Functional Analysis
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Lecture No. 21
Unbounded Operators, Adjoints – Part 1

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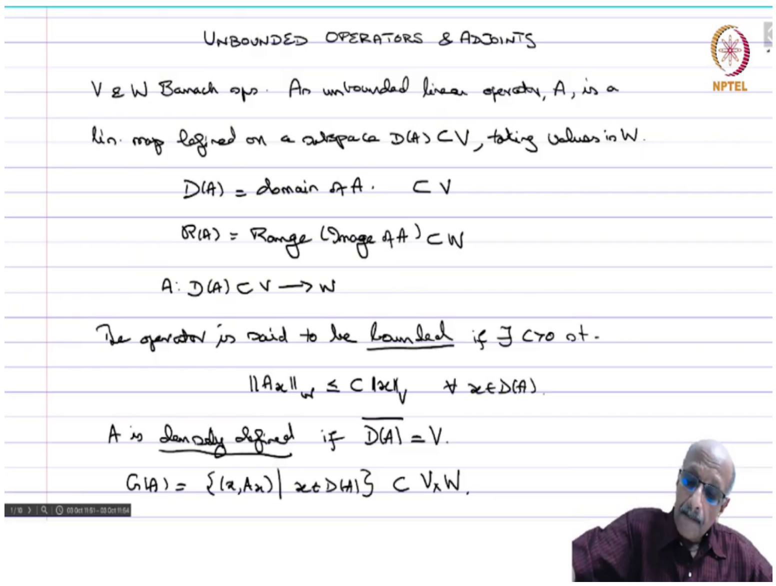

We will now discuss unbounded operators and adjoints. So, we will now look at linear maps, which are not necessarily defined on the entire vector space and also which may not be bounded or continuous in the sense that we have known up to now. The continuous linear operators of bounded linear operators which we have studied so far will come as a subclass of this, as a particular case.

So, let us have V and W Banach spaces, so an unbounded operator, linear operators A is a linear map defined on a subspace $D(A)$ contained in V taking values in W. So, such a map so, it is a linear map, but it can be defined only on a subspace, not necessarily the entire space. So, we say $D(A)$ is the domain of A, we denote by $R(A)$ is the range or image of A. So, we say that A is from $D(A)$ contained in V into W.

So, $D(A)$ this will be contained in V and $R(A)$ this will be contained in W. The operator is bounded, is said to be bounded if there exists C positive, such that $\|Ax\|_W \leq C\|x\|_V$, only now we have for every $x \in D(A)$ because that is where the operator is defined. So, the, then it is said to be densely defined, A is densely defined, if $D(A)$ is dense in V, that is $\overline{D(A)} = V$. Then we define the graph of A, $G(A)$ is set of all $(x,\ Ax)$, where $x \in D(A)$ and this, of course is contained in $V \times W$.

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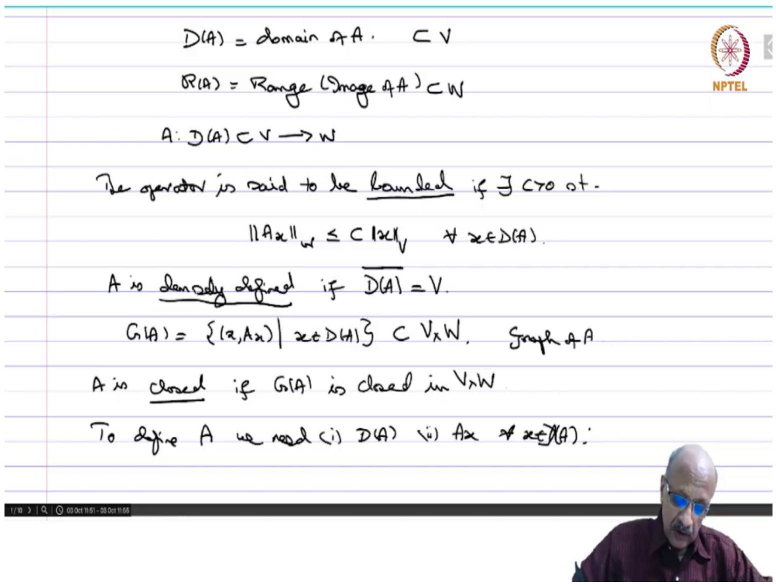

