Functional Analysis
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Lecture No. 22
4.6-Unbounded Operators, Adjoints-Part 2

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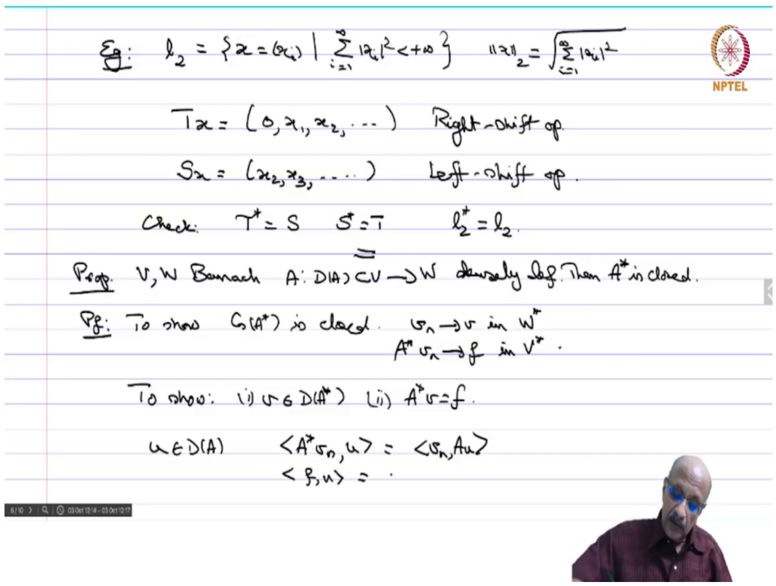

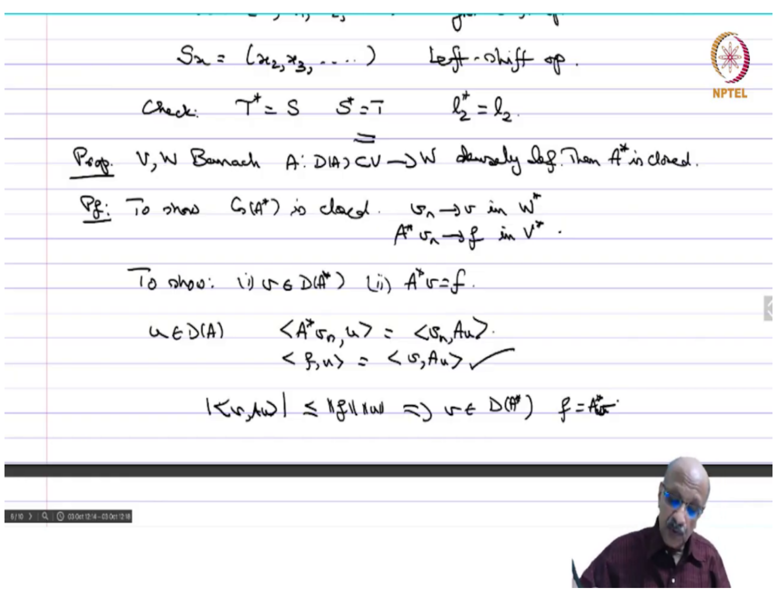

So, another example.

Example: So, let us take $l_2 = \{\ x = \left(x_i\right):\ \sum_{i=1}^{\infty} \left|x_i\right|^2 < \infty\}$. And what is the $||x||_2$? Is $\sqrt{\sum_{i=1}^{\infty} \left|x_i\right|^2}$. Define two operators, so $T(x) = (0, x_1, x_2, \cdot, \cdot, \cdot,)$. So, you push it to the right, put a zero in the front so, this is called the right shift operator. Similarly, $S(x)$ we are going to define as $S(x) = (x_2, x_3, \cdot, \cdot, \cdot,)$. So, you push it to the left and you get rid of the first coordinate x_1 and this is called the left shift operator. Both of them are continuous linear operators because $||Tx|| = ||x||$ in fact, and $||Sx|| \leq ||x||$. So, we have these two are continuous linear operators. So, one can easily check that $T^{\star} = S$ and $S^{\star} = T$. Sure, of course, we are identifying $l_2^{\star}$ is the same as l_2. So, that is why they are in the same spaces.

So, now, we will look at some properties of this adjoint. So, the first important property so,

Proposition: V, W Banach, $A: D(A) \subset V \rightarrow W$ is densely defined. So, that the adjoint is defined, then $A^{\star}$ is closed.

