Functional Analysis
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IMSc
Examples of Normed Linear Spaces

So, let us recapitulate.

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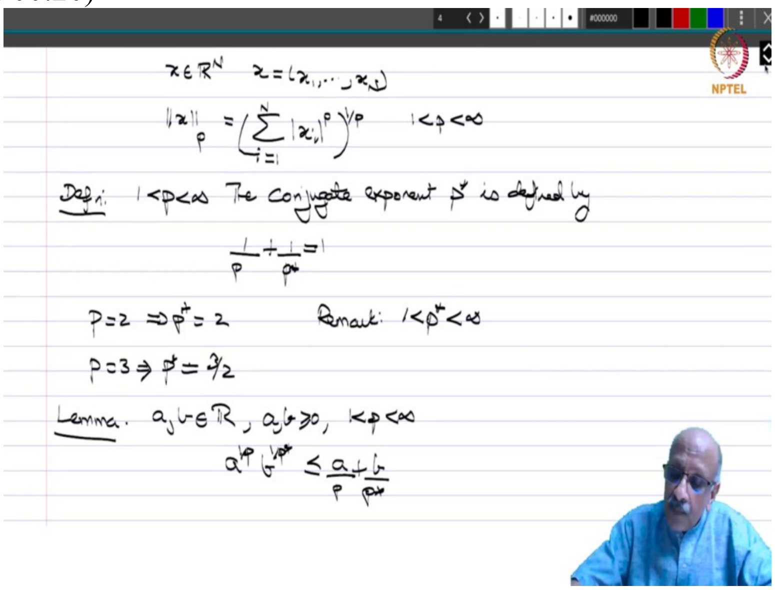

We have $x \in R^N$, $x = (x_1, x_2, \ldots, x_N)$, and we have $\|x\|_p = \left(\sum_{[i=1,2,\ldots N]} |x_i|^p \right)^{1/p}, 1 < p < \infty$. We want to show that $\|.\|_p$ satisfies the triangle inequality so that this will define a norm. We also saw that if this defines a norm, then R^N with this norm is in fact a complete normed linear space; or in other words a Banach space.

We make the following definition:

If $1 < p < \infty$, then the conjugate exponent $p^{¿}$ is defined by $\frac{1}{p} + \frac{1}{p^{¿}} = 1$. For instance if $p = 2$, then we have that $p^{¿} = 2$; if $p = 3$, then $p^{¿} = \frac{3}{2}$ and so on.

Remark. $1 < p^{¿} < \infty$.

Now, we have following lemma.

Lemma. Let $a, b \in R, a, b \geq 0, 1 < p < \infty$. Then $a^{\frac{1}{p}} b^{\frac{1}{p^{¿}}} \leq \frac{a}{p} + \frac{b}{p^{¿}}$.

So, if you look at this, if $p=2$, then $p^{¿}=2$; and therefore $a^{\frac{1}{p}}b^{\frac{1}{p^{¿}}}$ is nothing but the square root of ab, and the right hand side becomes $\frac{a+b}{2}$. So, this is nothing but the arithmetic mean greater than or equal to the geometric mean. Therefore, the above lemma is a generalization of this particular inequality; so let us try to prove the lemma.

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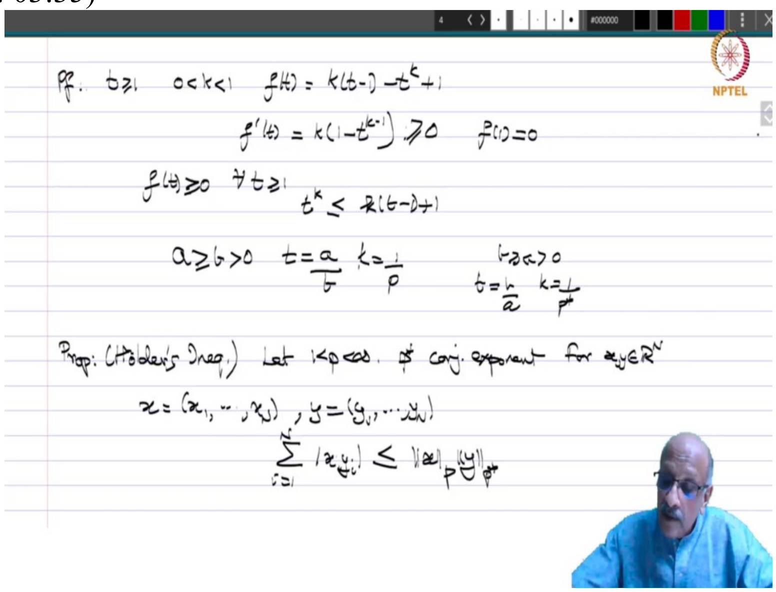

