Functional Analysis
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Lecture No. 24
Exercises

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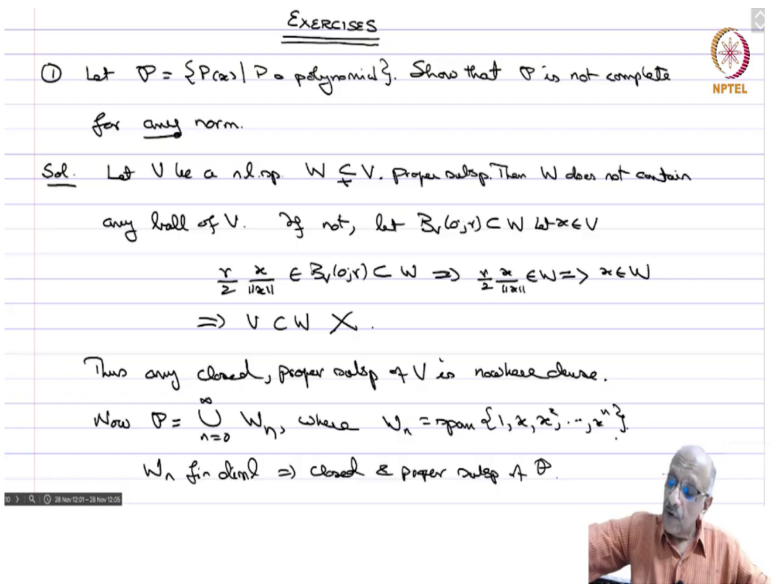

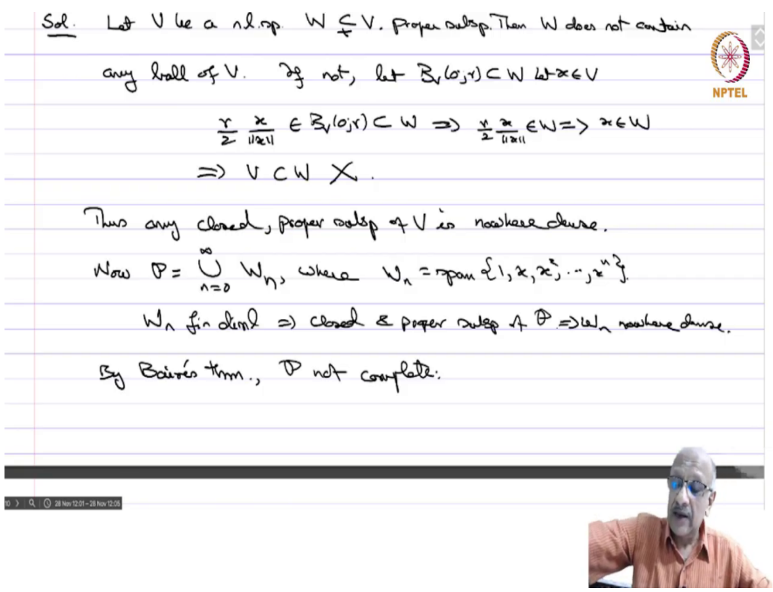

We will now do some exercises. First

Problem 1: Let P be the collection of all polynomials in 1 variable with real coefficients. Show that P is not complete for any norm. So, whatever norm you may put on P you cannot make it into a complete normed linear space.

Solution: Let V be a norm linear space and W a proper subspace, proper subspace. Then W does not contain any ball of V. So, let if possible, if not let $B_V(0, r)$ be contained in W. Because if it contains any ball you can translate it and therefore you can always have a ball centered at the origin. Then let $x \in V$. So, then you take $\frac{r}{2}\frac{x}{||x||} \in B_V(0, r)$ which is contained in W which implies that $\frac{r}{2}\frac{x}{||x||} \in W$ and this implies that $x \in W$, this will imply that $V \subset W$ which is a contradiction.