Proof: So, to show $G(A^{\star})$ is closed. So, we take (v_n) converges to v in $W^{\star}$ and $(A^{\star}v_n)$ converges to f in $V^{\star}$. And what must we show, we must show? We have to show two things now, because we are dealing with unbounded operators, which may not be defined everywhere but only on a certain domain. So, first we have to show, that $v \in D(A^{\star})$ and second, after that we can show that $A^{\star}v = f$. So, we have to show both these things. So, but what is, take any $u \in D(A)$, what is the definition of the adjoint? So, $\langle A^{\star}v_n,\ u\rangle = \langle v_n,\ Au\rangle$. Now you pass to the limit so, you get $\langle f,\ u\rangle = \langle v,\ Au\rangle$. So, $|\langle v,\ Au\rangle| \leq ||f||||u||$. So, this implies that $v \in D(A^{\star})$. And $\langle f,\ u\rangle = \langle v,\ Au\rangle$ here, which is true for all $u \in D(A)$ tells you that $f = A^{\star}$, because it satisfies the defining condition for $A^{\star}v$. So, this proves this proposition.

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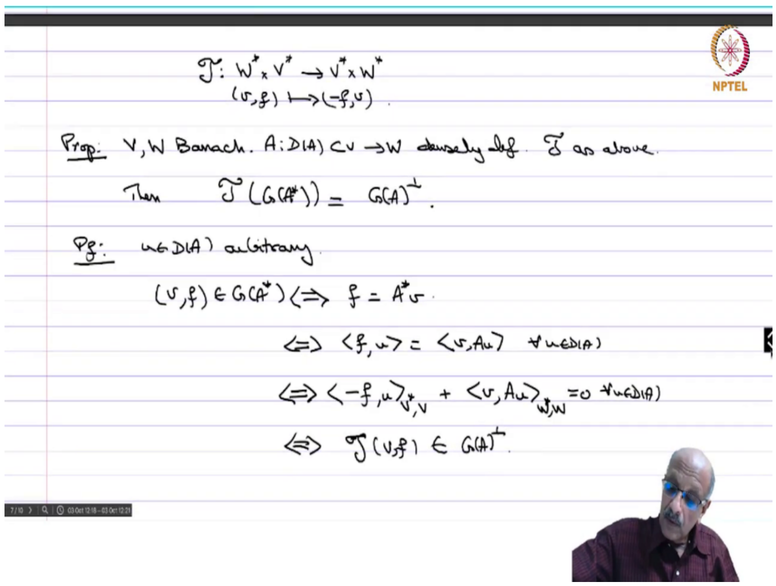

Now the graph of A and graph of $A^\star$ are related by a very simple relationship. So, let us now define I: $W^\star \times V^\star \to V^\star \times W^\star$. So, we take (v, f) and map it to $(-f, v)$. So, we are just flipping the coordinates and putting a minus sign in front. And proposition,

Proposition: V, W Banach, $A: D(A) \subset V \to W$ is densely defined, I as above then $I\left(G\left(A^\star\right)\right) = G(A)^\perp$.

So, you see the, annihilator and the graphs and annihilators they all come together. So, proof two lines.

Proof: So, let $u \in D(A)$ arbitrary. So $\langle v, f \rangle \in G\left(A^\star\right)$, this means what? $f = A^\star(v)$. And that implies $\langle f, u \rangle = \langle v, Au \rangle$, this is for every $u \in D(A)$. I am not putting the subscripts, we know where these things are acting and, therefore. So, this is equivalent to saying $\langle -f, u \rangle_{V^\star, V} + \langle v, Au \rangle_{W^\star, W} = 0$ for every $u \in D(A)$. Now, (u, Au) is a typical element of the graph of A and this tells you that $(-f, v)$ is killing all of them. So, that means that $I(v, f) \in G(A)^\perp$ and all these implications go, work both ways. And therefore, that proves this proposition.

So, now, we come back to the question of bounded operators.

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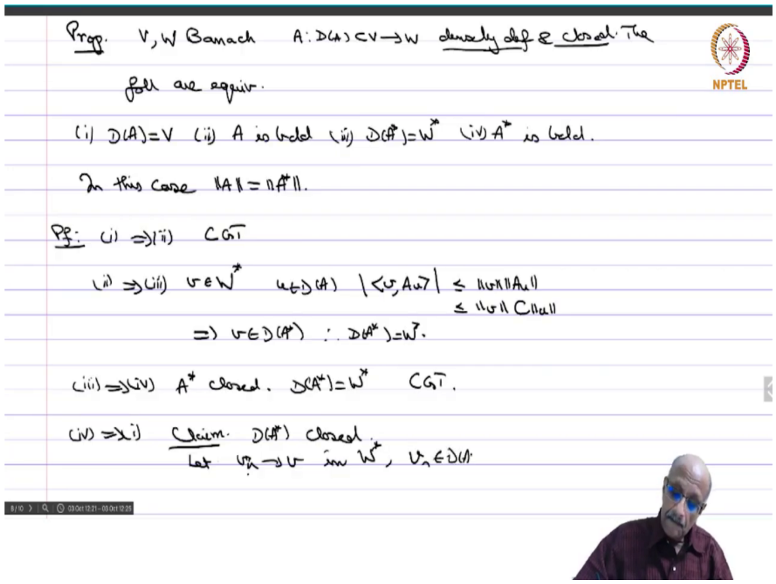

