Functional Analysis
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Lecture No. 23
Orthogonality relations

We will now investigate relationship between the range and kernel of an operator and it is adjoint. So, we have the following proposition.

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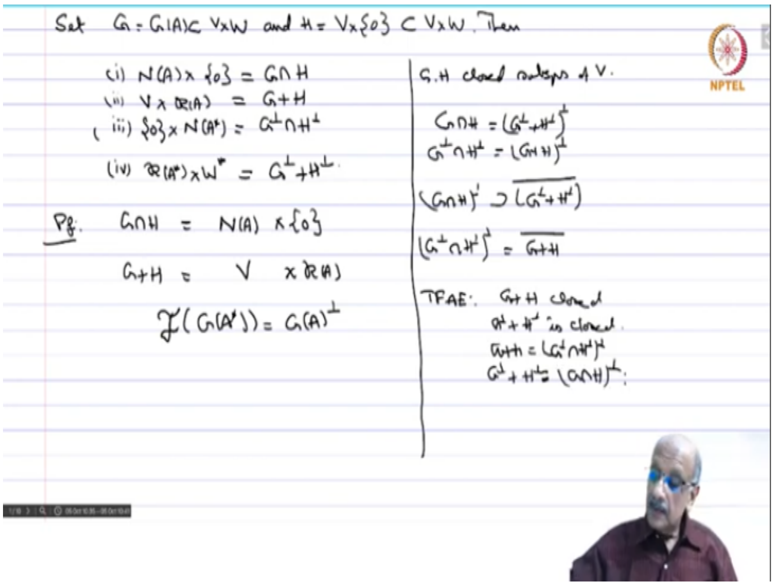

Proposition: So, let V, W Banach, A from $D(A)$ contained in V taking values in W, densely defined, linear operator. So, we set $G = G(A)$ the graph of A, this is a subspace of $V\times W$ and $H = V\times\{0\}$ again contained in $V\times W$, these are subspaces. Then

1. $N(A)\times\{0\} = G\cap H$;
2. $V\times R(A) = G + H$;
3. $\{0\}\times N(A^*) = G^\perp\cap H^\perp$; and
4. $R(A^*)\times W^* = G^\perp + H^\perp$.

This is almost an immediate thing, so proof.

Proof: So, let us first consider $G\cap H$. So, H has a second coordinate $\{0\}$, so $G\cap H$ will be something cross $\{0\}$ that is clear. Now, what is in G, the first coordinate is the domain of A and the second coordinate if it has to be $\{0\}$ then this has to be the null space of A. So, that is why we get that. So, now secondly for $G + H$, so, if you take $G + H$, so, the second coordinate has $\{0\}$ in H and so, if you add anything you will only get the second coordinate of $G(A)$ and that is in the range of A, so you will get so $G + H$ will be something cross range of A, and the first coordinate in H is V, so whatever you add to it you only get V so you will get $V \times R(A)$. So similarly you do the other two yourself, only you use the fact $I\left(G\left(A^{*}\right)\right) = G(A)^{\perp}$ you must remember this relationship and in the same way you can do 3 and 4. So, that is this proposition.

So while we are here, let me note down some of the old results which we did which we will be using. So if G and H closed subspaces of a Banach space V, Then you have $G \cap H = \left(G^{\perp} + H^{\perp}\right)^{\perp}$ and $G^{\perp} \cap H^{\perp} = (G + H)^{\perp}$, these we did when we were in the annihilator section and then you have from these two if you take the perp again, $(G \cap H)^{\perp} \supset \overline{\left(G^{\perp} + H^{\perp}\right)}$ and $\left(G^{\perp} \cap H^{\perp}\right)^{\perp} = \overline{G + H}$. And then we also had a theorem which says that the following are equivalent. The following are equivalent, you have that

(i) $G + H$ is closed,

(ii) $G^{\perp} + H^{\perp}$ is closed and then

(iii) $G + H = \left(G^{\perp} \cap H^{\perp}\right)^{\perp}$ and

(iv) $G^{\perp} + H^{\perp} = (G \cap H)^{\perp}$.

This is another theory we treat. So, let us just keep that in mind, so that.

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Cor. V, W Banach $A : D(A) \subset V \to W$ densely def & closed.

Then $N(A) = R(A^*)^\perp$

$N(A^*) = R(A)^\perp$

$N(A)^\perp \supset \overline{R(A^*)}$ ✓

$N(A^*)^\perp = \overline{R(A)}$ ✓

Pf. $G = G(A) \subset V \times W$, $H = V \times \{0\} \subset V \times W$.

$G \cap H = (G^\perp + H^\perp)^\perp$

$N(A) \times \{0\}$.

Cor. V, W Banach $A : D(A) \subset V \to W$ densely def & closed.

