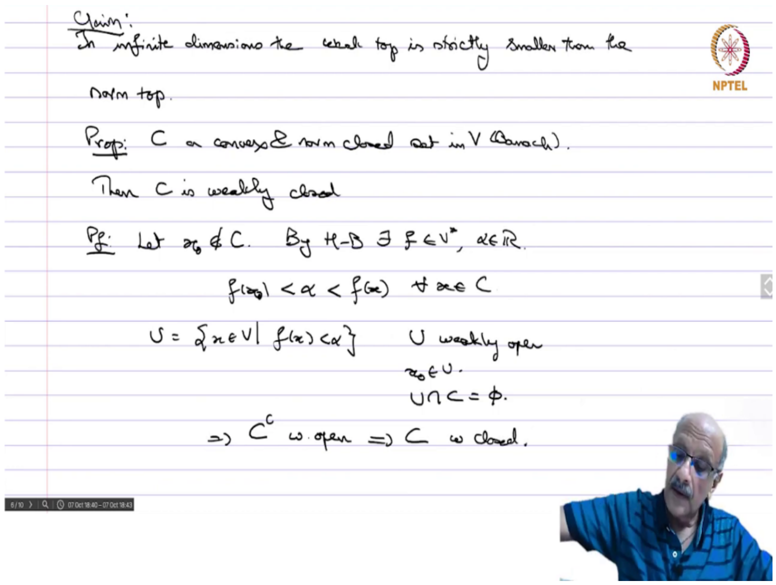

Claim: So, in infinite dimensions the norm, weak topology is strictly smaller than the norm topology.

We will prove the statement, this is only a claim. Before that, so we will show examples of sets which are norm closed but which are not weakly closed, which are norm open, which are not weakly open. So, then obviously the two things are same.

Proposition: C a convex and norm closed set in V, of course V is Banach, then C is weakly closed.

So, as I said, we will show that there exists weak norm close sets which are not weakly close to infinite dimensions. However, convex closed sets are also weakly closed. So, for convex sets, the example which we will take will be non convex obvious. So, let us take C.

Proof: So let $x_0 \notin C$. Then by Hahn-Banach there exists an $f \in V^*$ and $\alpha \in R$ such that $f(x_0) < \alpha < f(x)$. In fact, C is a closed convex set $\{x_0\}$ is a compact convex set, so you can strictly separate them and therefore, you have $f\left(x_0\right) < \alpha < f(x)$ for all $x \in C$. So, now you take $U = \{x \in V: f(x) < \alpha\}$. Then U is weakly open, $x_0 \in U$, $U \cap C = \phi$. So, the complement contains every point. So this means, that C^c is weakly open imply C is weakly closed.

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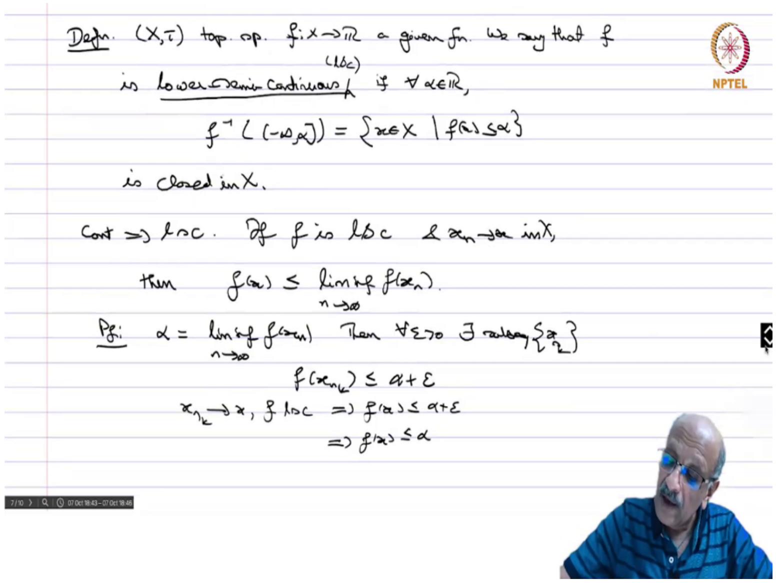