Proof of lemma. So, let $t\geq 1$ and $0<k<1$. Then, you look at the function $f(t)=k(t-1)-t^{k}+1$. Then $f^{'}(t)=k(1-t^{k-1})\geq 0$ (as $t\geq 1$, $0<k<1$). So, f is an increasing function. You also have that $f(1)=0$ and therefore $f(1)\geq 0$, $\forall t\geq 1$. This implies that $t^{k}\leq k(t-1)+1$. So, now if you look at the inequality which we want to prove, you have that, if a or b is 0 then there is nothing to prove. Thud, we can assume that a and b are not 0; so in particular let me assume that $a\geq b>0$. Then I put $¿\frac{a}{b},k=\frac{1}{p}$ and we would get the inequality which we wanted. If $b\geq a>0$, then we put $t=\frac{b}{a},k=\frac{1}{p^{¿}}$; and then we would get the required inequality. This proves the lemma.

Using this Lemma, we now prove an important result which is called Holder's inequality.

Holder's inequality. Let $1<p<\infty$, $p^{¿}$ is the conjugate exponent of p. Then, for $x,y\in R^{N}$ with $x=(x_1,x_2,\ldots,x_N)$, $y=(y_1,y_2,\ldots,y_N)$, we have

$$\sum_{i=1,..,N} |x_i y_i| \leq \|x\|_p \|y\|_{p^{¿}}.$$

This is a very important inequality which we will frequently come across in the future; and therefore we will take some time to prove this.

Proof of Holder's inequality. Again if x or y is 0; there is nothing to prove in this inequality.

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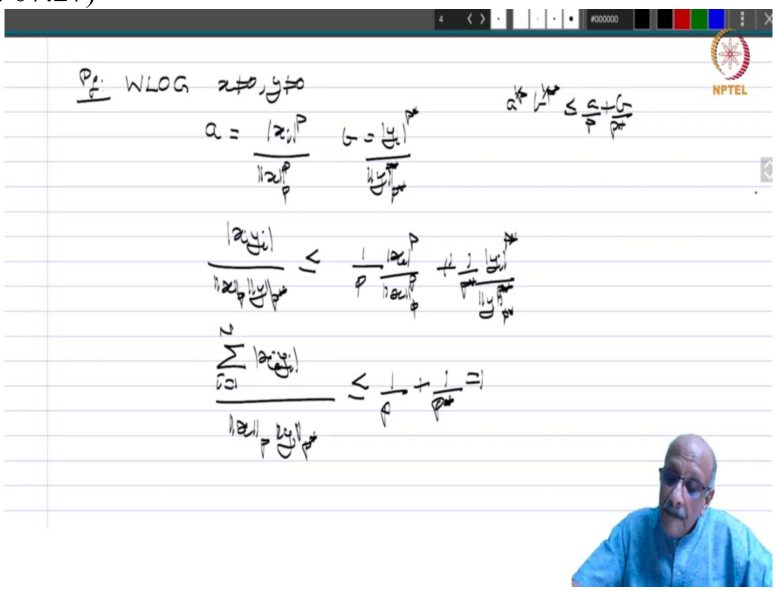

So, without loss of generality, we can assume that $x \neq 0, y \neq 0$. We now set $a = \frac{|x_i|^p}{\|x\|_p}; a = \frac{|y_i|^{p^{¿}}}{\|y\|_{p^{¿}}}$. And we apply the previous inequality $a^{\frac{1}{p}} b^{\frac{1}{p^{¿}}} \leq \frac{a}{p} + \frac{b}{p^{¿}}$. This gives

$$¿ x_i y_i \vee \frac{¿}{\|x\|_p \|y\|_{p^{¿}}} \leq \frac{1}{p} \frac{|x_i|^p}{\|x\|^p{}_p} + \frac{1}{p^{¿}} \frac{|y_i|^{p^{¿}}}{\|y\|_{p^{¿}}^{p^{¿}}} ¿ .$$

We sum over $i = 1,2,\ldots N$ and get

$$\frac{\sum_{i=1,..N} ¿ x_i y_i \vee ¿}{\|x\|_p \|y\|_{p^{¿}}} \leq \frac{1}{p} \frac{\sum_{i=1,..,N} |x_i|^p}{\|x\|^p{}_p} + \frac{1}{p^{¿}} \frac{\sum_{i=1,2,\ldots,N} |y_i|^{p^{¿}}}{\|y\|_{p^{¿}}^{p^{¿}}} = \frac{1}{p} + \frac{1}{p^{¿}} = 1 ¿.$$

Now we just cross multiply to have the result. So, this proves the inequality known as Holder's inequality.