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$G \cap H = (G^\perp + H^\perp)^\perp$

$N(A) \times \{0\} = R(A^*)^\perp \times \{0\}$

$G^\perp \cap H^\perp = (G + H)^\perp$

$\{0\} \times N(A^*) = \{0\} \times R(A)^\perp$

(i) $N(A) \times \{0\} = G \cap H$

(ii) $V \times R(A) = G + H$

(iii) $\{0\} \times N(A^*) = G^\perp \cap H^\perp$

(iv) $R(A^*) \times W^* = G^\perp + H^\perp$.

Pf. $G \cap H = N(A) \times \{0\}$

$G + H = V \times R(A)$

$\mathcal{J}(G(A^*)) = G(A)^\perp$

G, H closed subsps of V.

$G \cap H = (G^\perp + H^\perp)^\perp$

$G^\perp \cap H^\perp = (G + H)^\perp$

$(G \cap H)^\perp \supset \overline{(G^\perp + H^\perp)}$

$(G^\perp \cap H^\perp)^\perp = \overline{G + H}$

TFAE: $G + H$ closed

$G^\perp + H^\perp$ is closed.

$G + H = (G^\perp \cap H^\perp)^\perp$

$G^\perp + H^\perp = (G \cap H)^\perp$.

So, now we have corollary to the previous proposition,

Corollary: So, V, W Banach and A from $D(A)$ contained in V taking values in W, densely defined and closed. So, we are adding another hypothesis then

$N(A) = R(A^{\star})^{\perp}$, $N(A^{\star}) = R(A)^{\perp}$, $N(A)^{\perp} \supset \overline{R(A^{\star})}$, and $N(A^{\star})^{\perp} = \overline{R(A)}$.

So, the first last two relationships if you take $N(A)^{\perp}$ that will be $(R(A^{\star})^{\perp})^{\perp}$ that is in a dual space, so, that should contain the $\overline{R(A^{\star})}$. Then $N(A^{\star})^{\perp}$ is $(R(A)^{\perp})^{\perp}$ and this is in the basic space and therefore, it will be $\overline{R(A)}$. So, these two conditions follow from the first two and therefore, we would need not to. So, we only have to prove the first one. So

Proof: So as usual, let us take $G = G(A)$ contained in $V \times W$ and $H = V \times \{0\}$ as before contained in $V \times W$. So, then we already saw, what was one of the relationships we saw?

We saw that, $G \cap H = (G^{\perp} + H^{\perp})^{\perp}$ that relationship here. So, we have $G \cap H = (G^{\perp} + H^{\perp})^{\perp}$. Now, what is $G \cap H$? $G \cap H$ is $N(A) \times \{0\}$ and therefore, we have here $N(A) \times \{0\} = (G^{\perp} + H^{\perp})^{\perp}$. $G^{\perp} + H^{\perp}$ is $R(A^{\star}) \times W^{\star}$. So, if you take the perp the $W^{\star}$ it will give you $\{0\}$ and here you have $R(A^{\star})^{\perp}$. And then if you just compare this you get the first relationship namely $N(A) = R(A^{\star})^{\perp}$. Similarly, you have $G^{\perp} \cap H^{\perp} = (G + H)^{\perp}$. So, $G^{\perp} \cap H^{\perp} = (G + H)^{\perp}$. So, what is $G^{\perp} \cap H^{\perp}$? That is $\{0\} \times N(A^{\star}) = (G + H)^{\perp}$, $G + H$ is $V \times R(A)$. So, the perp of it will be $\{0\} \times R(A)^{\perp}$ and therefore, you have $N(A^{\star}) = R(A)^{\perp}$ that proves the second part of the perps. So, we have a relationship where the null space of an operator is nothing but the annihilator of the range of the adjoint. Similarly, the null space of that joint is nothing but the annihilator of the range of the original operator. So, this is a connection between now we have a little more.

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Thm V, W Banach. $A : D(A) \subset V \to W$ closed & densely def.

The foll. are equiv.

(i) $R(A)$ closed in W $\iff$ $G + H$ closed in $V \times W$

(ii) $R(A^*)$ closed in V^* $\iff$ $G^\perp + H^\perp$ closed in $V^* \times W^*$

(iii) $R(A) = N(A^*)^\perp$ $\iff$ $G + H = (G^\perp \cap H^\perp)^\perp$

(iv) $R(A^*) = N(A)^\perp$ $\iff$ $G^\perp + H^\perp = (G \cap H)^\perp$

Rem. V, W fin. dim. $R(A) = N(A^*)^\perp$ Fredhölm alternative.

$$Ax = b$$

$$A^* y = 0 \qquad (y, b) = (y, Ax) = (A^* y, x) = 0$$

$$b \in R(A) \implies b \in N(A^*)^\perp$$

$$N(A)^\perp \supset \overline{R(A^*)} \checkmark$$

$$N(A^*)^\perp = \overline{R(A)} \checkmark$$

Pf. $G = G(A) \subset V \times W$, $H = V \times \{0\} \subset V \times W$.

$$G \cap H = (G^\perp + H^\perp)^\perp$$

$$N(A) \times \{0\} = R(A^*)^\perp \times \{0\}$$

$$G^\perp \cap H^\perp = (G + H)^\perp$$

$$\{0\} \times N(A^*) = \{0\} \times R(A)^\perp$$

Thm V, W Banach. $A : D(A) \subset V \to W$ closed & densely def.