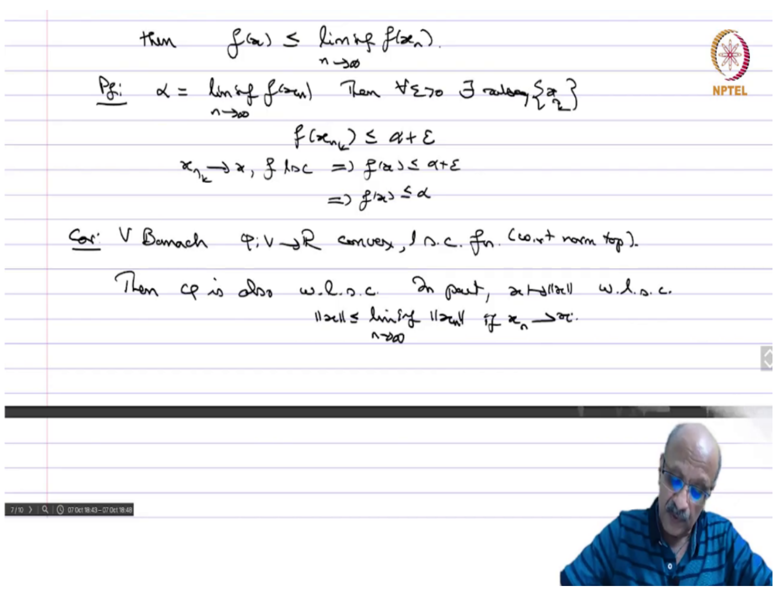

So now we have a definition from topology you might have already seen.

Definition: So, (X, τ) is a topological space and $f: X \rightarrow R$ a given mapping, given function. We say that, f is lower semi continuous if for every $\alpha \in R$ we have $f^{-1}((-\infty, \alpha]) = \{x \in X: f(x) \leq \alpha\}$

is closed in X. So, such a function is called lower semi continuous. So, of course, continuous always implies lower semi continuous. So, Lower Semi Continuous abbreviation is LSC, that is the abbreviation for lower semi continuous.

And therefore, any continuous mapping is lower semi continuous. So, if f is lower semi continuous then we have and $x_n \to x$ in X, then $f(x) \leq f(x_n)$. So, this is a useful result to know. So, why is this true?

Proof: So proof of this statement, $\alpha = f(x_n)$. Then by the definition of lim inf then for the every $\epsilon > 0$ there exists a subsequence $\{x_{n_k}\}$ such that $f\left(x_{n_k}\right) \leq \alpha + \epsilon$ because what is lim inf it is a sup of the inf of the $f(x_n)$. Sup over all n, inf of $m \geq n$, $f(x_n)$. So then, if the sup is equal to α then there must be for $\alpha + \epsilon$ the infs must be between α and $\alpha + \epsilon$. So, this is what we are saying and therefore, this will be true. Now, but then since $x_{n_k} \to x$ and f is lower semi continuous, this implies that $f(x) \leq \alpha + \epsilon$. Now, ϵ is arbitrary, so this implies $f(x) \leq \alpha$ which is the limit. So, this is.

So, now we have a corollary.

Corollary: V Banach and $\phi: V \to R$ convex and lower semi continuous functions with respect to norm topology. So, it is lower semi continuous in the norm topology. Then, ϕ is also weakly lower semi continuous, weakly low semi continuous, that means lower semi continuous in the weak topology.

So, in particular $x \mapsto \|x\|$ is a convex and continuous function, therefore it is lower semi continuous. And therefore, it is weakly lower semi continuous. And of course, $\|x\| \leq \|x_n\|$, if $x_n \rightharpoonup x$. So, this is another proof of the same thing. We already proved the statement and now we have a second proof of the same statement.

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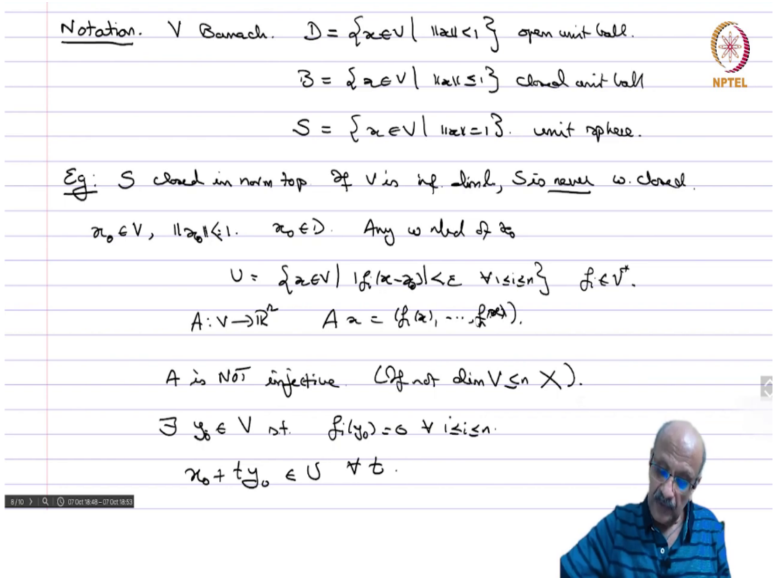

