Functional Analysis
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Lecture No. 26
Weak Topology – Part 1

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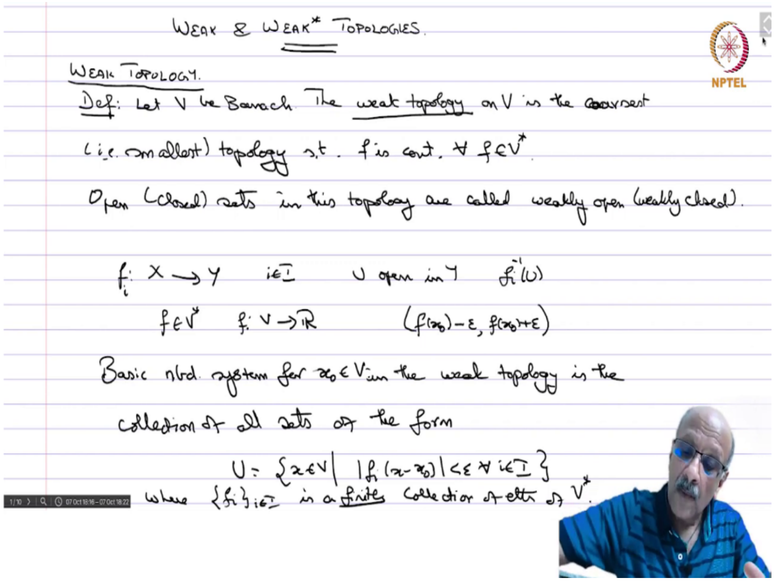

We will now start a new chapter. So, we are going to study weak and weak star topologies. So, up to now we have been dealing with the norm topology. So, this is symmetric topology. So, now we want to study other kinds of topologies on a Banach space and these have a lot of very nice applications. So, we will start with defining.

So, first we will study the weak topology.

Definition: So, let V be Banach. The weak topology on V is the coarsest that is smallest topology such that f is continuous for every $f \in V^*$. So, we are going to economize on the number of open sets. When I say smallest topology, I mean the least number of open sets possible, so that the members of the dual space are still continuous. So, open respectively closed sets in this topology are called weakly open, similarly weakly closed.

So, how do you construct such a topology? So, this is well known in topology how to construct given a set and a set of mappings $f_i : X \to Y$. So then, i in some index set I. So what is the smallest topology such that all these f_i are continuous. So, if U is open in Y then $f_i^{-1}(U)$ must be open in X. And then every open set in the topology here must contain all finite intersections of this and all arbitrary unions of finite intersections of these.

So, that is how the topology on X is constructed. So therefore, a basic open neighbourhood. So, what is here we are taking $Y = R$. So, we have $f \in V^{\star}$, so f goes from V to R or C it is whatever I am saying for R works for C also. So, we want to know what is a neighbourhood here. So, if I take the neighbourhood, so if I take x_0, then x_0 goes to $f(x_0)$.

So, we will take a neighbourhood $(f\left(x_0\right) - \epsilon, f\left(x_0\right) + \epsilon)$ and you all inverse images of this should be open and then you must take finite intersection of such sets and therefore, that will constitute a basic neighbourhood system at each point and then from this you can construct the rest of the open set.

So, the basic neighbourhood system for $x_0 \in V$ in the weak topology is the collection of all sets of the form. So, U will be set of all $x \in V$ such that $f(x)$ should belong to $(f\left(x_0\right) - \epsilon, f\left(x_0\right) + \epsilon)$ and therefore, $\left|f_i(x - x_0)\right| < \epsilon$, for all $i \in I$ where $\{f_i\}_{i \in I}$ is a finite collection of elements of $V^{\star}$.

So when I want to take an intersection of sets of this form, I take a finite number of functionals and then take the inverse image of all that and take the intersection. So, this should be true for all $i \in I$ and so, this is form, forms a neighbourhood system for this, so given any point x_0 a typical neighbourhood in the weak topology will look like this.