So, given any ball you do not have any proper subspace it cannot contain a ball. Thus, any close subspace, closed proper subspace of V is nowhere dense. Now, $P = \cup_{n=0}^{\infty} W_n$ where W_n is the span of $\{1, x, x^2, \cdot, \cdot, \cdot, x^n\}$ that is it is a collection of all polynomials of degree less than or equal to n. So, then W_n is finite dimensional implies closed and proper subspace of P. And therefore, by the Bair category theorem, P is the countable union of nowhere dense sets. So, this implies W_n nowhere dense. Therefore, P is not complete.

So, this can be extended to any vector space, which has a countable basis, a basis means and set up in linearly independent elements. So, set every member of X can be written as a finite linear combination from this set. So, if you have a countable basis, then such a space by Bair category theorem can never be made into a complete space.

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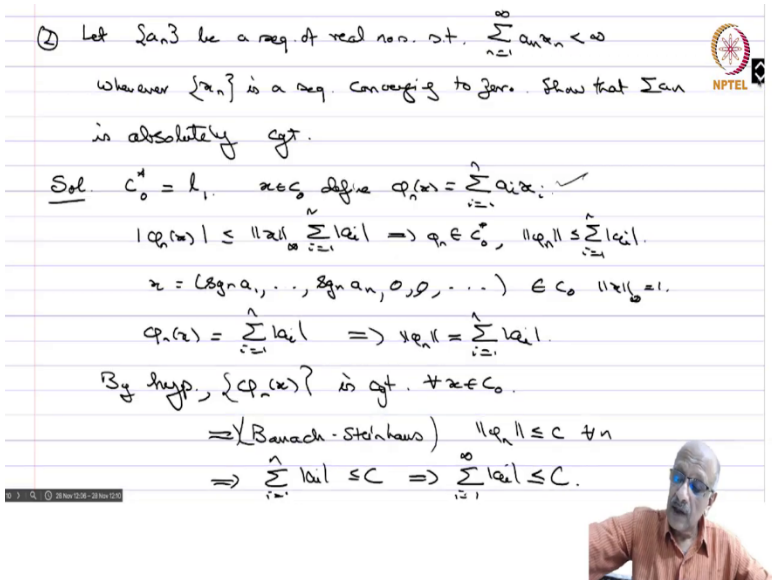

Two, it is a very nice exercise.

Problem 2: Let $\{a_n\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n x_n$ is convergent whenever $\{x_n\}$ is a sequence converging to 0. Show that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

So, you take an and you multiply it pointwise, component wise with the sequence which converges to 0 and that new series, the series which you just get is always supposed to be convergent. And therefore, you want to show now that an is absolutely convergent, so solution.

Solution: So, converging to 0 means what, C_0 is space and you know that $C_0^{\star} = l_1$. So, for $x \in C_0$ define $\phi_n(x) = \sum_{i=1}^{n} a_i x_i$. Then of course, ϕ_n is well defined; it is only a finite sum. And now you have $\left|\phi_n(x)\right| \leq \|x\|_{\infty} \sum_{i=1}^{n} \left|a_i\right|$. So, this implies that $\phi_n \in C_0^{\star}$ and you have $\|\phi_n\| \leq \sum_{i=1}^{n} \left|a_i\right|$. Now, if you take $x = (sgn\,(a_1),\, sgn\,(a_2),\, \cdot,\, \cdot,\, \cdot,\, sgn\,(a_n),\, 0, 0, \cdot, \cdot, \cdot)$, this belongs to C_0 and $\phi_n(x)$ will be in fact $\sum_{i=1}^{n} \left|a_i\right|$. And then $\|x\|_{\infty} = 1$ and therefore, this implies that $\|\phi_n\| = \sum_{i=1}^{n} \left|a_i\right|$.

Now, by hypothesis, $\{\phi_n(x)\}$ is convergent for every x in C_0, because that is what it says, $\sum_{i=1}^{n} a_i x_i$ these are the partial sums of $\sum_{n=1}^{\infty} a_n x_n$. So, ϕ_n and therefore, these converge, that is the hypothesis. And therefore by Banach Steinhaus, $\|\phi_n\| \leq C$ for all n, this implies $\sum_{i=1}^{n} |a_i| \leq C$, this implies that $\sum_{i=1}^{\infty} |a_i| \leq C$ and therefore $\{a_i\}$ is absolutely convergent.