Proof: So you take $\alpha \in R$ and $\phi^{-1}((-\infty, \alpha])$. So, this ϕ is convex so the set is convex. And it is also closed since ϕ is lower semi continuous, in a norm topology. So, convex and closed in the norm topology. And by the previous proposition this implies this is weakly closed and therefore, ϕ is weakly lower semi continuous. So, now some notation.

Notation: We will take V Banach and then we will say $D = \{x \in V: \|x\| < 1\}$, this is the open unit disk. So open unit ball. So, $B = \{x \in V: \|x\| \leq 1\}$ is the closed unit ball. And $S = \{x \in V: \|x\| = 1\}$ is the unit sphere.

Example: S is closed in norm topology if V is infinite dimensional, S is never weakly closed. So, you have a set which is closed in a norm topology which will be never be which will never be close to the weak topology if we see infinite dimensional. So, in the infinite dimensional our claim is established namely, the two topologies are strictly different. So, let us take $x_0 \in V$ and such that $\|x_0\| < 1$. So, $x_0 \in D$.

So, now take any neighbourhood U of x_0 any, weak neighbourhood of x_0. So, what is it, it is $U = \left\{x \in V: \left|f_i(x - y)\right| < \epsilon,\ \forall 1 \leq i \leq n\right\}$. So, some finite set them taking, so I am as a, so $f_i \in V^\star$. So, $\epsilon > 0$ and so. So, now you consider the map $A: V \to R^n$. So,

$Ax = (f_1(x), \cdot, \cdot, \cdot, f_n(x))$, we have done this kind of mapping before. Then A is not injective, because if A were injective then you have a one one map from V into R^n this, because if not dimension of V will be less than or equal to n which is a contradiction because V is infinite dimensional. So, this is not possible. So V, so there exists x, so let me call it y_0, $y_0 \in V$ such that you have that $f_i(y_0) = 0$ for all $1 \leq i \leq n$. Then you take $x_0 + ty_0$, this belongs to U for all t because $f_i(y_0) = 0$, so if I just added it nothing will change. So, if $x_0 \in U$, then you have, therefore this is also there.

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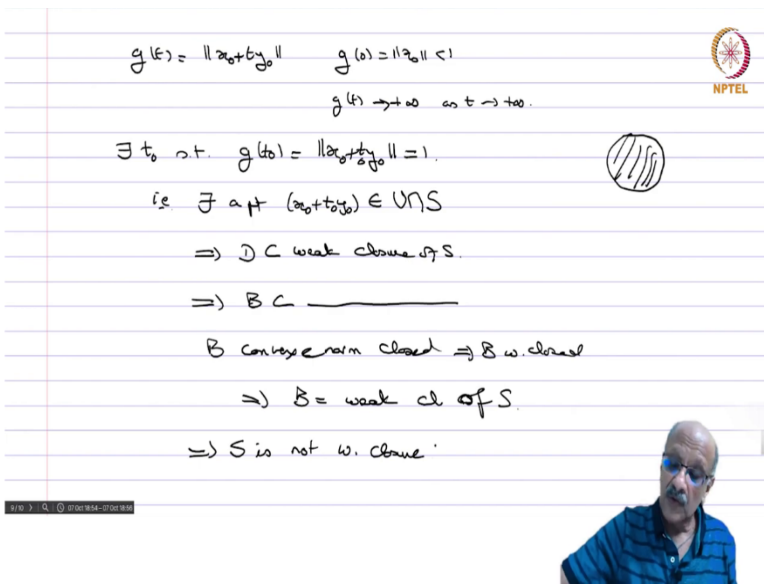

