Functional Analysis
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Lecture No. 20
Complemented Subspaces

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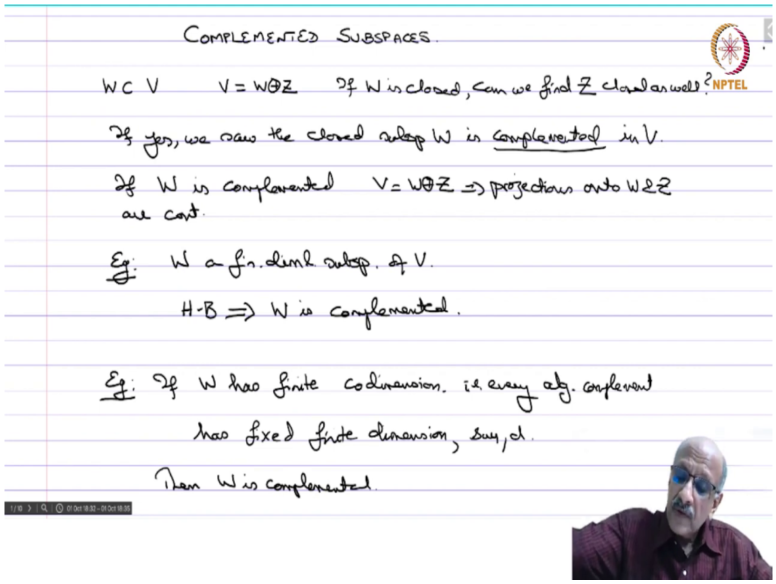

We will now look at complemented subspaces. If V is a vector space and W is a subspace of V, then in linear algebra we know by extending a basis we can always write $V = W \oplus Z$. Now, the question is, if W is closed, can we find Z closed as well? So, this is the question which we want to see. If yes, we say the closed subspace W is complimented in V. So, the question is, whether every close subspace has a compliment or not?

So, if W is complimented, then we have $V = W \oplus Z$, both of them are closed. Then, we saw by an application of the open mapping theorem that the projections onto W and Z are continuous. This was done last time when we used the open mapping theorem. So, let us take some examples.

Example 1. Let W be a finite dimensional subspace of V. Then W is automatically closed and we saw as an application of the Hahn Banach theorem that W is complemented.

Example 2. If W has finite co-dimension i.e., every algebraic complement has a fixed finite dimension say d then W is complemented. Why? Because any, any algebraic complement has finite dimension. So, it is automatically closed and therefore, trivially this is an acceptable decomposition. So, how do these finite co-dimension spaces happen?

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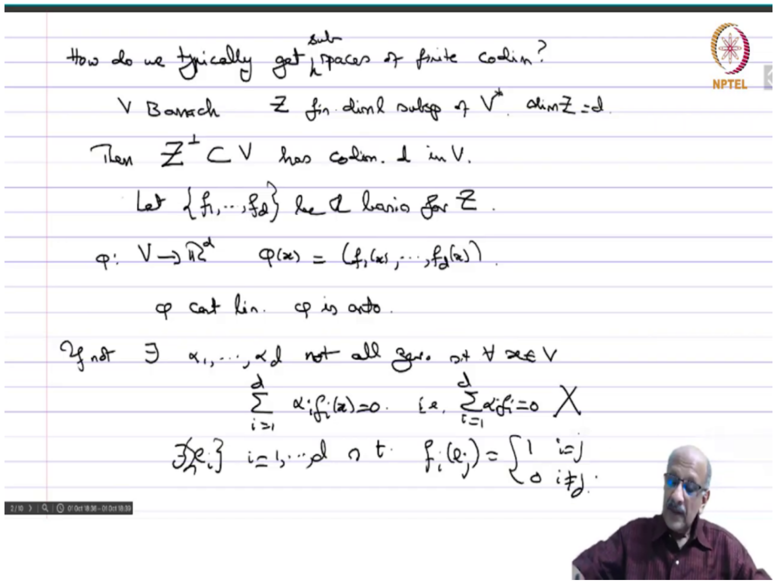