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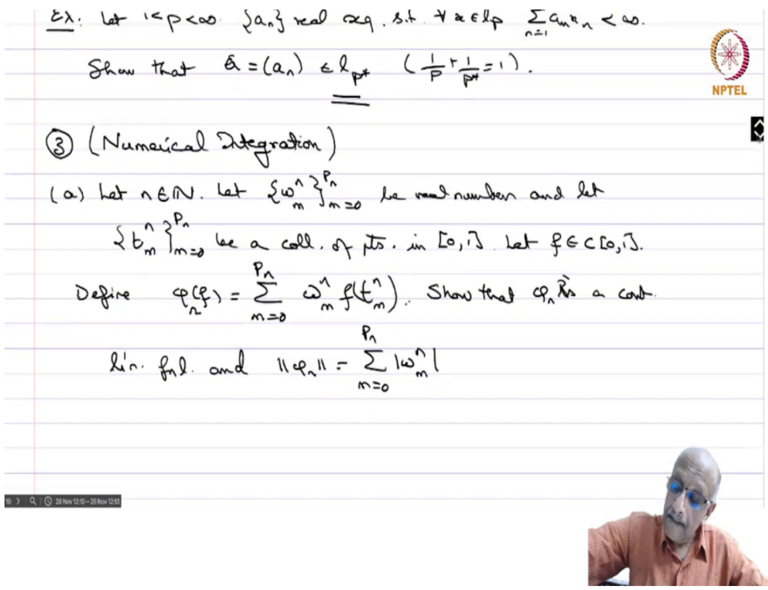

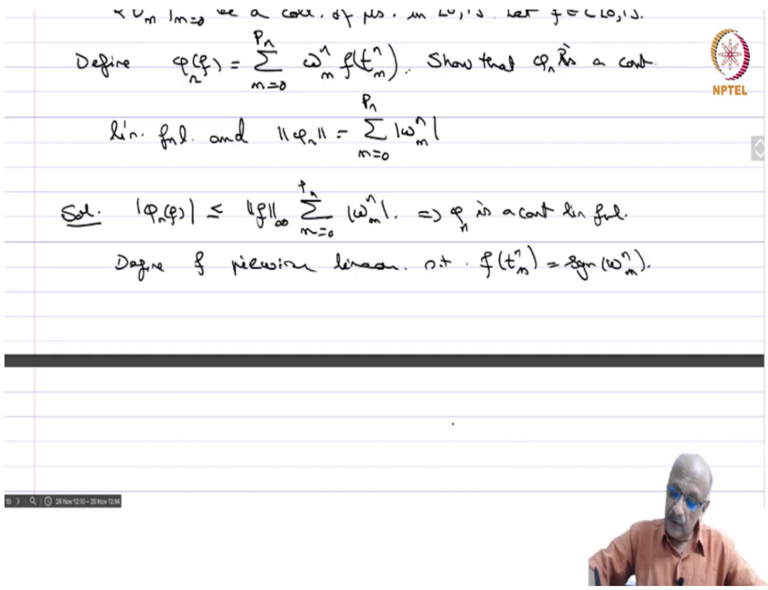

Three, before that, so try this as an exercise.

Exercise: Let $1 < p < \infty$, $\{a_n\}$ real sequence such that for all $x \in l_p$, $\sum_{n=1}^{\infty} a_n x_n$ is convergent. Show that $a = a_n \in l_{p^\star}$ where of course, $p^\star$ is the conjugate exponent, same proof you can imitate.

So third, so this is application of function analysis to numerical integration.

Problem 3: (Numerical Integration) (a) Let $n \in N$ and let $\{w_m^n\}_{m=0}^{P_n}$ be real numbers and let $\{t_m^n\}_{m=0}^{P_n}$ be a collection of points in $[0, 1]$. Let $f \in C([0, 1])$. Define $\phi_n(f) = \sum_{m=0}^{P_n} w_m^n f(t_m^n)$. So, you are evaluating the function at some points multiplying it by some weights and adding it. Show that ϕ_n is a continuous linear functional and $\|\phi_n\| = \sum_{m=0}^{P_n} \left|w_m^n\right|$.

