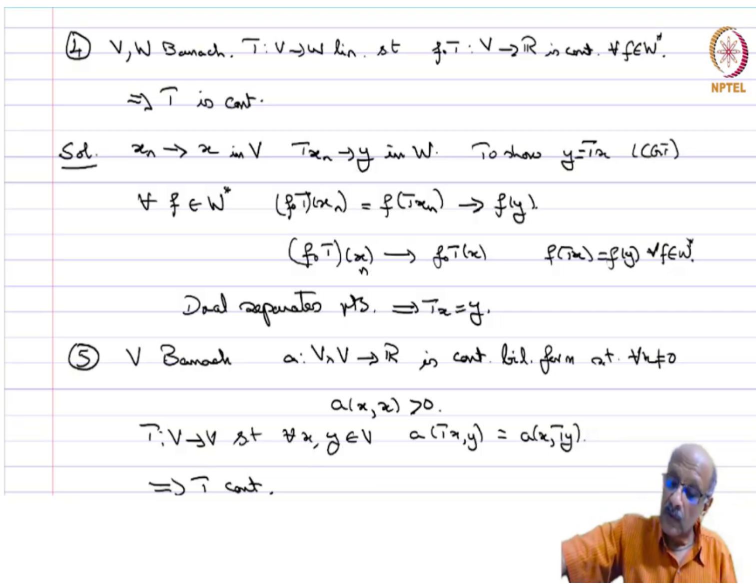
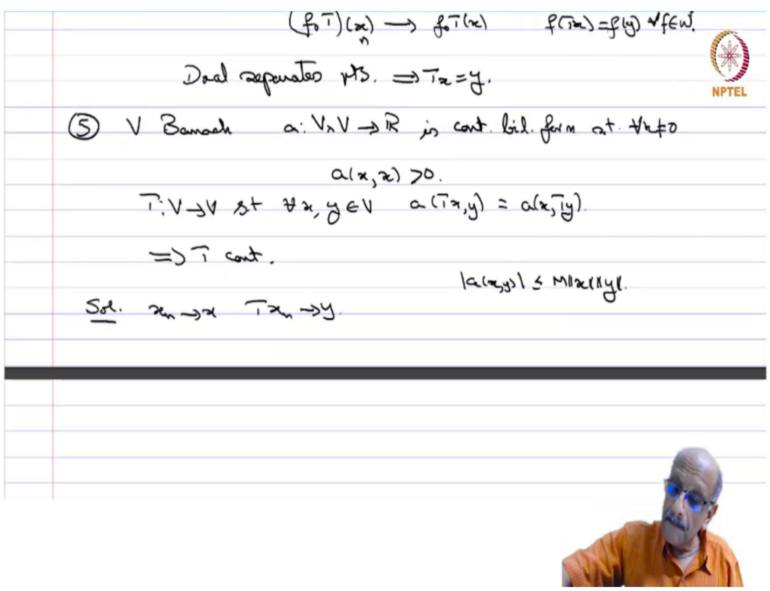
Functional Analysis
Professor. S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No. 25
Exercises (Contd…)

(Refer Slide Time: 00:16)





Problem 4: So V, W Banach and $T: V \rightarrow W$ linear such that $f \circ T: V \rightarrow R$ is continuous for all $f \in W^{\star}$. So, you take any $f \in W^{\star}$, $T: V \rightarrow W$ and then you apply f so you get a linear functional. So $f \circ T$ is a linear function. Show that T is continuous.

Solution: So, we will try to use the close graph there. So, let us take $x_n \to x$ in V and $T(x_n) \to y$ in W. So, to show $y = Tx$, so this is what we want to show and then the graph is closed and then we will apply the closed graph theorem, then we can apply the closed graph. So, now we know that f composed so, for all $f \in W^*$ we have $(f \circ T)(x_n)$, so this is f composed with T acting on x_n is $f(Tx_n)$. $T(x_n) \to y$, so $f(Tx_n) \to f(y)$ but $x_n \to x$. So, you have this is also equal to, so $(f \circ T)(x_n) \to (f \circ T)(x)$. So, $f(Tx) = f(y)$ for all $f \in W^*$. And since we know that the dual separates points, so dual separates points by the Hahn Banach theorem, so this implies that $Tx = y$. So, by the closed graph theorem.