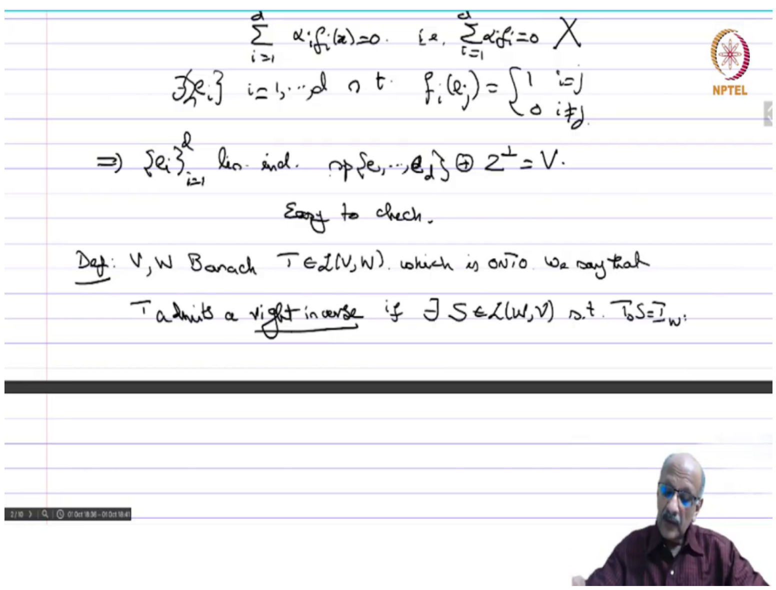

How do we typically get spaces of finite co-dimension? Let V be Banach and let us take Z finite dimensional subspace of V^*. Let us say dim $dim\, Z\ = d$. Then, $Z^{\perp} \subseteq V$ has co-dimension d in V. So, this is the typical way in which they occur. So, how do we have this? So, we define the map from $\phi: V \mapsto R^d$ as follows:

Let $\{f_1, \ldots f_d\}$ be a basis for Z, then you define $\phi(x) = (f_1(x), \ldots f_d(x))$. Then, of course, ϕ is continuous linear and in fact, ϕ is onto, because if it is not onto, then the image of ϕ is a subspace of R^d, which has strictly less dimension. Therefore, by the Hahn Banach theorem, there exists a continuous linear functional on R^d, which is not identically 0 but which vanishes on the image.

So, what is a continuously, in functional Rd? Again it is just d scalars not vanishing means, not all constants are 0, so there exists $\alpha_1, \ldots \alpha_d$ not all 0 such that it vanishes completely on this image of ϕ. So, for every $x \in V$, we have $\sum_{i-1,2,\ldots d} \alpha_i f_i(x) = 0$, that is $\sum_{i-1,2,\ldots d} \alpha_i f_i = 0$ which is a contradiction since the f_i's are linearly independent, they have taken to be a basis and therefore, these are linearly independent. Since this is onto, I can always find an x such that any given d dimensional vector, I have a solution. Therefore, in particular there exists $\{e_i\}_{i=1,2,\ldots d}$, such that $f_i(e_j) = 1$ for $i = j$, $f_i(e_j) = 0$ for $i \neq j$. Then it is clear that $\{e_i\}_{i=1,2,\ldots d}$ are linearly independent and $Span\{e_i\}_{i=1,2,\ldots d} \oplus Z^{\perp} = V$ (easy to check). Therefore, $Z^{\perp}$ has finite co-dimension.

So, this is how you have subspaces of finite co-dimension. So, now I am going to make a definition.

Definition. Let V, W Banach and $T \in L(V, W)$ which is onto. We say that T admits a right inverse if there exists $S \in L(W, V)$ such that $T \circ S = Id_W$ is identity.

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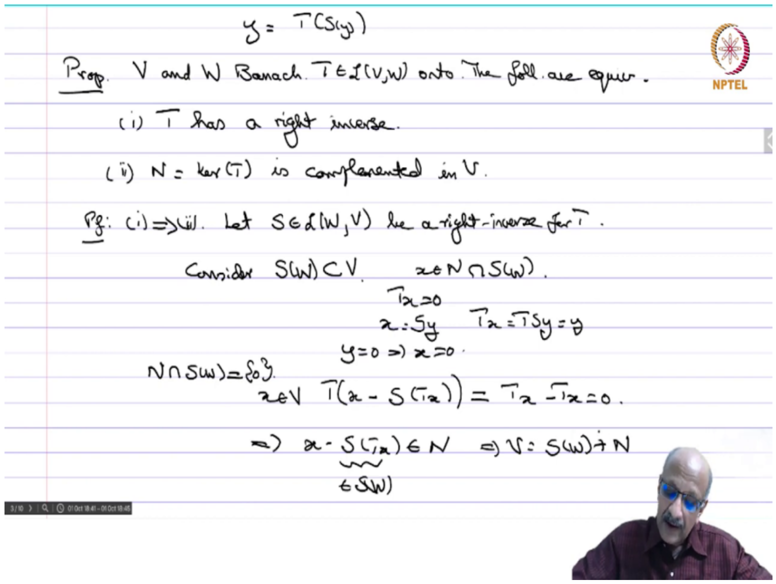