Solution: So, $\left|\phi_n(f)\right| \leq \|f\|_\infty \sum_{m=0}^{P_n} \left|w_m^n\right|$. So, this is a trivial implies ϕ_n is a continuous linear functional. And now define f piecewise linear such that $f\left(t_m^n\right) = sgn(w_m^n)$.

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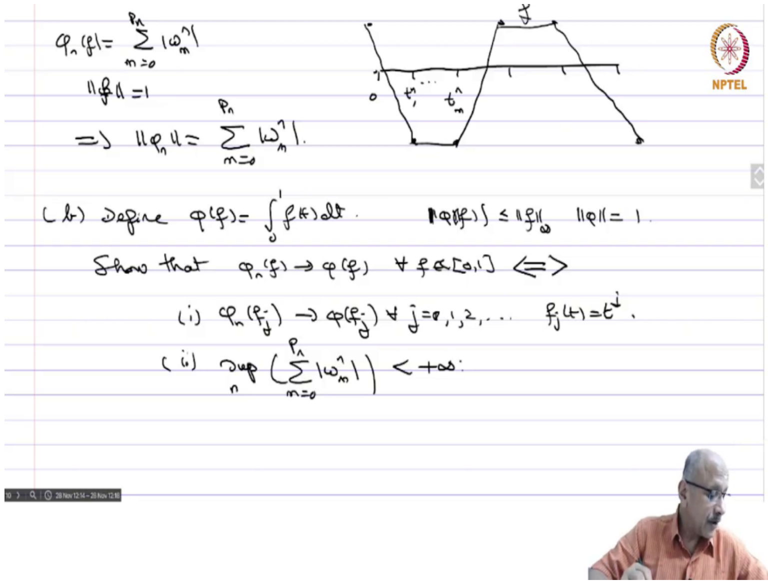