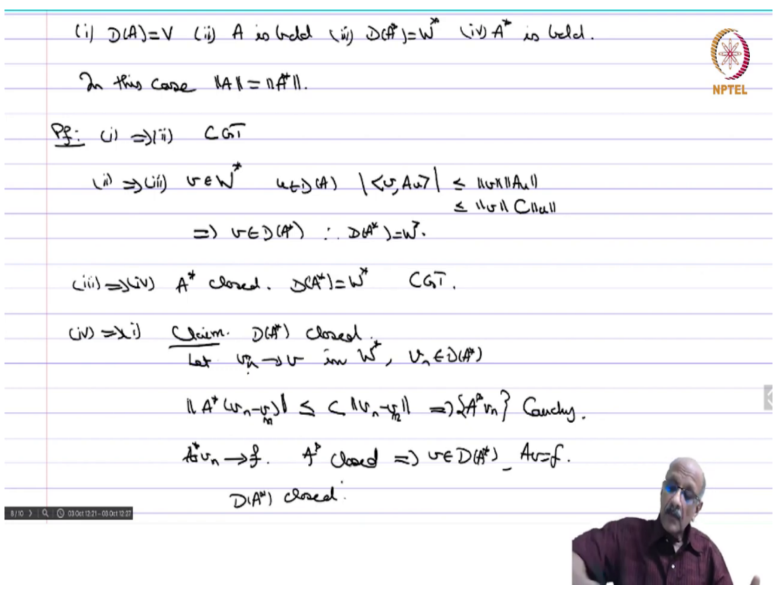

So, we have the following important proposition.

Proposition: V, W Banach, $A: D(A) \subset V \to W$ is densely defined and closed. So, we are making an extra hypothesis on this. Then the following are equivalent:

(i) $D(A) = V$, (ii) A is bounded, (iii) $D\left(A^{\star}\right) = W^{\star}$, and (iv) $A^{\star}$ is bounded.

All these and in this case we have, everything is a continuous linear operator now, so $||A|| = ||A^{\star}||$.

Proof: $(i)\Rightarrow(ii)$ $D(A) = V$. So A is defined on the entire space, they are all Banach spaces. And A is closed. That means the graph is close. So, this is just a closed graph theorem. So, that is it.

Now $(ii)\Rightarrow(iii)$ A is bounded. Therefore, we want to $D(A)$ so. So, let $v\in W^{\star}$ we have already done this calculation a few minutes ago. So, if you take $u\in D(A)$ and then you take $|\langle v, Au\rangle| \leq ||v||||Au|| \leq ||v||C||u||$. So, this implies that $v\in D\left(A^{\star}\right)$ by definition and therefore, $D\left(A^{\star}\right) = W^{\star}$. We did this a little earlier already.

Now $(iii)\Rightarrow(iv)$ We know that $A^{\star}$ is closed and, $D\left(A^{\star}\right) = W^{\star}$. Therefore, again by the close graph theorem, we have that $A^{\star}$ is a bounded linear operator or continuous linear operator. So, now we only have to show

$(iv)\Rightarrow(i)$ So, first claim, $D\left(A^{\star}\right)$ is closed. So, if $A^{\star}$ is bounded, then $D\left(A^{\star}\right)$ is closed. So, let $v_n\rightarrow v$ in $W^{\star}$ and $v_n \in D\left(A^{\star}\right)$. Then $||A^{\star}(v_n - v_m)|| \leq C||v_n - v_m||$ because $A^{\star}$ is bounded. So, this implies that $\{A^{\star}v_n\}$ is Cauchy. So, $\left\{A^{\star}v_n\right\}$ converges to some f, but then so, $\{v_n\}$ converges to v, $\left\{A^{\star}v_n\right\}$ goes to f therefore, when $A^{\star}$ is closed and therefore, this implies that $v\in D\left(A^{\star}\right)$ and $Av = f$. Therefore, we have the $D\left(A^{\star}\right)$ is closed.