And the operator A is closed, if $G(A)$ is closed, so the graph of A should be, so, $G(A)$ is called the graph of A. So, it is a closed subspace of $G \times W$, then you say the operator is closed. So, to define an operator, we need to specify two things. So, to define A, we need 1: $D(A)$. So, we have to say what, where it is defined and 2: what its action is? $A(x)$ for every $x \in D$. So, we have to specify both these things in order to completely define an operator A.

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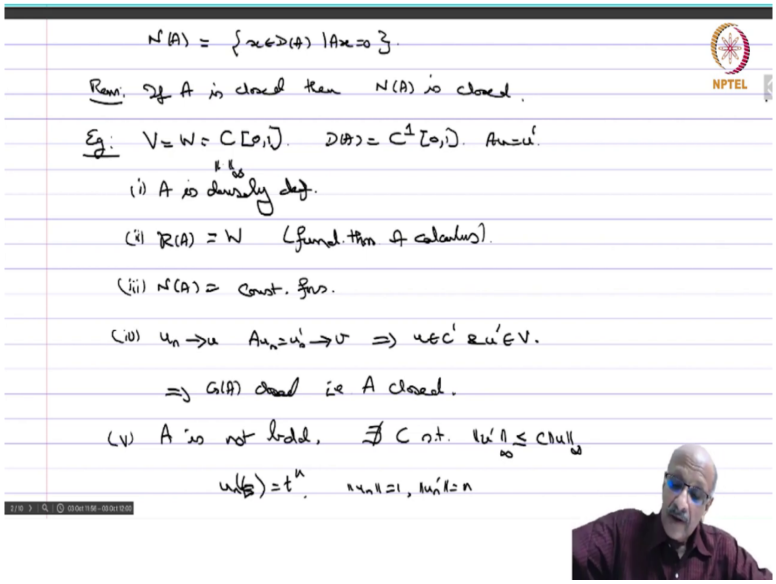

So, then the null space of A or $N(A)$ equals set of all $x \in D(A)$ such that $A(x) = 0$. Unlike norm linear, continuous linear operator you cannot say that $N(A)$ is closed. However

Remark: If A closed, you can check this very easily, then $N(A)$ is closed.

Now, all this terminology is a little bit unfortunate. Normally, logically, we should have first considered these operators and then considered the continuous linear operators or bounded linear operators, which we have been studying (06:04) as a subclass, but historically this is how it has been evolving. And so we will use this terminology.

Henceforth, of course, I will say continuous linear operator to distinguish between bounded linear operators which may not be defined on the whole space. So, for me a continuously operated means $D(A) = V$, and A is bounded in the usual, as we have been studying up to now. Otherwise, when I say bounded operator then it may be just defined on $D(A)$ and not on the whole space.

So, let us give an example.

Example: Let us take $V = W = C([0, 1])$, and let us take $D(A) = C^1[0, 1]$, continuously differentiable functions on the closed interval $[0, 1]$. And then you define $A(u) = u'$, which is the first derivative of this. So, if u is in $D(A)$, then u, it is a C^1 function. So, its derivative is continuous and that is exactly where we are. So, now, first of all, A is densely defined. Because $C^1[0, 1]$ is dense $C([0, 1])$. In particular, you have the (07:33) approximation theorem. So, then it is, range $R(A)$ is entire W i.e., $C([0, 1])$. Because if you take any continuous function, then it is indefinite integral is a differentiable function and its derivative is the given function. So, by the fundamental theorem of calculus you have, so this is nothing but the fundamental theorem of calculus. Then what is $N(A)$, this is just constant functions, so that is precisely the thing.