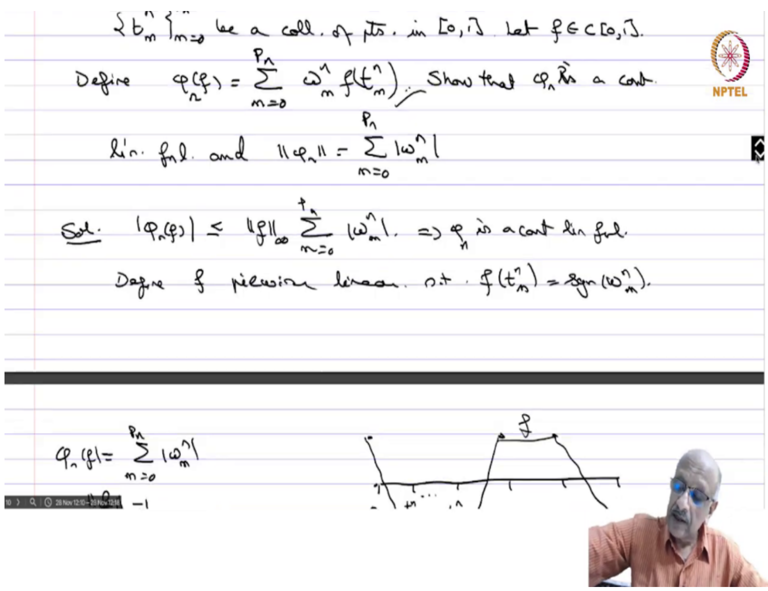

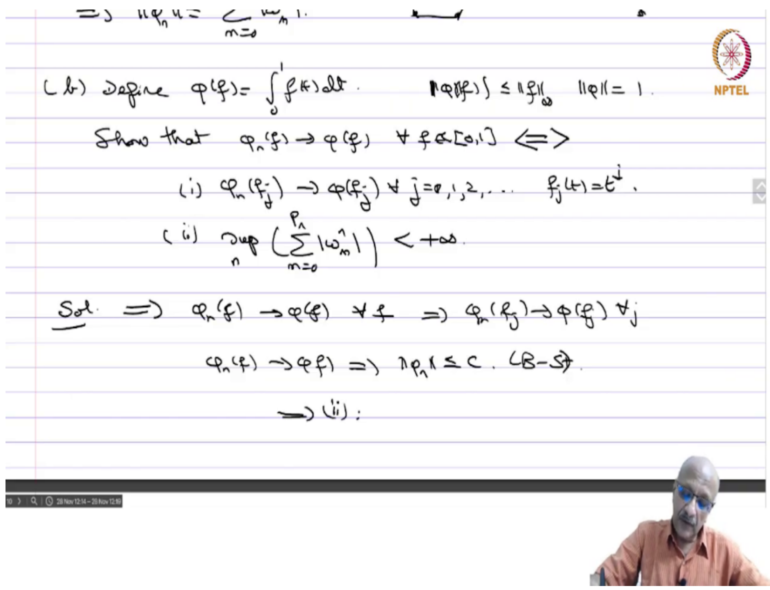

So, you have $[0, 1]$ here, so you have the points, various points. So, t_m^n, so this is t_1^n, t_m^n and so on. So, you have these points. So, at each point you are taking the sign, so the function has takes the value $+ 1$ or $- 1$. So, you have $+ 1$, let us say $- 1$ here, $- 1$ here, $+ 1$ here, $+ 1$ here, $- 1$ here, and so on. So, you have the function which is piecewise linear like this.

So, this is the function f. So, you fix the values at these points and in between you just join them by means of straight line segments. Then what is $\phi_n(f)$? Is precisely $\sum_{m=0}^{P_n} \left|w_m^n\right|$ and $\|f\| = 1$ because it takes a maximum value or $+ 1$ or $- 1$ maximum, minimum values. And therefore, this implies that $\|\phi_n\| = \sum_{m=0}^{P_n} \left|w_m^n\right|$.

(b) So now define $\phi(f) = \int_0^1 f(t)\, dt$. Then ϕ is also continuous linear functional, in fact $\|\phi\|$ is, is equal to in fact 1 again because $|\phi(f)| \leq \|f\|_\infty$ and therefore, $\|\phi\| \leq 1$ and in fact you can show very easily that $\|\phi\| = 1$. So, show that $\phi_n(f) \to \phi(f)$ for every $f \in C([0, 1])$ if and only if,

(i) $\phi_n\left(f_j\right) \to \phi(f_j)$ for all $j = 0, 1, 2, \cdot, \cdot, \cdot,$ and where $f_j = t^j$.

(ii) sup $sup\left(\sum_{m=0}^{P_n}\left|w_m^n\right|\right) < \infty$ they are all bounded.

So, as I said so, this is nothing but numerical integration because you are replacing $\phi(f)$ which is the integral off by means of a sum which is like $\phi_n(f)$. So, that is by integration formula and therefore, this gives you the numerical integration. So, this is what you do with numerical answers.

And now we are giving a necessary and sufficient condition for the numerical integration scheme to converge. So solution,

Solution: So let us assume this way. So, $\phi_n(f)\to\phi(f)$ for all f so implies $\phi_n\left(f_j\right)\to\phi(f_j)$ for all j, that is trivial. And then since $\phi_n(f)$ converges to $\phi(f)$ so this implies that $\|\phi_n\|\leq C$ and that is exactly by Banach Steinhaus and therefore, that is exactly implies to. So, that is the second condition.

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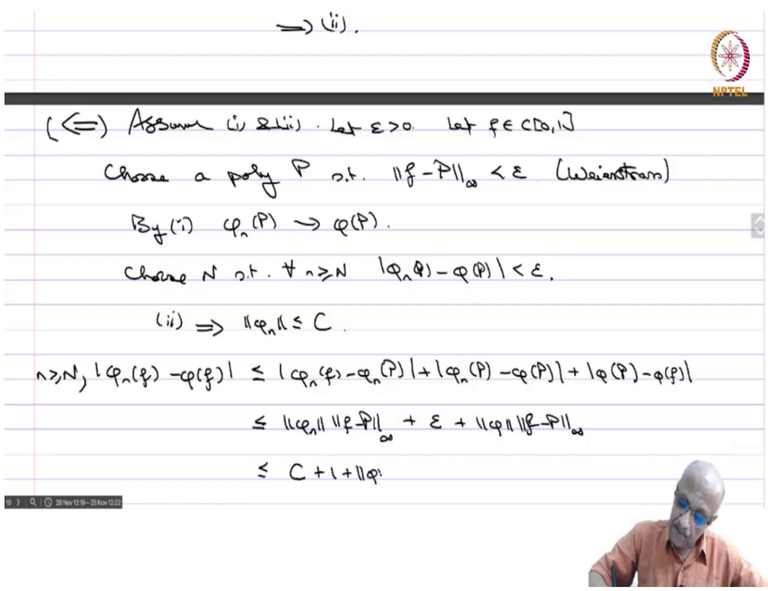