Set $G = G(A) \subset V \times W$ and $H = V \times \{0\} \subset V \times W$. Then

(i) $N(A) \times \{0\} = G \cap H$
(ii) $V \times R(A) = G + H$
(iii) $\{0\} \times N(A^*) = G^\perp \cap H^\perp$
(iv) $R(A^*) \times W^* = G^\perp + H^\perp$.

Pf. $G \cap H = N(A) \times \{0\}$

$$G + H = V \times R(A)$$

$$\mathcal{J}(G(A^*)) = G(A)^\perp$$

G, H closed subsps of V.

$$G \cap H = (G^\perp + H^\perp)^\perp \checkmark$$

$$G^\perp \cap H^\perp = (G + H)^\perp \checkmark$$

$$(G \cap H)^\perp \supset \overline{(G^\perp + H^\perp)}$$

$$(G^\perp \cap H^\perp)^\perp = \overline{G + H}$$

TFAE:
$$\left.\begin{array}{l} G + H \text{ closed} \\ G^\perp + H^\perp \text{ is closed.} \\ G + H = (G^\perp \cap H^\perp)^\perp \\ G^\perp + H^\perp = (G \cap H)^\perp. \end{array}\right\}$$

So,

Theorem: V, W Banach and A from $D(A)$ contained in V taking values in W which is closed and densely defined. So, the previous corollary is true. So, in particular then the following are equivalent:

1. $R(A)$ is closed.
2. $R(A^{\star})$ is closed.

So, this $R(A)$ is closed to in W. $R(A^{\star})$ is closed in $V^{\star}$ and then

3. $R(A) = N(A^{\star})^{\perp}$ and
4. $R(A^{\star}) = N(A)^{\perp}$.

So, now, if you look at the G and H definitions, which we have given before $G = G(A)$, $H = V\times\{0\}$ and then we had various relationships between the two in this particular corollary.

So, if you go back and check all this then this statement $R(A)$ closed is the same as saying $G + H$ is closed in $V\times W$ because, what is $G + H$? $V\times R(A)$. So, if this is closed then the product is closed and therefore, you have that. So, in the same way $R(A^{\star})$ is closed will be saying the $G^{\perp} + H^{\perp}$ is closed in $V^{\star} \times W^{\star}$ and then the third one is saying that $G + H = (G^{\perp} + H^{\perp})^{\perp}$ and the fourth one is saying $G^{\perp} + H^{\perp} = (G\cap H)^{\perp}$.

But, this theorem, we have already shown. The following are equivalent. So, we have these four statements here and that is exactly what we had. So, that completes the proof, there's no need to prove it again. Now,

Remark: Of course, all these theorems are naturally true for continuous linear operators because they are automatically closed, densely defined because they are defined everywhere.

In particular, so if you take, so, remark again,

Remark: Suppose V, W finite dimensional, then all operators are continuous linear operators even all subspaces are closed and this is so we have that $R(A) = N(A^{\star})^{\perp}$ this is called the usual Fredholm alternative and for instance, if you want to solve the equation $A(x) = b$

where A is a matrix say $m\times n$ matrix, so, when will this have a solution? So, let us assume that $A^{\star}y = 0$. Then if you take the inner product $\langle y, b\rangle$, this is $\langle y, Ax\rangle$ by definition this is $\langle A^{\star}y, x\rangle$, here the duality bracket is the same as the inner product in finite dimensional spaces and then this is 0. So, if $b\in R(A)$ this implies that $b\in N\left(A^{\star}\right)^{\perp}$. And then by a dimension argument you can show this in fact, it is it works both ways and therefore, you have this Fredholm alternative.

So, the annihilator of the kernel of the adjoint is precisely the range. So, this is in fact the compatibility condition that b must satisfy in order that this equation has a solution. So, this equation has a solution if b is in the range of A and for that it needs to satisfy some condition and it is precisely saying that $\langle y, b\rangle = 0$ for every $A^{\star}y = 0$. So, that is and that result is called the Fredholm alternative.