Now, 4: suppose, what is convergence. So, what is the norm here, it is a usual soup norm, which is the norm infinity and therefore, convergence in this norm means uniform convergence and therefore, if you take $u_n \to u$ and $A(u_n)$ which is u_n' converges to v, then we know that is u_n converges to uniformly and u_n' converges uniformly, then we know from real analysis course on uniform convergence that $u \in C^1[0, 1]$ and $u' \in V$. And therefore, this implies that $G(A)$ is closed, that is A is closed.

5: A is not bounded. We have already seen this, that is you cannot have a constant, so that does not exist C such that $\|u'\|_\infty \leq C\|u\|_\infty$. And for this we took the sequence, we have already seen this $u_n(t) = t^n$. So, then norm u_n, $\|u_n\|_\infty$ is always 1, $\|u_n'\|_\infty$ is n and therefore, you can never have a constant like this, because C is a finite number. So, this is an example.

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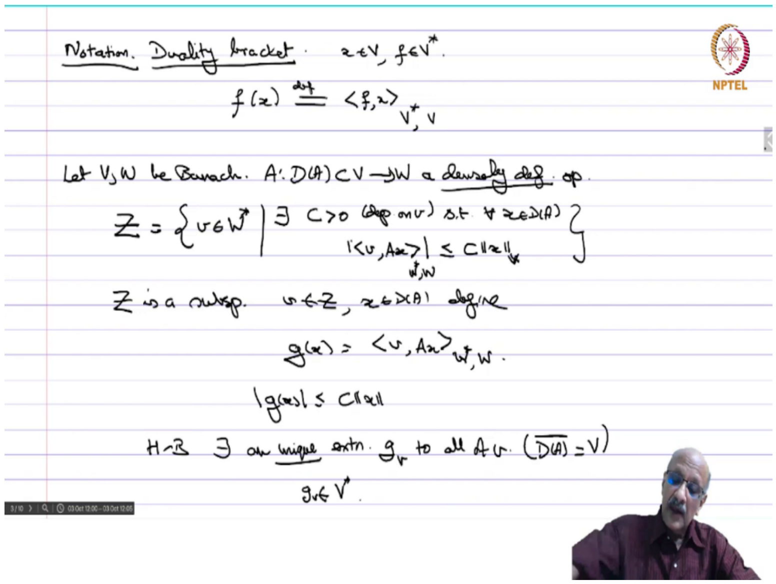

So, now we are going to make an important definition and for that we need some notation. This is called the duality bracket. So, we take $x \in V$ and let us take $f \in V^{\star}$. So, up to now the evaluation I have denoted as $f(x)$, now I am going to define as $\langle f,\ x\rangle$, I should probably write it the other way around, does not matter. So, this is $\langle f,\ x\rangle$, and to ensure where we are working, so I will say $\langle f,\ x\rangle_{V^{\star},V}$. So, it is a bracket, the first element in the bracket will be the linear functional and x is the point where it is being evaluated and this bracket is just the evaluation of these two. And to say, where we are working. So, we are saying $V^{\star}$ and V and therefore, this is a notation for the duality bracket.

So, now, let V, W be Banach and A from $D(A)$ contained in V taking values in W, a densely defined operator, this is important. So, now, we are going to make a definition of a subspace.

Definition: So, Z is equal to set of all, $v \in W^{\star}$, such that there exists a constant C positive, which of course, depends on v, such that for all $x \in D(A)$, we have $|\langle v,\ Ax\rangle| \leq C\|x\|$. So, here $\langle v,\ Ax\rangle$ is $\langle v,\ Ax\rangle_{W^{\star},W}$, because $v \in W^{\star}$, $Ax \in W$, so, this duality bracket there and $\|x\|$ of course, is a norm $\|x\|_{V}$.

So, this is Z, so then Z is a subspace that you can easily check. So, if $v \in Z$ and $x \in D(A)$, define $g(x) = \langle v,\ Ax\rangle$ again, this is $\langle v,\ Ax\rangle_{W^{\star},W}$. Then $|g(x)| \leq C\|x\|$. So, g is a continuous linear functional. So, by Hahn Banach there exists an …… extension, g_v to all of v. Why did they leave some space here? I want to write a unique extension.