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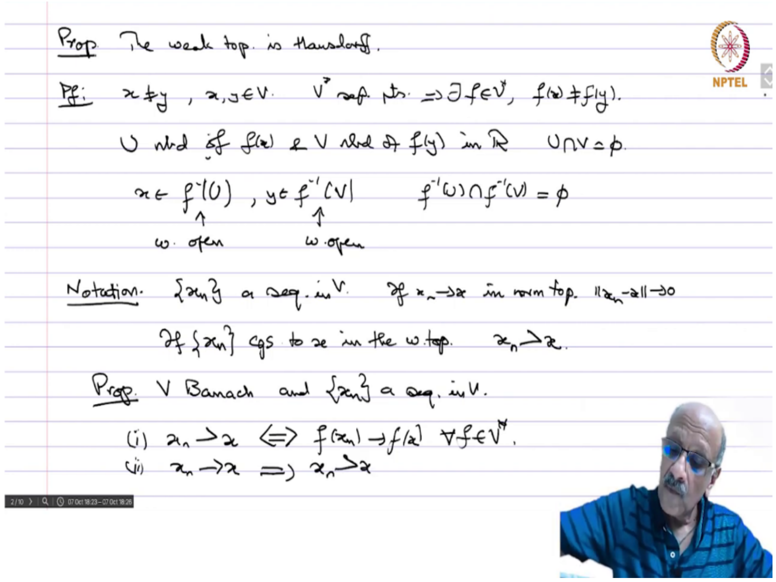

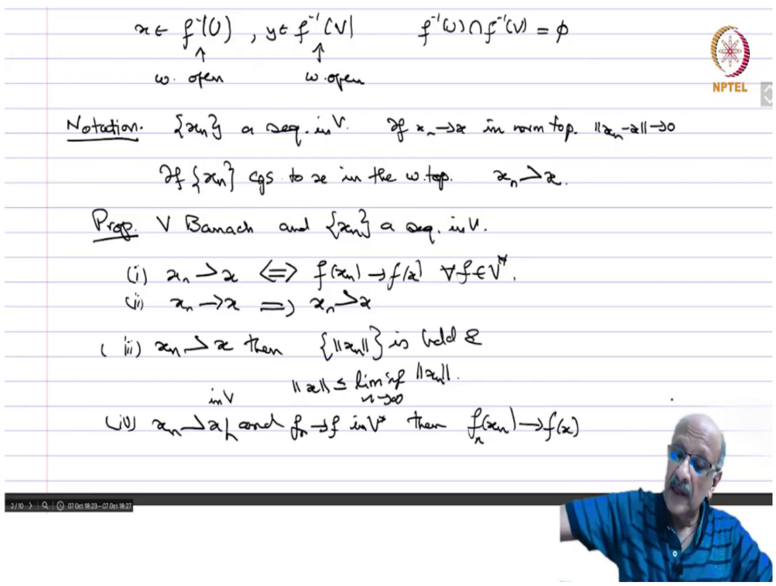

So, first proposition

Proposition: The weak topology is Hausdorff. So, it is important because you know, when we define even a topological vector space, we always want the topology to be Hausdorff.

Proof: So, let us take $x \neq y,\ x, y \in V$. So, we want to have disjoint neighbourhoods. Therefore, $V^{\star}$ separates points implies there exists an $f \in V^{\star}$ such that $f(x) \neq f(y)$. So, let us take U neighbourhood of $f(x)$ and V neighbourhood of $f(y)$ in R or C or whichever you want and then $U \cap V$ is empty because the real lines or complex plane is Hausdorff anyway and therefore, you can always find this. And therefore, you if you take $f^{-1}(U)$, $x \in f^{-1}(U)$ and $f^{-1}(V)$, $y \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V)$ will be empty and these are all open, weakly open, this is also weakly open because you and we are open and f is continuous in the weak topology and therefore, these are weakly open.