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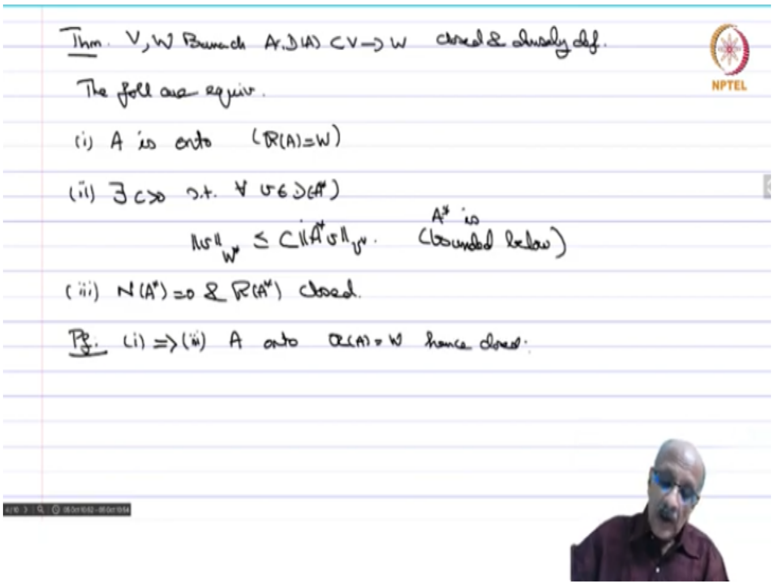

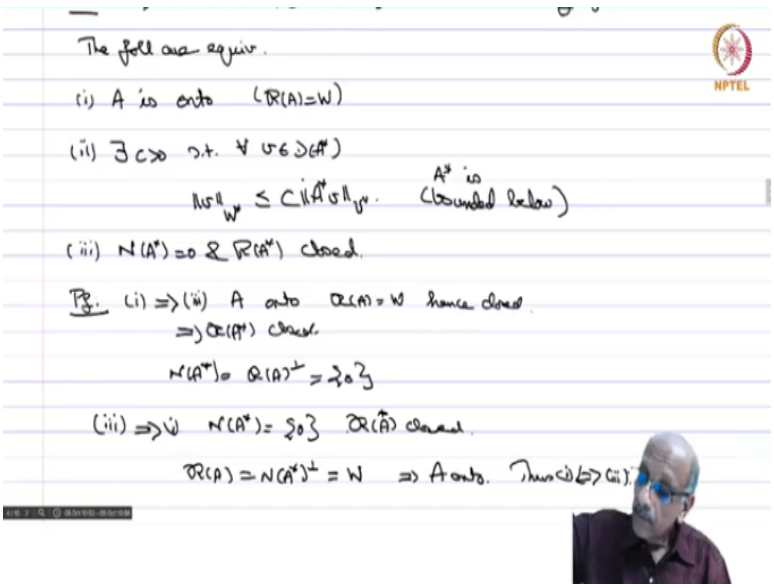

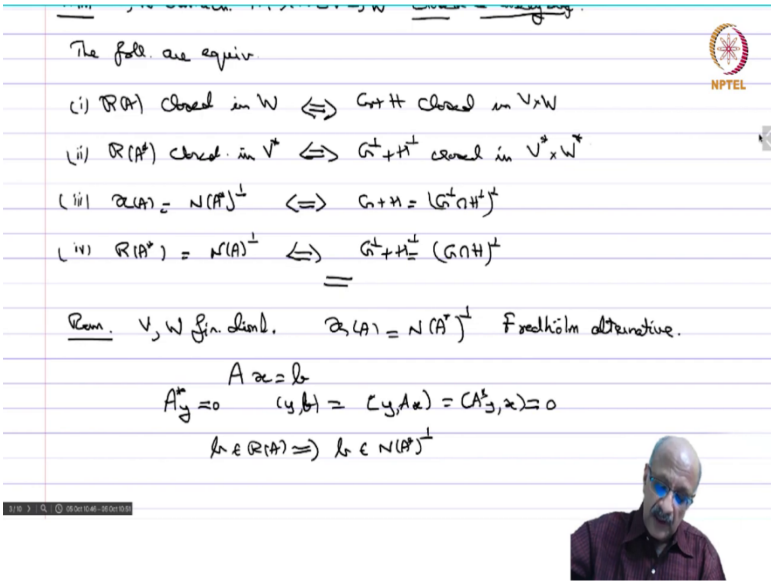

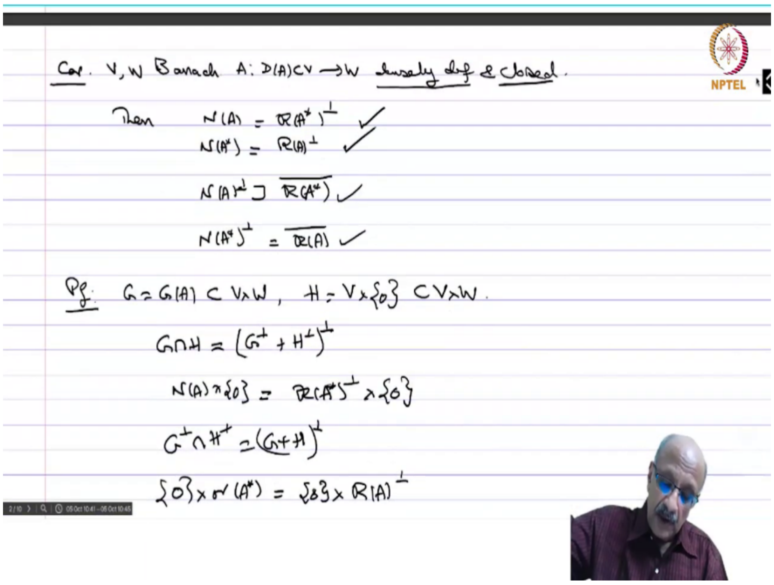

Finally, we have a theorem on surjective maps. So, theorem, this is a very nice theorem So, we say that