So, you see how easily we can use the close graph here. So, this and the next exercise again,

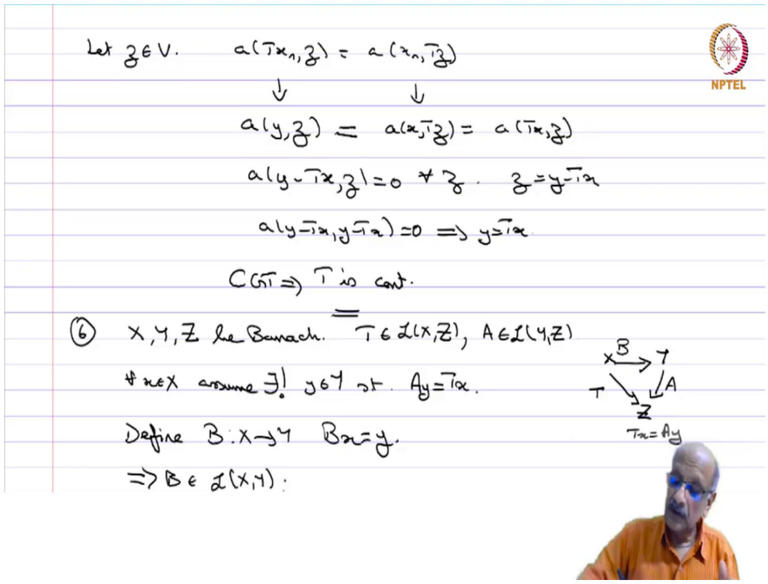
Exercise: So, V is Banach and $a: V \times V \to R$ is a continuous bilinear form such that for all $x \neq 0$, $a(x, x) > 0$. $T: V \to V$ such that for all $x, y \in V$, we have $a(Tx, y) = a(x, Ty)$. So, this implies T is continuous.

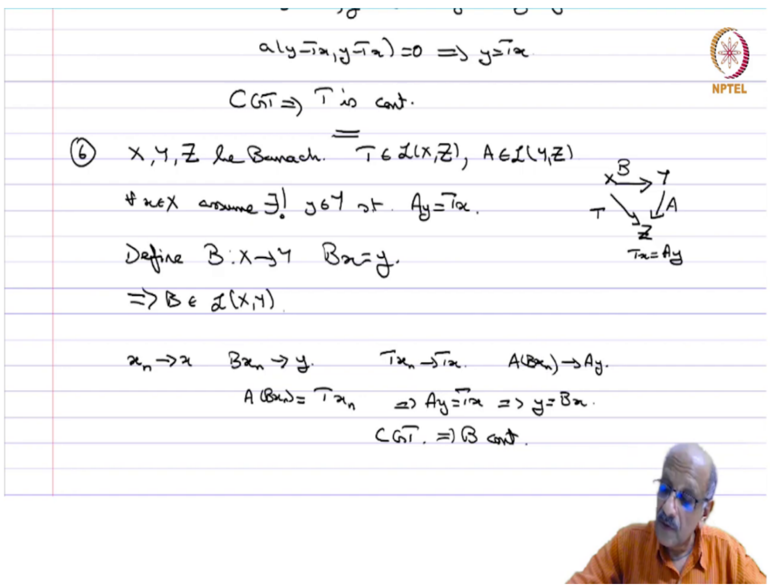
So, you see here we have a completely algebraic statement that there is no convergence, no analysis, no continuity nothing is said we have a bilinear form and with respect to the bilinear form T is symmetric. So, this is purely an algebraic statement and we are going to deduce the continuity. Now, just a word about what is a continuous bilinear form, that means $a(x, y) \leq M\|x\|\|y\|$. And then, so if you fixed one variable then it is continuous in the other variable, that is what this tells you.