We can use the Holder's inequality to prove the triangle inequality for $\|x\|_p$. We go to the next result.

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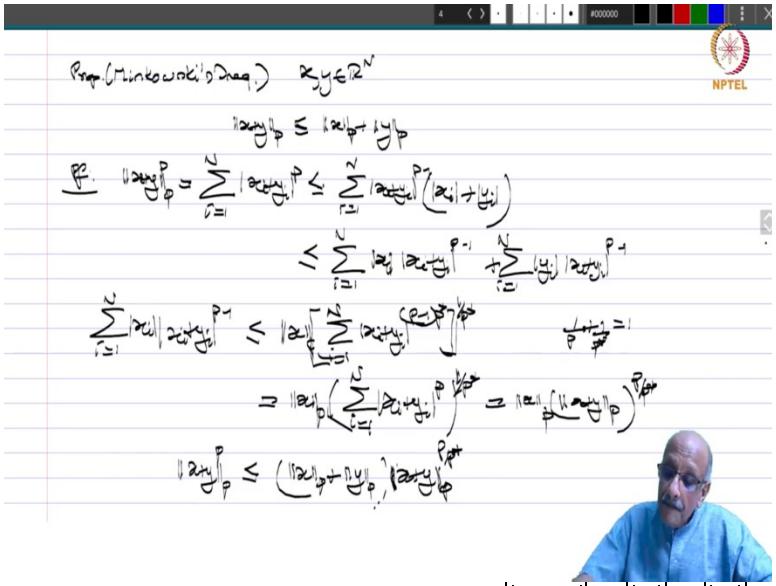


Minkowski's inequality. If $x, y \in R^N$, then we have the $\|x+y\|_p \leq \|x\|_p + \|y\|_p$.

Proof of Minkowski's inequality. Let us write

$$\|x+y\|_p^p = \sum_{i=1,\ldots N} |x_i + y_i|^p \leq \sum_{i=1,\ldots N} |x_i + y_i|^{p-1} \left(|x_i| + |y_i|\right)$$

$$= \sum_{i=1,\ldots N} \text{¿} x_i \vee |x_i + y_i|^{p-1} + \sum_{i=1,\ldots N} \text{¿} y_i \vee |x_i + y_i|^{p-1}$$

Now, to each of these terms we are going to apply Holder's inequality.

$$\sum_{i=1,\ldots N} \text{¿} x_i \vee |x_i + y_i|^{p-1} \leq \|x\|_p \left[\sum_{i=1,\ldots N} |x_i + y_i|^{(p-1)p^{\text{¿}}}\right]^{\frac{1}{p^{\text{¿}}}} = \|x\|_p \left[\sum_{i=1,\ldots N} |x_i + y_i|^p\right]^{\frac{1}{p^{\text{¿}}}}$$

$$\text{¿} \|x\|_p \|x+y\|^{\frac{p}{p^{\text{¿}}}}.$$

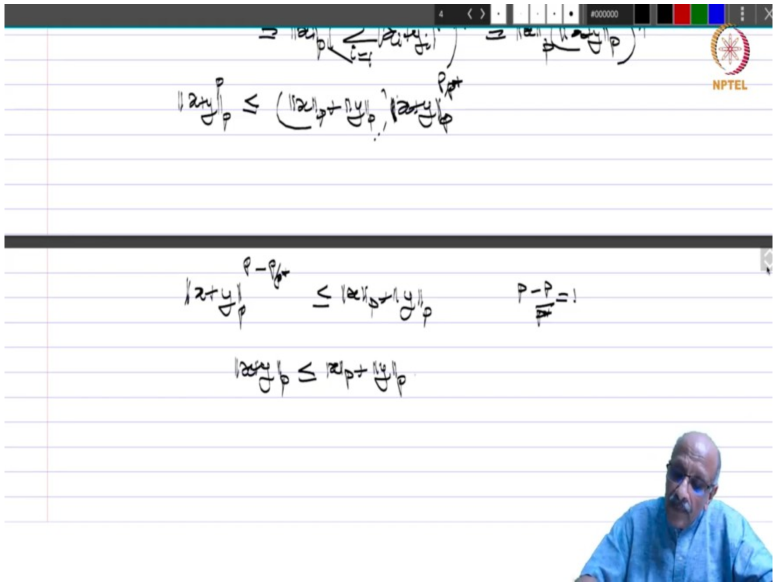
One can do the same thing for the second term. Therefore, we get

$$\|x+y\|_p^p \leq \left(\|x\|_p + \|y\|_p\right) \|x+y\|^{\frac{p}{p^{\text{¿}}}}.$$

Now, if $x+y=0$, we have nothing to prove in the Minkowski's inequality.