Theorem: V, W Banach and $A: D(A)\subset V\to W$ is closed and densely defined. The following are equivalent.

1. A is onto. So, there is a map, it is surjective. So, that means $R(A) = W$.
2. there exists a constant $C > 0$ such that for all $v\in D\left(A^\star\right)$ we have

$$\|v\|_{W^\star}\leq C\,\|A^\star v\|_{V^\star}.$$

So, such a map is called bounded below. A star is bounded below that means, you have a constant going in the opposite of the usual thing you have norm A bounded means, $\|A^{\star}v\| \leq C\|v\|$, but here having $\|A^{\star}v\| \geq C\|v\|$ and that is called bounded below.

3. You have that $N(A^{\star}) = \{0\}$ and $R(A^{\star})$ is closed.

So, these are characterizations of a surjective map. So, once you have a surjective map that means, you can solve the equations. That is the important thing, why we are interested in such things. So, proof,

Proof: Let us prove that $1 \Rightarrow 3$. So, A is onto, so $R(A)$ is W and hence closed. So, we can apply the previous theorem because we have closed bounded operators So, $R(A)$ is closed, so you have so many things. So, $R(A^{\star})$ will be $N(A)^{\perp}$. And in fact, $R(A^{\star})$ will be closed. So, $R(A)$ is closed implies $R(A^{\star})$ is closed and you also have $N(A^{\star})$ is $R(A)^{\perp}$. And since $R(A)$ is whole of W all that we have proved even much before $N(A^{\star})$ is $R(A)^{\perp}$ and therefore, and since $R(A)$ is whole of W, $R(A)^{\perp}$ has to be $\{0\}$. So, this proves 3. $R(A^{\star})$ is closed and this. So let us say $3 \Rightarrow 1$. That means $N(A^{\star})$ is $\{0\}$ and $R(A^{\star})$ closed. So, then you have $R(A) = N\left(A^{\star}\right)^{\perp}$ by the previous theorem, $R(A) = N\left(A^{\star}\right)^{\perp}$, that is number 3 and therefore $N(A^{\star}) = \{0\}$ and therefore $N\left(A^{\star}\right)^{\perp}$ has to be W. So implies A onto. So, we approved that. So, thus 1 is equivalent to 3. So, now we want to show the 2 and 3 are also equivalent.

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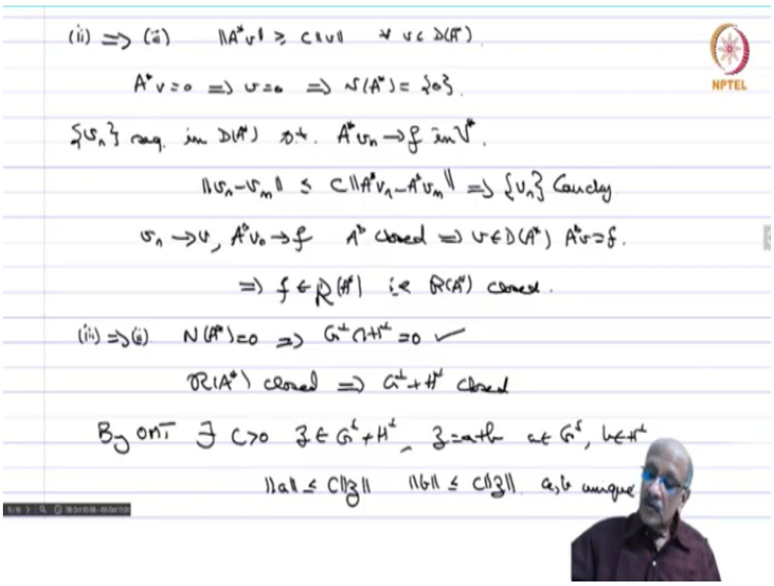