And then, so (())(04:42) is strictly and bilinear means linear in each variable, that is all. So now again, we are going to try to use the closed graph theorem. So, solution.

Solution: So, $x_n \to x$ and let us assume the $T(x_n) \to y$.

(Refer Slide Time: 05:07)





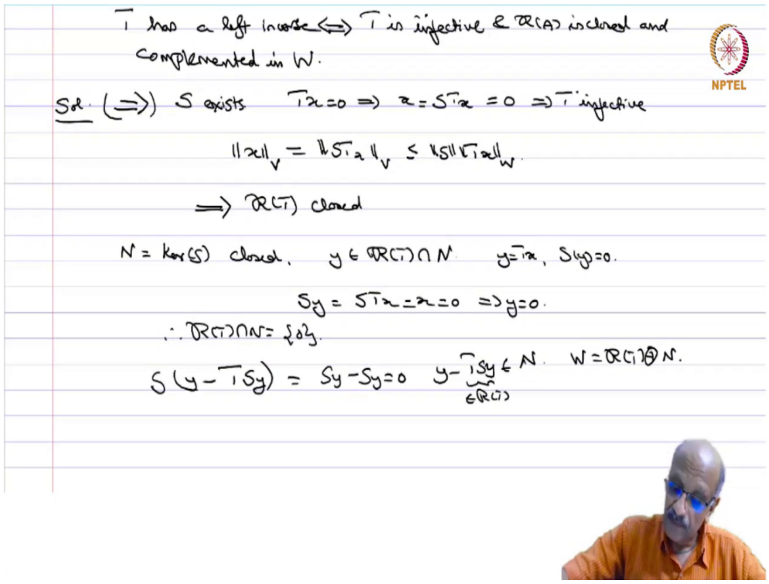
So, let $z \in V$. So, then you have $a\left(Tx_n, z\right) = a(x_n, Tz)$. So, now you pass to the limit. So, it is linear in each variable and because of the continuity condition it is continuous when the second variable is, each variable is fixed $Tx_n \rightarrow y$, so $a\left(Tx_n, z\right) \rightarrow a(y, z)$ and $a\left(x_n, Tz\right) \rightarrow a(x, Tz)$. So, these two are equal, and then this is equal to $a(Tx, z)$ by the given condition, $a(x, Tz) = a(Tx, z)$.

So, you have that $a(y - Tx, z) = 0$ for all z. So, you take $z = y - Tx$, so you get $a(y - Tx, y - Tx) = 0$. And this is possible only if $y = Tx$ because if it were not 0, $a(y - Tx, y - Tx)$ will be strictly positive. So, that is the given condition and therefore, by the CGT we have that T is continuous.

Problem 6: So let X, Y, Z be Banach and T a continuous linear operator from X to Z and A is continuous linear operator from Y to Z. For every $x \in X$ assume there exists a unique $y \in Y$, such that $Ay = Tx$. So, you have X here, you have Y here, so here I have a map from each of them into Z. So, X to Z is T and Y to Z is A. So, for given any x, I bring Tx here and then that should be equal to Ay. And then I define B. So, define B from X to Y given by $B(x) = y$. Then $B \in L(X, Y)$. B is a continuous (())(08:12) So, B is linear that is all no problem. So, we will just have to show continuity.

Solution: So, again we are going to use the close graph theorems one more application. So, $x_n \rightarrow x$ and then you have $B(x_n) \rightarrow y$. So, $x_n \rightarrow x$, so $Tx_n \rightarrow Tx$ and $A(Bx_n) \rightarrow Ay$. But what do you know, $A(Bx_n)$ by definition is equal to Tx_n that is how B was defined $Ay = Tx$, so Bx_n this. So, Bx_n goes to Ay so this implies that $Ay = Tx$. And by uniqueness this implies that $y = Bx$. And that is exactly what we want to know. So, by the closed graph theorem, we have B is continuous, so you can lift the map and that becomes continuous.