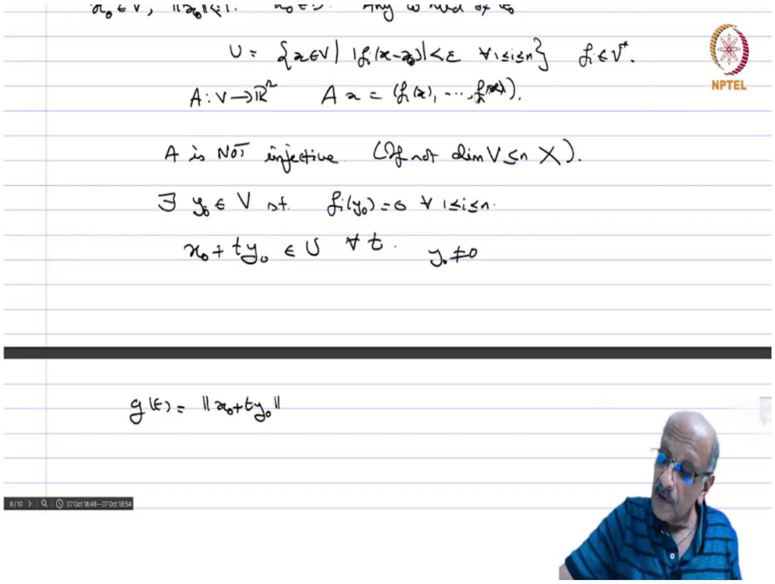

Now, you will look at $g(t) = \|x_0 + ty_0\|$. So, this is y_0, $y_0 \neq 0$, because it is not injective. So, you have a (())(14:34). So, then $g(0) = \|x_0\| < 1$ and $g(t) \rightarrow +\infty$ by the triangle inequality as $t \rightarrow +\infty$. So there exists a t_0 such that $g(t_0) = \|x_0 + t_0 y_0\| = 1$.

What does this mean? That this there exists a point $x_0 + t_0 y_0 \in U \cap S$. So, any neighbourhood of x_0 will intersection, intersect S and therefore, D is contained in the weak closure of S. And S is already there this means that B is contained in the weak closure of S, but then B is convex and norm closed implies B is weakly closed therefore, B equals weak closure of S.

So, you have the unit sphere in the weak topology whose closure is in fact the entire closed ball. So, so, this is a very strange situation and therefore, you have that S is not weakly closed because its closure is much bigger than, So, this shows that the two are entirely different.

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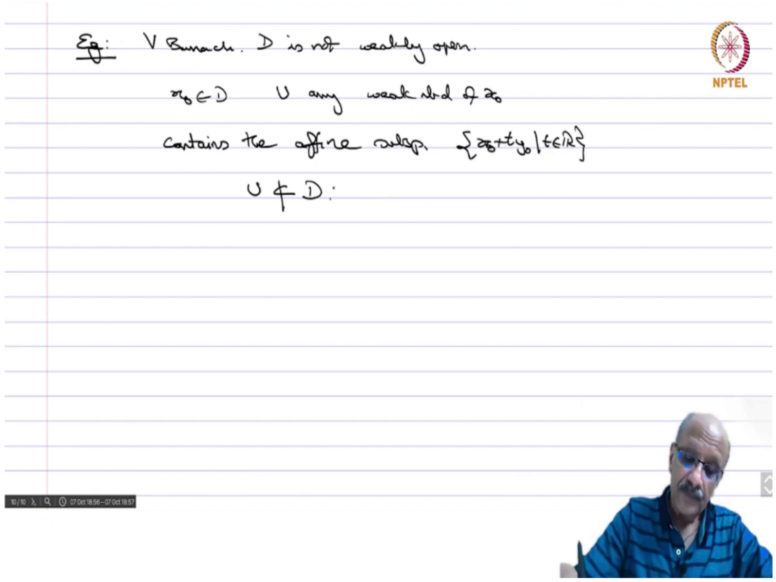

So, another example which is almost the same as the previous one.

Example: So, V Banach then D is not weakly open. So, you know, in the norm topology, the ball is the fundamental thing. So, if here you are saying the unit, open unit ball, which is a norm open set, that is the basic kind of open set the ball, this is not weakly open, why is it not weakly open.

So, if you took $x_0 \in D$, then you have any neighbourhood any weak neighbourhood of x_0 contains the affine subspace $\{x_0 + ty_0 : t \in R\}$. So, it contains the whole line, it contains the whole line passing through x_0 and it is a very big set. So, open neighbourhoods are very large and therefore, it does not contain and therefore U is not contained in D. So, D cannot contain any weak neighbourhood and therefore D is not weakly open.