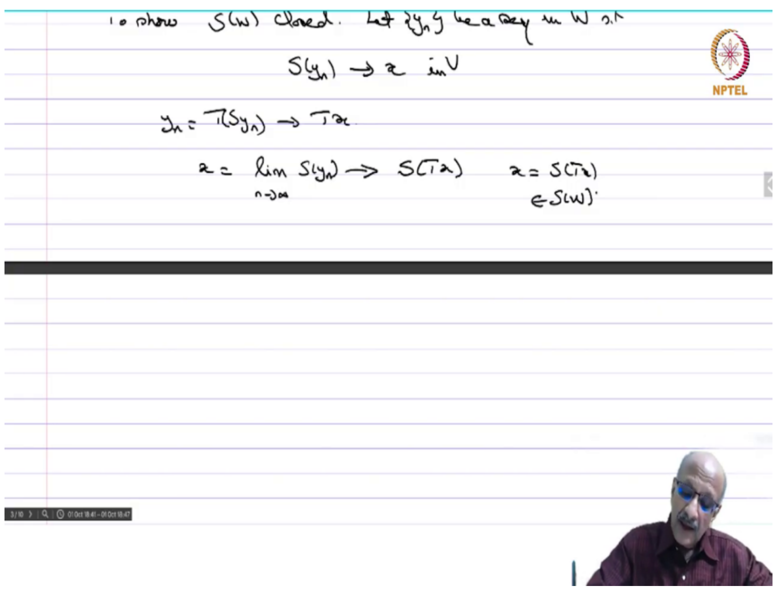

So, what does it mean? Any y will be equal to $T(S(y))$. So, necessarily if you have right inverse, the mapping has to be subjective. So, now we are going to characterize mappings which have right inverses,

Proposition. Let V and W be Banach and $T \in L(V, W)$ be onto. Then the following are the equivalent.

1: T has the right inverse.

2: $N = Ker(T)$ is complemented in V.

So, having a right inverse is equivalent to say that the subspace N is complemented. N is closed since it is the kernel of a continuous linear mapping.

Proof. 1 $\Longrightarrow$ 2: Let T has a right inverse. Let $S \in L(W, V)$ be a right inverse. You consider the space $S(W) \subseteq V$. I claim that $S(W)$ is in fact the complement for N. Let $x \in N \cap S(W)$. Since $x \in N$, $T(x) = 0$ and $x \in S(W)$ implies $x = S(y)$. So, $T(x) = T(S(y))$. But $T \circ S = Id_V$ and therefore, $y = 0$. This implies that $x = 0$. Therefore, we have that $N \cap S(W) = \{0\}$.

Next, let us take any $x \in V$. Then $T(x - S(T(x))) = 0$. Therefore, $x - S(T(x)) \in N$. Notice that $S(T(x)) \in S(W)$ and therefore, every x can be written as a vector in N plus a vector in $S(W)$ and therefore, this implies that $V = N \oplus S(W)$.

Finally, we have to show that $S(W)$ is closed. Let (y_n) be a sequence in W such that $S\left(y_n\right) \to x \in V$. Now, $y_n = T(S\left(y_n\right))$ and therefore, that should converge to $T(x)$. Therefore, $x = S(y_n) = S(T(x)) \in S(W)$. So, $S(W)$ is closed. This shows the $S(W)$ is closed and it is a complement algebraicall. So, the kernel is indeed complimented.

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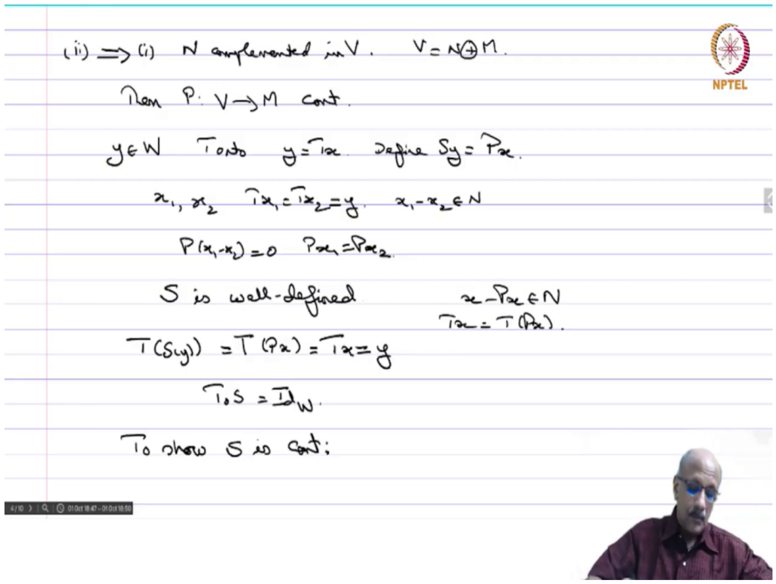

Now, 2 $\Longrightarrow$ 1: Let N be complimented in V. Let $V = N \oplus M$. So, M will be closed and N is already closed. Then the projection $P: V \mapsto M$ is continuous. We have to show that there is a right inverse, so let us define the map. Let $y \in W$. Since T is onto, $y = T(x)$. Define $S(y) = P(x)$. So, we have made a definition. Now, this definition depends on the pre-image of y. If we have x_1, x_2 such that $T(x_1) = T(x_2) = y$, then we have $x_1 - x_2 \in N$. Since P is the projection to the complement, $P(x_1 - x_2) = 0$. Therefore, $P(x_1) = P(x_2)$ and therefore, it does not depend, whatever maybe the pre-images and therefore, S is well defined.