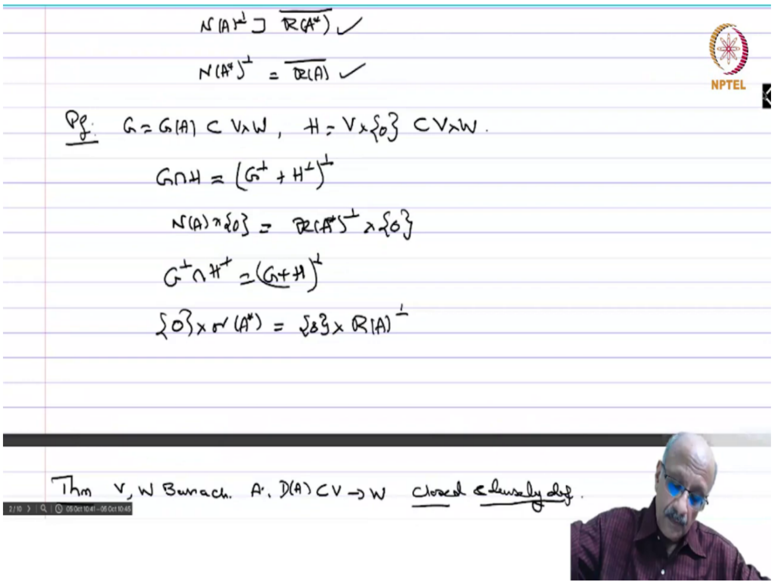

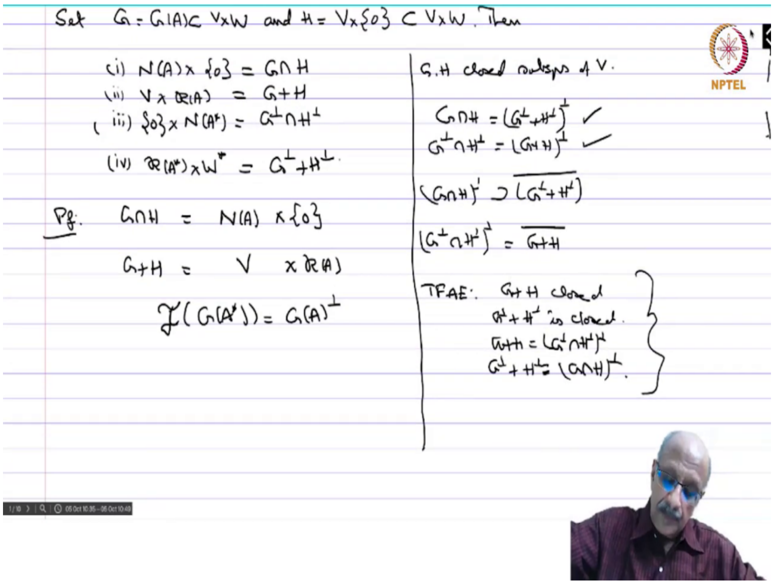

$2 \Rightarrow 3$. So, the condition that $\|A^\star v\| \geq C\|v\|$ for all $v \in D(A^\star)$. So, if $A^\star v = 0$ automatically this means so $A^\star v = 0 \Rightarrow v = 0$. So, this implies that $N(A^\star) = \{0\}$. And then let us take a sequence $\{v_n\}$ in $D(A^\star)$ such that $A^\star v_n \to f$ in $V^\star$, then again by the boundedness below we have $\|v_n - v_m\| \leq C\|A^\star v_n - A^\star v_m\|$. $\{A^\star v_n\}$ is Cauchy. So, implies $\{v_n\}$ is Cauchy. We are in the Banach space. So, $v_n \to v$, $A^\star v_n \to f$ and therefore, $A^\star$ is closed, this is always true and therefore, this implies that $v \in D(A^\star)$ and $A^\star(v) = f$. Therefore, if $\{A^\star v_n\}$ goes to some f, $f = A^\star(v)$, so, this implies $f \in D(A^\star)$, that is $R(A^\star)$ is closed. So, then finally, this proves 3 completely.

So, now we want to show $3 \Rightarrow 2$. So, we go back to the notation which we had earlier about G and H, $G = G(A)$, $H = V\times\{0\}$ and then we, we have $N\left(A^{\star}\right) = \{0\}$ implies that $G^{\perp}\cap H^{\perp} = \{0\}$. So, because what is $G^{\perp}\cap H^{\perp}$ we computed it somewhere. $G^{\perp}\cap H^{\perp}$ number 3 is $\{0\}\times N\left(A^{\star}\right)$. If $N\left(A^{\star}\right) = \{0\}$ then $G^{\perp}\cap H^{\perp}$ is in fact $\{0\}$, so we have. So then and you also have that $R\left(A^{\star}\right)$ closed implies $G^{\perp} + H^{\perp}$ is closed again. What is $G^{\perp} + H^{\perp}$? $G^{\perp} + H^{\perp} = R\left(A^{\star}\right)\times W^{\star}$. So, if number 4 and therefore, if $R\left(A^{\star}\right)$ is closed you have the $G^{\perp} + H^{\perp}$ is closed. So, we are in the situation where the sum of two close subspaces is closed and the intersection is $\{0\}$. So, this is in fact a direct sum. So, then by a proposition which we proved by the open mapping theorem there exists a $C > 0$ such that if you have $z\in G^{\perp} + H^{\perp}$ then $z = a + b$, $a\in G^{\perp}$, $b\in H^{\perp}$ and $||a||\leq C||z||$, $||b||\leq C||z||$ and you have two subspaces you have this, but now, since $G^{\perp}\cap H^{\perp} = \{0\}$ in fact a and b are unique, there is no choice of decomposition, there is only one decomposition available. So, if we find it that is the correct one and it will automatically satisfy the condition.