So this proves, so given any point I found two disjoint neighbourhoods and therefore, you have this. So, notation,

Notation: So $\{x_n\}$ a sequence in V. So, if $x_n \to x$ in norm topology, what do you mean that is $\|x_n - x\| \to 0$ as $n \to \infty$. Then we write $x_n \rightharpoonup x$ if $\{x_n\}$ converges to x in the weak topology. So, we call it weakly convergent, then we write x_n with a half arrow like this except weakly converges to x.

So, now we have another proposition.

Proposition: V Banach and $\{x_n\}$ a sequence in V.

1. $x_n \rightharpoonup x$ if and only if $f\left(x_n\right) \to f(x)$ this in the real line or complex plane for every $f \in V^{\star}$.
2. $x_n \to x$, it converges weakly as well. So norm topology is stronger.
3. If $x_n \rightharpoonup x$ then $\|x_n\|$ is bounded and $\|x\| \leq \|x_n\|$. And then
4. We have if $x_n \rightharpoonup x$ in V and $f_n \to f$ in $V^{\star}$. Then $f_n(x_n) \to f(x)$. So, remember you have two sequences here. So, if one is strong and the other is weak then the limit will be the correct one.

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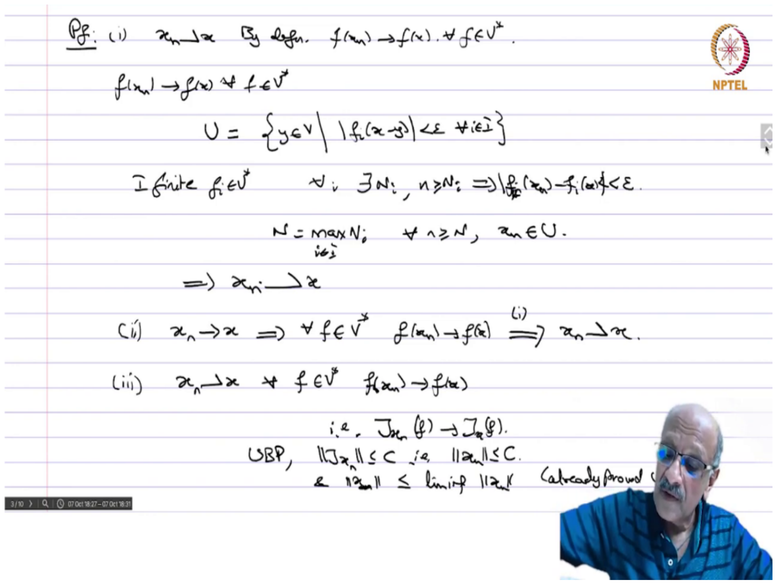

So then, so proof,

Proof: 1. So what $\{x_n\}$, so goes to x weakly, then by definition, since all the f's are continuous for this topology $\{f(x_n)\}$ has to converge to $f(x)$. So, now, let us assume that $f(x_n) \to f(x)$ for every $f \in V^*$. Then how does the neighbourhood of x look like? So, U is a neighbourhood of x if you remember. This all the way $y \in V$ such that $|f_i(x - y)| < \epsilon$ for all $i \in I$ and I is finite, and f of course belongs to U^*. So, this is how a neighbourhood would look like. Now, if $\{f(x_n)\}$ converges to $f(x)$, then for every i you have there exists N_i such that $n \geq N_i$ implies $|f_i(x_n) - f_i(x)| < \epsilon$ because you have $f(x_n) \to f(x)$ for every $f \in V^*$. Now, you take N to be the maximum of the N_i, I is a finite set you can take that and therefore, for all $n \geq N$ you have that $x_n \in U$. So, after some time the sequence belongs to any neighbourhood and therefore, so this implies that $x_n \rightharpoonup x$. So, given any weakly, weak neighbourhood of x you have that after N you have $x_n \in U$ and that is precisely the definition of convergence.