Hahn Banach extensions is not, not unique, but now it will be unique because $\overline{D(A)} = V$. So, if you have something even in Hahn Banach theorem, we have seen if some, if you were, what is the test of density? If something vanishes, linear function vanishes on the space, then it vanishes everywhere. So, that automatically tells you that if you have g which is defined on a dense subspace, then it has a unique extension to the, to all of V. So, g_v is again a continuous linear functional. So, now $g_v \in V^{\star}$. So, starting with something in $W^{\star}$, we have defined something in $V^{\star}$ provided it is in Z. So, we have, we have taken $v \in Z$.

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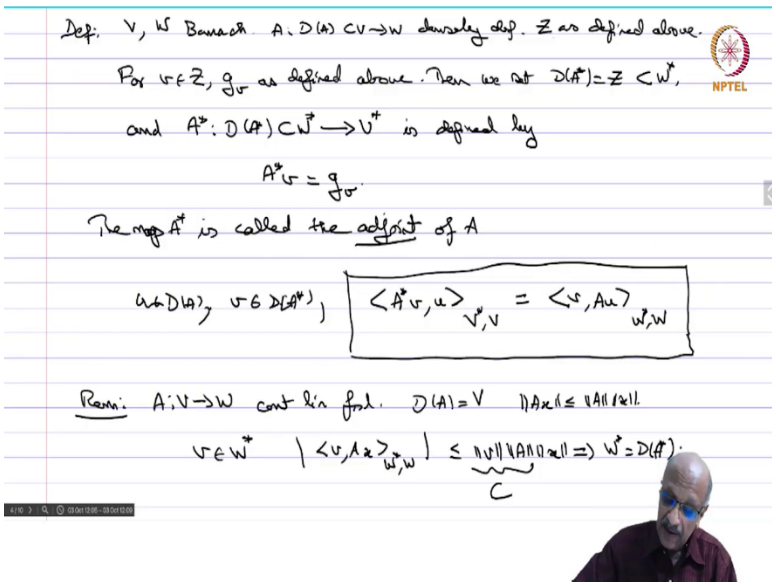

So, now we make a definition.

Definition: So, V, W Banach, A from $D(A)$ in V taking values in W, densely defined, Z as defined above and for $v \in Z$, g_v as defined above. Then we set $D(A^\star) = Z$ which is contained in $W^\star$ and $A^\star$: $D(A^\star) \subset W^\star \rightarrow V^\star$ is defined by $A^\star v = g_v$. The map $A^\star$ is called the adjoint of A. So, this is the so, the adjoint is defined for densely defined operators, of course, $A^\star$ may not be densely defined it is defined on Z or $D(A^\star)$ which is some subspace that is all we know, we have no information other than that.

So, the important characterization is, if $u \in D(A)$ and $v \in D(A^\star)$, you have, from the definition of g_v, what is the definition of g_v? Here it is $\langle v, Ax \rangle$. So, we have $\langle A^\star v, u \rangle$, this is $\langle A^\star v, u \rangle_{V^\star, V}$ because $A^\star v \in V^\star$, this $u \in V$, this is nothing but by the definition $\langle v, Au \rangle$ and this is $\langle v, Au \rangle_{W^\star, W}$. So, this is the fundamental defining relationship for the adjoint, and if any map, any continuous linear functional in $V^\star$ defines this relationship then it, by the density and uniqueness we know that it has to be $A^\star v$.

So, let us briefly look at continuous linear functionals. So, remark,

Remark: Suppose $A: V \rightarrow W$continuous linear functional that means, $D(A) = V$ and $\|Ax\| \leq \|A\|\|x\|$. So, this we have. Now, we will take any $v \in W^{\star}$, you have that $\left|\langle v,\ Ax \rangle_{W^{\star},W}\right| \leq \|v\|\ \ \|Ax\| \leq \|v\|\|A\|\|x\|$. And therefore, you have, so $\|v\|\|A\|$ will be the constant C and therefore, you have that entire $W^{\star}$ is the domain $D(A^{\star})$. So, the adjoint is defined on the entire space. So, this is information. So, if you have continuous linear operators, in fact, we will see a little more about this in a theorem at the end of this session.