(Refer Slide Time: 09:48)



Now, we saw in the lectures about left, right inverses and we proved the theorem about compliment spaces and the right inverse. The kernel is compliment, not the image, Kernel was complimented, that is what we saw.

Problem 7: So now, we have V, W Banach and $T \in L(V, W)$. So, S is a left inverse, means $S \in L(W, V)$ and now you have $S \circ T = I_V$. So, this is left in the, previously it was T composed with S identity in W that is the right inverse, now the inverse lives on the left and so this is called a left inverse.

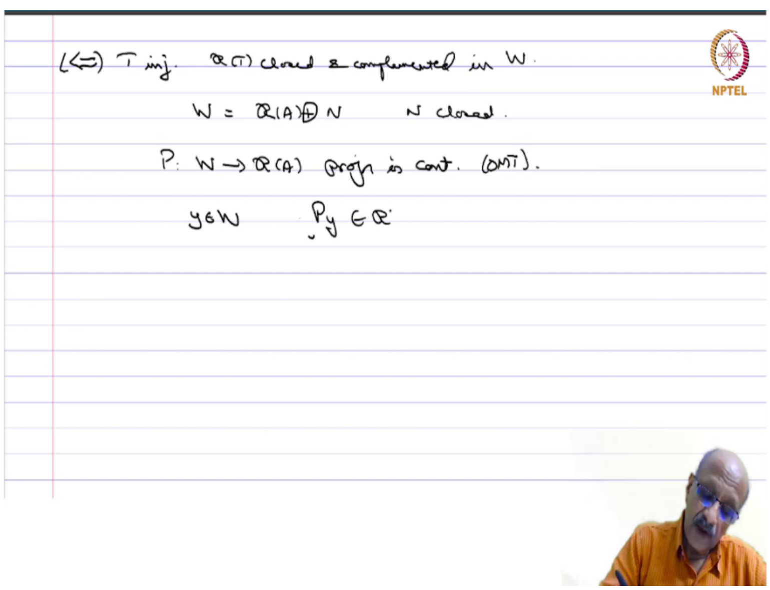
So, T has a left inverse if and only if T is injective that is one-one, T injective and $R(A)$ is closed and complemented in W. So, these are necessary and sufficient conditions for the existence of a left inverse. So solution,

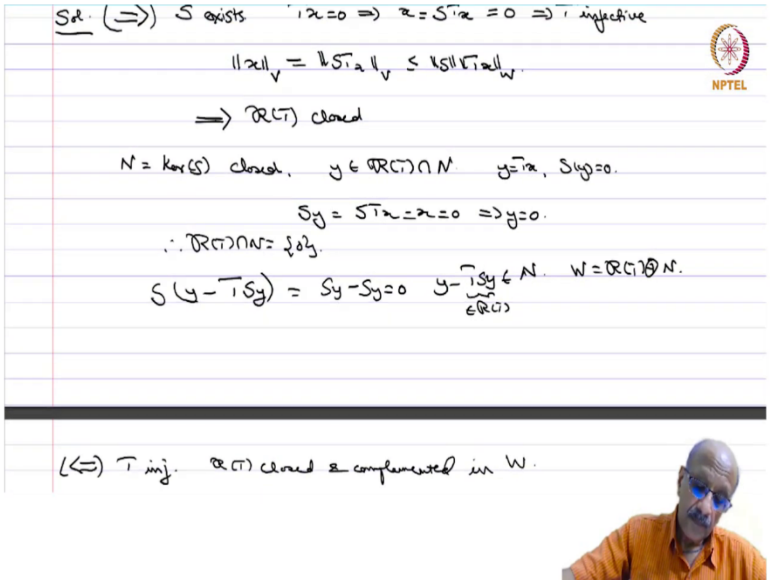
Solution: Let us prove one way. So, we assume that T has a left inverse. So, S exists. So, $Tx = 0$ implies $x = STx$ because ST is identity in V and that is equal to 0. Therefore, T is injective.