2. So $x_n \to x$, so implies for every $f \in V^*$ we have $f(x_n) \to f(x)$ and this implies by one you have that $\{x_n\}$ converges to x weakly.

3. So $x_n \rightharpoonup x$ in V that means, for every x, for every $f \in V^*$, we have $f(x_n) \to f(x)$, that is $J_{x_n}(f) \to J_x(f)$. And this implies that by the uniform boundedness principle we have that $\|J_{x_n}\| \leq C$ that is $\|x_n\| \leq C$. So, this shows that the sequence is bounded.

Then we also showed when you have a sequence in which you have every point converges then the limit transformation satisfies this condition. So, you have and $\|x\| \leq \|x_n\|$ already proved. Uniform boundedness chapter we have already seen this proof.

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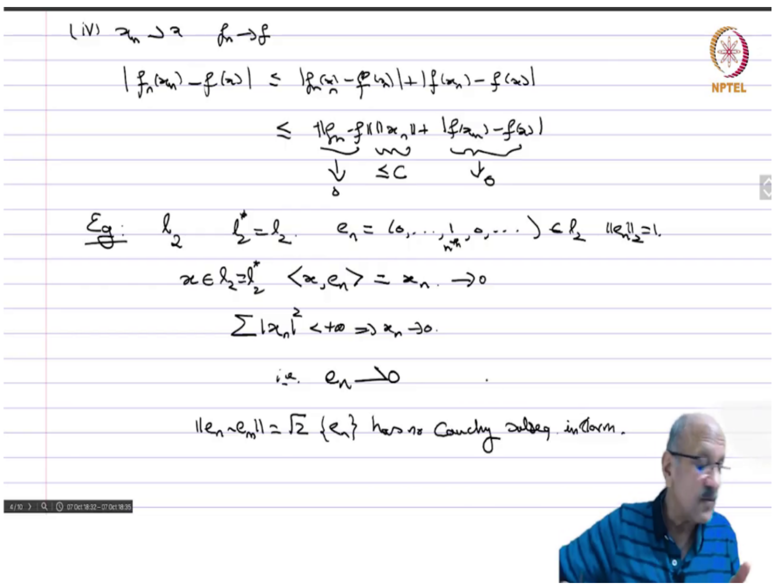

And the fourth one. 4. So, we have $x_n \rightharpoonup x$ and $f_n \to f$ in V^*. So, we want $\left|f_n(x_n) - f(x)\right| \leq \left|f_n(x_n) - f(x_n)\right| + \left|f(x_n) - f(x)\right|$. The first one is $\left|f_n(x_n) - f_n(x)\right| \leq \|f_n - f\|\|x_n\|$. Now, by the first part of this proposition $\left|f(x_n) - f(x)\right| \to 0$ since $x_n \rightharpoonup x$ and then $\|f_n - f\| \to 0$ and $\{x_n\}$ is bounded by the part 3 and therefore, you have the whole thing goes to 0. And we have what we want.

So, this proves this proposition completely. So, now, let us look at an example.

Example: You look at l_2, so this is the set of all square summable sequences with the corresponding norm and then you have that $l_2^{\star}$ is identified with the l_2 in the usual way. So, now you look at $e_n = \{0, \cdot, \cdot, \cdot, 1, 0, \cdot, \cdot, \cdot\} \in l_2$, $\|e_n\|_{l_2} = 1$. So, now if you take any $x \in l_2$ then you will look at $\langle x, e_n \rangle$ or rather take l_2 as $l_2^{\star}$. So, $\langle x, e_n \rangle$ and that is equal to x_n. But then what do you know about x, $x \in l_2$ so $\sum x_n^2 < \infty$. So, this implies that $x_n \to 0$. For $\langle x, e_n \rangle \to 0$ for every $x \in l_2$, which is the dual space and therefore, you have that $e_n \rightharpoonup 0$. For every linear functional $\langle x, e_n \rangle \to 0$ and therefore, $e_n \rightharpoonup 0$. But remember, that $\|e_n - e_m\| = \sqrt{2}$ and therefore, $\{e_n\}$ has no Cauchy subsequence in norm. Therefore, for no subsequence $\{e_n\}$ will converge in norm and therefore, we know that norm convergence implies weak convergence but the converse is not true, weak convergence does not imply norm convergence.