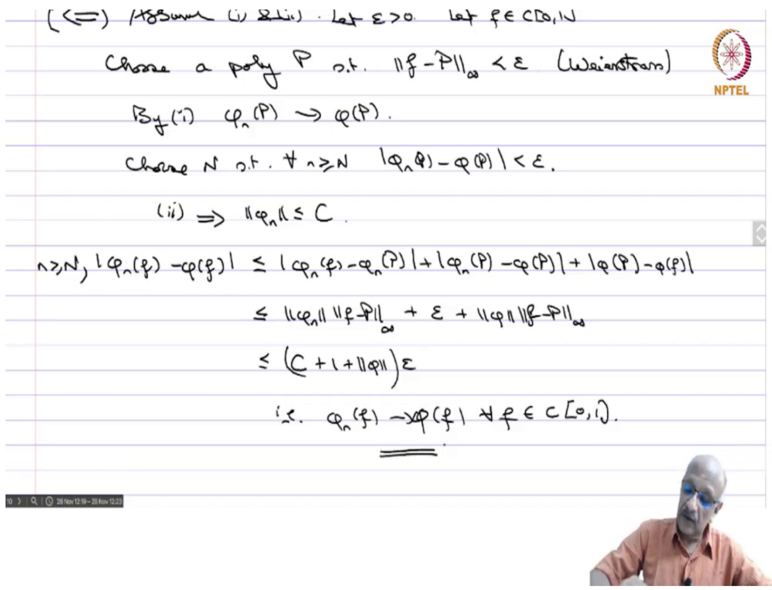

Now, conversely let us assume, so conversely assume (i) and (ii). So choose, so let ϵ be positive. A polynomial P such that $\|f - P\|_{\infty} < \epsilon$, so let $f \in C([0, 1])$. So, this is you can do by Weierstrass, you can always find a polynomial which is this. Now by (i), since $\phi_n\left(f_j\right) \to \phi\left(f_j\right)$ that is t^j for every j, it means, by linearity we have that $\phi_n(P) \to \phi(P)$. So, choose N such that for all $n \geq N$, $\left|\phi_n(f) - \phi(f)\right| < \epsilon$.

Now, what does (ii) imply we have that $\|\phi_n\| \leq C$. So,

$$\left|\phi_n(f) - \phi(f)\right| \leq \left|\phi_n(f) - \phi_n(P)\right| + \left|\phi_n(P) - \phi(P)\right| + |\phi(P) - \phi(f)|.$$

Now, $\left|\phi_n(f) - \phi_n(P)\right| \leq \|\phi_n\| \|f - P\|_{\infty}$. The second one, so if $n \geq N$, $\left|\phi_n(P) - \phi(P)\right| < \epsilon$ and the third one is $\|\phi\| \|f - P\|_{\infty}$. But $\|f - P\|_{\infty} \leq C$, so

$$\left|\phi_n(f) - \phi_n(P)\right| + \left|\phi_n(P) - \phi(P)\right| + |\phi(P) - \phi(f)| \leq (C + 1 + \|\phi\|)\epsilon,$$

and therefore, it goes to 0 and consequently you have shown that that is $\phi_n(f) \to \phi(f)$ for all $f \in C([0, 1])$. So, this tells you, gives you a necessary and sufficient condition for the convergence of a numerical integration scheme and this comes from the Banach's Steinhaus theorem.