Now, $T(S(y)) = T(P(x)) = T(x) = y$ (as $x - P(x) \in N$). So, $T \circ S = Id_W$. Therefore, we have found a candidate. Now, how to show that S is continuous?

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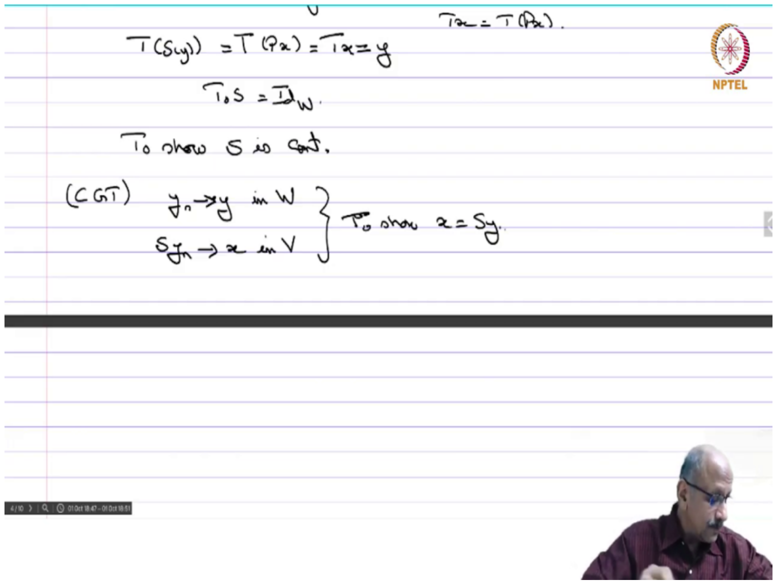

Let us do it by the close graph theorem. So, we will use the close graph theorem, we can do it with the open mapping theorem if you like so. Let us take $y_n \rightarrow y \in W$ and $S(y_n) \rightarrow x \in V$ and we have to show that $x = S(y)$. This is what we need to show.

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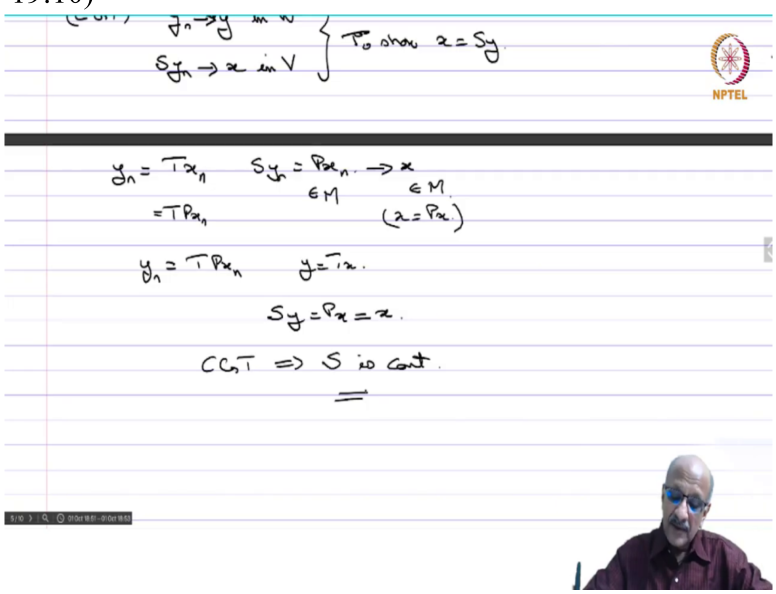

$y_n = T(x_n)$ for some $x_n \in V$. So, $S(y_n) = P(x_n) \rightarrow x$. Since the projection is always in M and M is closed, $x \in M$. So, in particular you have $x = P(x)$. Now, $y_n = T(P(x_n))$. So now, you pass to the limit to get $y = T(x)$.

So, $S(y) = P(x) = x$ and that is exactly what we want and consequently by the close graph theorem S is continuous.

So, the next thing we will do will be to look at adjoint of an operator and since, with a little extra cost, we can do it in a more general setting. So, we will take that up next. Thank you.