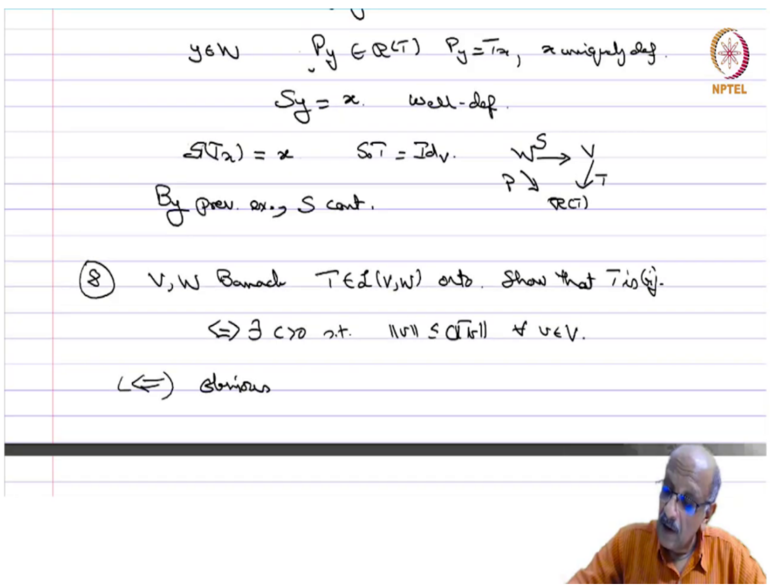
So, if you have a left inverse automatically the mapping has to be injected. So now, we, let us take $\|x\|_V = \|STx\|_V$, S is continuous, so this is $\|S\|\|Tx\|_W$. So, we have $\|STx\|_V \leq \|S\|\|Tx\|_W$ and we saw yesterday in one of the theorems if you have a bounded below then automatically $R(T)$ is closed. Why do you take any sequence in the range, $R(T)$ which converges that sequence is Cauchy here, which means x_n is also Cauchy and by continuity you will be able to complete the proof.

So, $R(T)$ is closed whenever you have bounded below. So, that is automatic. So, we have shown that T is injective range is close. Now, we want to show that the range is complemented. So, let us take $N = Ker(S)$. So, S is a continuous linear operator, so this is closed. So, let us assume that $y \in R(T) \cap N$. So, this means $y = Tx$ and then $Sy = 0$. So, $Sy = STx$, which is x and that is 0 and this implies that $y = 0$. Therefore, $R(T) \cap N$ is just the single term. Now, if you take $y - STy$ and apply S to this, so this you get $Sy - STSy$ which is $Sy - Sy$ so this is 0 and therefore $y - STy \in N$. And $STy \in R(T)$. And therefore, you have shown that $W = R(T) \oplus N$. So, it is complemented as well. So now, for the converse.

(Refer Slide Time: 14:57)