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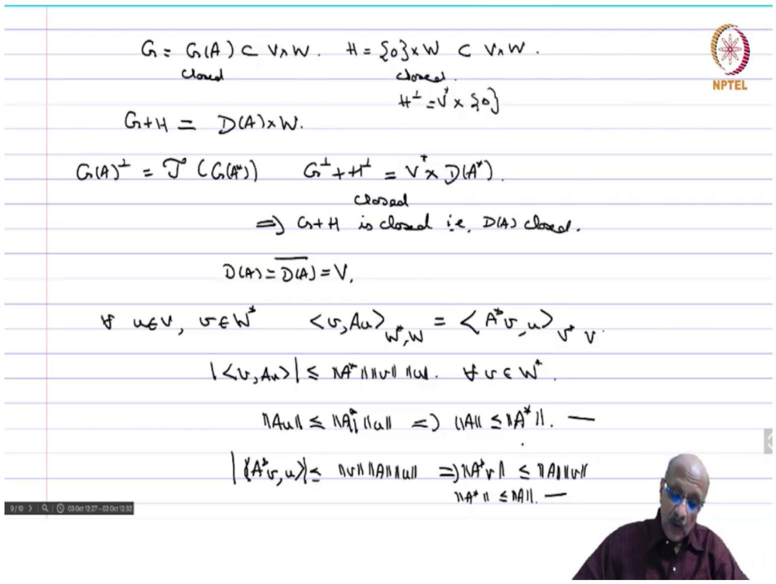

Now, we are going to, we will set G to be $G(A)$. Graph of A this is contained in $V \times W$ and H to be $\{0\} \times W$ which is again in $V \times W$. Now, by hypothesis $G(A)$ is closed, and $\{0\} \times W$ is trivially closed. So, there is no problem. And now what can you say about $G + H$? So you are taking an element in $G(A)$, How does an element in $G(A)$ look like? The first coordinate is from the domain, second component is $A(u)$. So, $(u,\ Au)$ and here they are taking 0 any element in W. So, when you add these two, you will get elements in the domain and any element in W, so this will be $D(A) \times W$. Now $G(A)^{\perp} = I\left(G\left(A^{\star}\right)\right)$ flipping and putting a minus sign and therefore we have $G^{\perp} + H^{\perp}$. Now what is $H^{\perp}$? So, $H^{\perp}$, you want to kill everything here. So, the second one, if you want to kill all of W you will have to be 0 and if you want to kill 0 any, any element is fine. So, this will be $V^{\star} \times \{0\}$.

So, if you so, if you add these two, so you have $V^{\star}$ here in the first component and anything. So, this will be $V^{\star}$ cross second component is 0 and in $G\left(A^{\star}\right)$, the first component was $D\left(A^{\star}\right)$ that I have flipped it and so, in j, so this will be $D\left(A^{\star}\right)$. And then $G^{\perp} + H^{\perp}$ is given to be closed, because $D\left(A^{\star}\right)$ is closed and $V^{\star}$. So, $G^{\perp} + H^{\perp}$ will be closed and then we prove this theorem, that this implies, this is the only part of that theorem which I did not prove.

This implies that $G + H$ is closed, that is $G + H$ is $D(A) \times W$ therefore, $D(A)$ is closed therefore, $D(A)$ equals $\overline{D(A)}$, which will be equal to V and that is the first part of the theorem, which we wanted to prove. So, this proves the equivalence of all the statements. So, now, let us, so for all $u \in V$ and $v \in W^{\star}$ you have $\langle v, Au \rangle_{W^{\star},W} = \langle u, A^{\star}v \rangle_{V^{\star},V}$. So, $|\langle v, Au \rangle| \leq ||A^{\star}|| ||v|| ||u||$ for every element in the dual space so, for all $v \in W^{\star}$ and then we have from $|\langle v, Au \rangle|$, we saw corollary because what is the corollary of the Hahn Banach theorem, in fact $||Au||$ is a max namely, the norm of the vector is sup of $||v||$ less than equal to 1, $|\langle v, Au \rangle|$. So, $||Au||$ will be less than or equal to $||A^{\star}|| ||u||$ and this implies that $||A|| \leq ||A^{\star}||$.

On the other hand, the same relationship will tell you $\left|\langle A^{\star}v, u \rangle\right| \leq ||v|| ||A|| ||u||$. And $A^{\star}v$ is now a dual element so, you just have to take the supremum over all u and therefore, this tells you that $||A^{\star}v|| \leq ||A|| ||v||$ and therefore, you have $||A^{\star}|| \leq ||A||$. So, we have both inequalities and that shows that the two norms are equivalent.

So, we will continue with some properties of the dual, such as the adjoints. So, we will connect what is the relationship between the range null space of the, of an operator and the range and null space of the adjoints and in between some annihilators will have to come in. So, we will see how these things work out.