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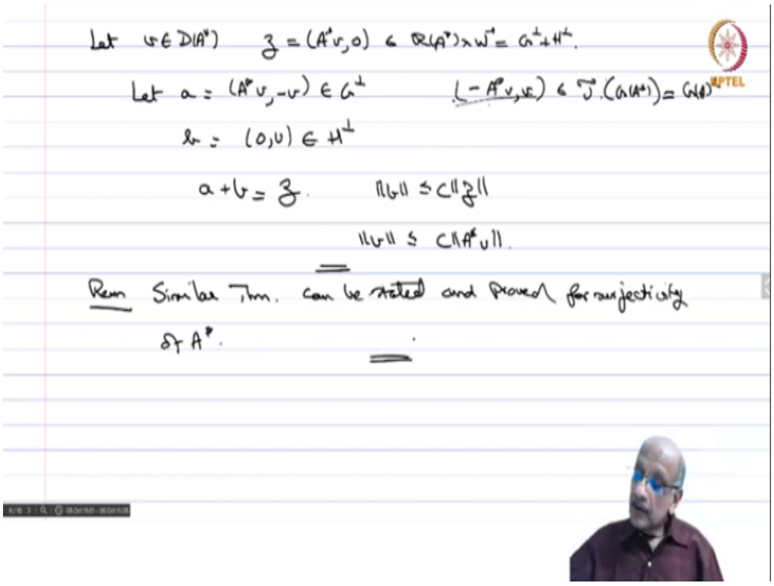

So, let us take $v\in D\left(A^{\star}\right)$ then you said $z = (A^{\star}v,\ 0)$. So, this belongs to $R\left(A^{\star}\right)\times W^{\star}$ which is in as we know $G^{\perp} + H^{\perp}$. So, now let $a = (A^{\star}v, -\ v)$. So, take $(A^{\star}v, -\ v)$. So, this if you take $(-\ A^{\star}v, v)$ this belongs to $I\left(G\left(A^{\star}\right)\right)$ but $G\left(A^{\star}\right)$ is a subspace. So, this also which is equal to $G(A)^{\perp}$. But then minus of $(A^{\star}v, -\ v)$ will also be in $G(A)^{\perp}$ because that is a subspace So,

$= \left(A^{\star}v, -v\right) \in G(A)^{\perp}$ or $G^{\perp}$. Now, you take $b = (0, v)$ and automatically that belongs to $H^{\perp}$ and then you just add you get $a + b$ is $\left(A^{\star}v,\ 0\right)$ which is equal to z and therefore, you have those two inequalities. So, in particular, you have that $||b|| \leq C||z||$ and that tells you that $||v|| \leq C||A^{\star}v||$ and that proves the remaining thing. So, this completes the proof. Fine.

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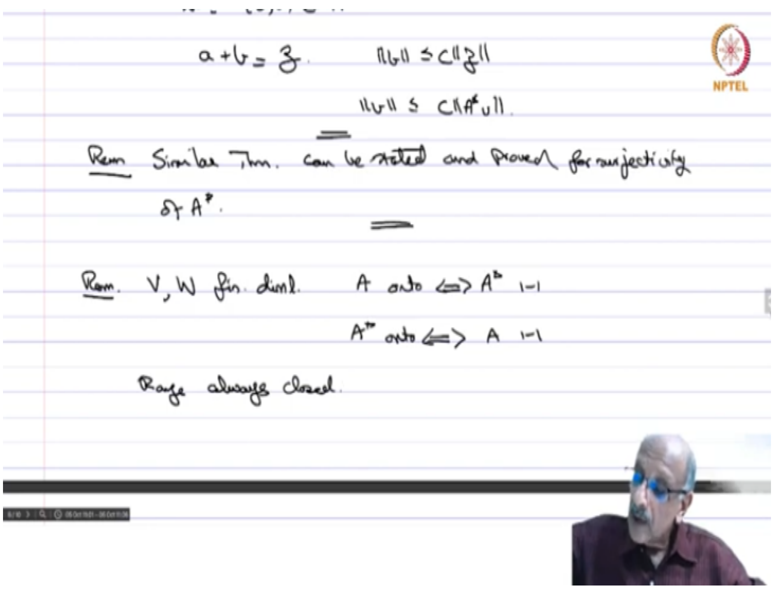

So, similar remark,

Remark: Similar statement, similar theorem can be stated and proved for surjectivity of $A^{\star}$. You can try it as an exercise, it is just identical. So, you can go ahead.