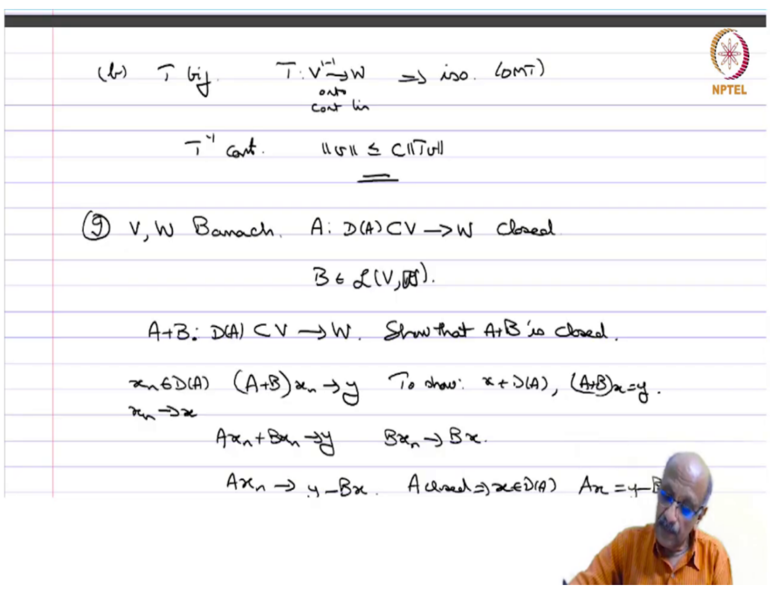
So, T injective and $R(T)$ closed and complement. So, you can take W will be some $R(T) \oplus N$, N closed. So, now you have P, W is a direct sum of two closed subspaces then by the open mapping theorem, we have that $P: W \to R(T)$ projection is continuous, it is a consequence of the open mapping theorem we have already seen this. So, then if you take, so we want to define Sy. So, you take $y \in W$ and you define Sy equals, so you take Py belongs to range of not range A but range of T, I hope we are not confused. The earlier page, everything is (())(17:07) range of T. So now, if you take Py that belongs to $R(T)$, so $Py = Tx$ and x is uniquely defined because we have T is injective. So, Py is in the range, so it has to be some Tx. So, you define $Sy = x$. So, well defined. So, if you take STx then Tx is already in the range. So, trivially this has to be only x. So, $S\circ T = I_V$. And now you have W and you have V and you have $R(T)$ this is closed so all three are Banach spaces and from W to $R(T)$ you have a map P from V to $R(T)$ you have the map T.

And then you have defined the map S is the unique x such that $Tx = Py$. And that is exactly the previous problem. So, by previous exercise we have S discontinuous and therefore, we have everything done.

Problem 8: V, W Banach and $T\in L(V, W)$ onto. Show that T is bijective if and only if there exists a $C > 0$ such that $||v||\leq C||Tv||$ for every $v\in V$.

So again, so the bijection if and only if it is bounded below. So one way, so this is obvious because if you have such an inequality, then automatically T is injective, so obvious and T is already subjective and therefore, you have that it is a bijection.

(Refer Slide Time: 20:11)



So, conversely T is a bijection. If T is a bijection, then you have $T: V\rightarrow W$ one-one onto and linear and it is, one-one, onto continuous linear. So, this implies by the open mapping theorem, this is an isomorphism. So, open mapping theorem, that means T^{-1} is continuous and that is exactly saying that $\|v\| \leq C\|Tv\|$. That is just the condition for T^{-1}. You write $T^{-1}Tv$ is v and therefore, you have the continuity that tells you that this is particularly true, so this is proved.

Then nine,

Problem 9: V, W Banach, $A: D(A) \subset V \rightarrow W$ closed operator, that means the graph is closed. Let $B \in L(V, W)$ it is a usual continuous linear operator. Then you can define $A + B$ again from the $D(A)$ because A is defined, B is defined everywhere.

So, $A + B$ is defined $D(A)$ contained in V taking values in W. So show that, $A + B$ is closed. So we, what do you mean close means let $x_n \in D(A)$, $(A + B)x_n \rightarrow y$ some y. So, to show $x_n \rightarrow x$. So, $x \in D(A)$ and $(A + B)x = y$, so this is just, these are the two conditions we have to show.

Solution: So, $Ax_n + Bx_n \rightarrow y$. But B is continuous So, $Bx_n \rightarrow Bx$ therefore, $Ax_n \rightarrow y - Bx$ and $x_n \rightarrow x$. So, A is closed given. So, this implies $x \in D(A)$ and $Ax = y - Bx$ and therefore, $(A + B)x = y$. So, this shows that this. Thank you.