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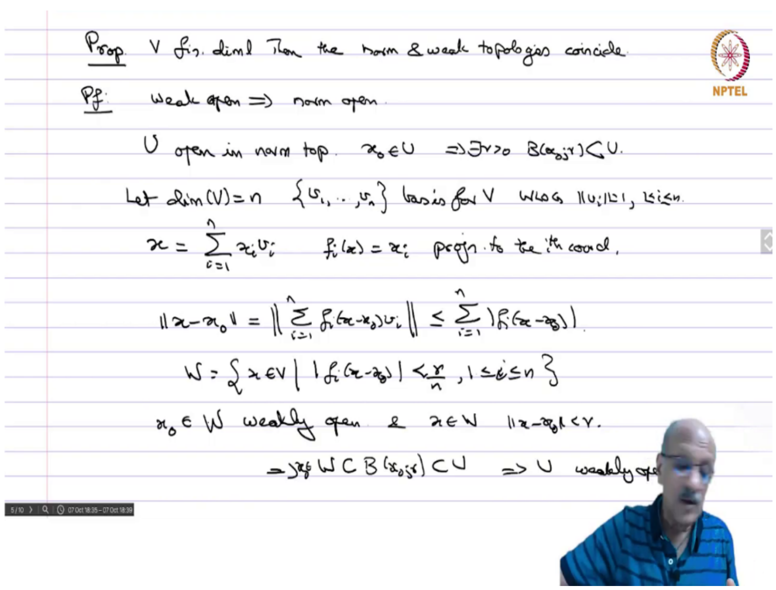

Proposition: V finite dimensional then the norm and weak topologies coincide. So, there is no difference so, if anything is interesting it is only in infinite dimensions.

Proof: So, weak topology weakly open implies norm open because the norm topology is much bigger, weak topology is a smaller topology, so every element of the weak topology is a member of the norm topology. So, now let us take U be open in norm topology.

So, let $x_0 \in U$ and let us then implies there exists an $r > 0$ such that the $B\left(x_0; r\right) \subset U$. So, so, let us assume dimension of V equal to n and then we want $\{v_1, \cdot, \cdot, \cdot, v_n\}$ the basis for V and without loss of generality we can assume $\|v_i\| = 1$ for all $1 \leq i \leq n$. So, then $x = \sum_{i=1}^{n} x_i v_i$. And so you take $f_i(x) = x_i$. So, projection to the i'th coordinate. So, then these are all continuous linear functionals. So, then you have $\|x - x_0\| = \|\sum_{i=1}^{n} f_i(x - x_0) v_i\|$. So, that is how the projections have been defined. $\|\sum_{i=1}^{n} f_i(x - x_0) v_i\| \leq \sum_{i=1}^{n} \left|f_i(x - x_0)\right|$. So, now you take $W = \{x \in V: \left|f_i\left(x - x_0\right)\right| < \frac{r}{n}, 1 \leq i \leq n\}$. So, then W is weakly open, this is a typical neighbourhood of x_0, $x_0 \in W$ and W is weakly open and then if all of these are less than $\frac{r}{n}$ then you have and if $x \in W$ we have $\|x - x_0\| < r$ and therefore, $W \subset B\left(x_0; r\right)$ and $x_0 \in W$. So, $B\left(x_0; r\right)$ contains a weak neighbourhood of every point in it implies which is of course, contained in U implies, U is weakly open. Because every point in it contains a weak neighbourhood inside the same set.

And therefore, every norm open set is weakly open. Every weak open is norm open therefore, the two topologies are one and the same. So, we only want to, if at all there is going to be a difference, we have it only infinite dimensions.