So, now, if you look at another remark,

Remark: Let us take V and W finite dimensional, then everything as I said is a continuous linear operator in particular close densely defined everything and therefore, you have that A is onto, if and only if $A^{\star}$ is one-one because we have the $R(A)$ equals $N(A^{\star})^{\perp}$. So, and this is equal so, you have similarly $A^{\star}$ onto if and only if A is one-one.

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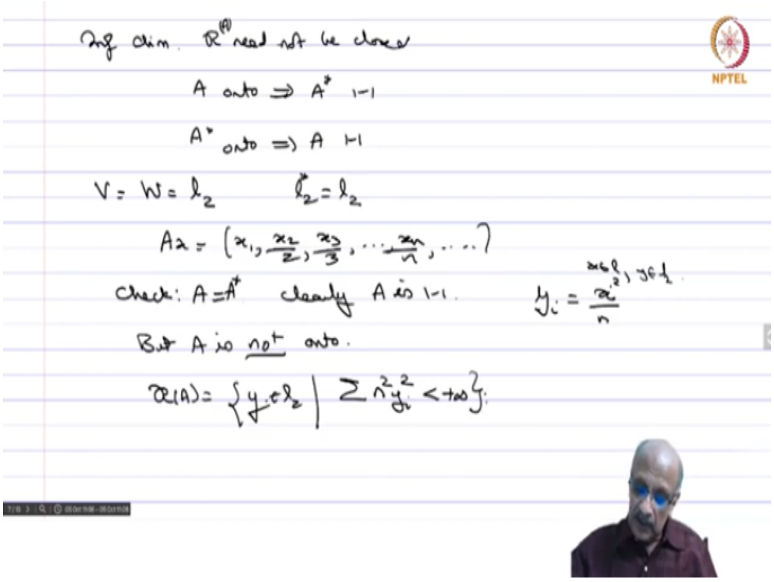

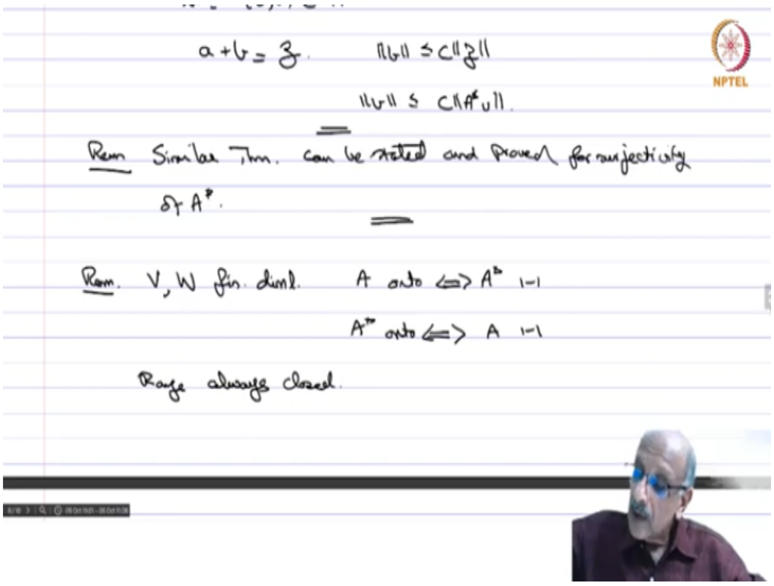

But here in infinite dimensions we have, we do not have, so here why is this true, because the range is always closed because in finite dimensional range is always closed. That is why these theorems work. But in infinite dimensions the $R(A)$ may not be closed and therefore, infinite dimensions range need not be closed. So, you can only say from the above theorem that A is onto implies A^* is one-one and then A^* onto implies A is one-one you cannot go back and say the thing unlike in the finite dimensional case.

Let us take an example,

Example: Let $V = W = l_2$ again remember these are sequences which are square integrable and therefore, you have l_2^* is the same as the l_2 and we know what is the duality bracket, it is just the usual inner product. So, let us take $A(x) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdot, \cdot, \cdot, \frac{x_n}{n}, \cdot, \cdot, \cdot,)$. So, then you can check $A = A^\star$, so this operator and then clearly A is one-one. But A is not onto because what is the range the $R(A)$ is set of all x in l_2 to such that x, so this is the range means it must be n times this must, so if you have let us say $y \in l_2$ to be clearer.

So, we have $y_i = \frac{x_i}{n}$ where $x \in l_2$ and $y \in l_2$. So, you have $x_i = ny_i$. So, $R(A) = \left\{ y \in l_2 \colon \sum n^2 y_i^2 < \infty \right\}$. So, this is a very stringent requirement. So, this is the range of A, so this is not an onto map and then you can also check then the range is not closed. So, with this we complete this chapter and we will do some exercises next.