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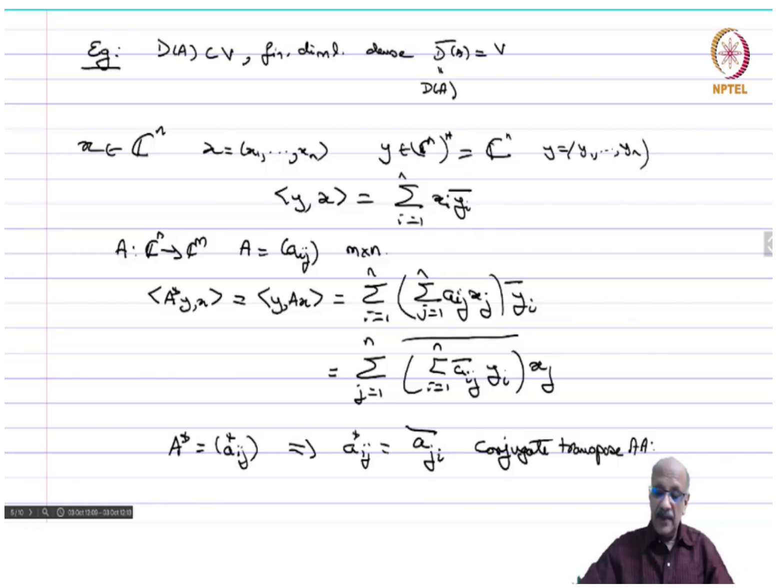

So, let us see some examples.

Example: So, let us look at finite dimensional spaces. So, in the finite dimensional space, all subspaces are closed. Therefore, in particular if you have $D(A)$ contained in V finite dimensional, densely defined, so, this means $\overline{D(A)} = V$, but $\overline{D(A)}$ is same as $D(A)$ because every subspace in finite dimensional space is automatically closed. Therefore, when you say in the finite dimensional case, that is it is always defined on the entire space, a densely defined operator is A.

So, we can and you know once you have this, we have already observed, so that you can define the adjoint on all of the dual space, which is again isomorphic to the same space. So, let us take C^n and then the dual is the same. So, if $x \in C^n$, namely $x = (x_1, x_2, \cdots, x_n)$ and if y belongs to $(C^n)^\star$, which is also C^n again and $y = (y_1, y_2, \cdots, y_n)$, then you have $\langle y, x\rangle$ is nothing but $\sum_{i=1}^{n} x_i \overline{y_i}$. This is the convention, if you have R^n to R^n, then this conjugate will not be there, it will be just $\sum_{i=1}^{n} x_i y_i$.

Now, let $A\colon C^n \to C^m$ be a linear transformation. So, then A will be represented by some matrix (a_{ij}) which is $m \times n$ matrix. So, m rows and n columns. So, what is $\langle A^\star y, x\rangle$ so this will be by definition $\langle y, Ax\rangle$ and therefore, that will be $\sum_{i=1}^{n} (\sum_{j=1}^{n} a_{ij} x_j) \overline{y_i}$. So, now, I want to write it in the form $\langle A^\star y, x\rangle$. So, I should bring out the x's outside and then bring in the y with, everything should be in conjugate form for the $A^\star y$. So, this will be equal to $\sum_{j=1}^{n} \overline{(\sum_{j=1}^{n} \overline{a_{ij}} y_i)} x_j$. And therefore, we have, if $A^\star$ is the matrix $(a_{ij}^\star)$, then it automatically follows that $a_{ij}^\star$ is nothing but $\overline{a_{ji}}$. And this is the usual conjugate transpose of the matrix A. So, this is how the adjoint works, in case of real, just the usual transpose of the matrix. There is no conjugation involved.