So, if $x+y \neq 0$, then we can divide by $\|x+y\|^{\frac{p}{p^{\text{¿}}}}$ and therefore you will get

$$\|x+y\|_p^{p-\frac{p}{p^{\text{¿}}}} \leq \left(\|x\|_p + \|y\|_p\right).$$



Since $p-\frac{p}{p^*}=1$, we get $\|x+y\|_p \le \|x\|_p + \|y\|_p$.

This proves the triangle inequality for $\|.\|_p$ and therefore they are all norms; and we also saw that, since Cauchy sequence means component wise Cauchy (coordinate wise Cauchy). Therefore, by the completeness of R, we also get the completeness of R^N for each of these norms. The case of $\|.\|_\infty$ is easier than all of $\|.\|_p$, so I will leave it as an exercise for you to do this. So, we have defined a whole family of norms and on finite dimensional spaces on R^N. You could have done it on C^N as well without any change in any of the proofs, and hence you have a lot of examples.
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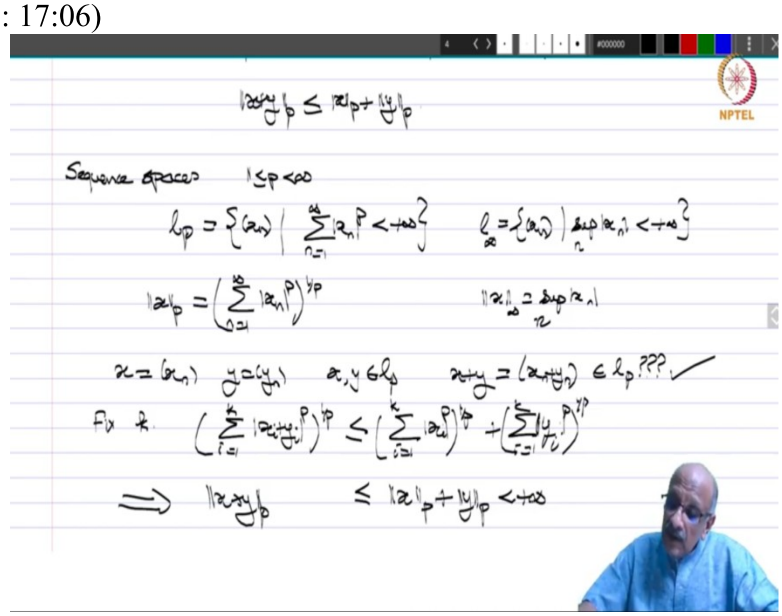

Now let us come to the question of sequence spaces; again we look at $1 \le p < \infty$,. And we define

$l_p = \left\{ (x_n) : \sum_{i=1,\ldots\infty} |x_n|^p < \infty \right\}$ and $l_\infty = \left\{ (x_n) : Sup_n |x_n| < \infty \right\}$.

So, l_∞ is the set of all bounded sequences, l_p is the set of all sequences that are p summable. Now we will define the norm on these spaces.

For $x=(x_n)\in l_p$, we define $\|x\|_p=\left(\sum_{n=1,\dots\infty}|x_n|^p\right)^{\frac{1}{p}}$ and that is finite and so it is well defined.

For $x=(x_n)\in l_\infty$, we define $\|x\|_\infty=Su\,p_n \vee x_n \vee ¿$ which again is well defined because the sequence is bounded.

So, I will leave it you to check (as we did it in the case of finite dimensions) that this satisfies all the properties of a norm. But, before it is a norm, we do not even know that l_p is a vector space. If you take a sequence $(x_n)\in l_p$ and multiply it by a number α, then of course the sequence $(\alpha x_n)\in l_p$ as well. But, if two if you have two sequences $(x_n),(y_n)\in l_p$ then, what is the guarantee that $(x_n+y_n)\in l_p$. This is the question which we should answer first, only then l_p will become a vector space; then only it will make sense for us to talk of a normed linear space.

We will do both together so again all the other properties of norm are obvious; in one stroke we will prove that l_p is a vector space and simultaneously $¿\vee.\vee¿_p$ satisfies the triangle inequality. Let us do that.

Let us take $x=(x_n),y=(y_n)\in l_p$; so the question is $x+y:=(x¿¿n+y_n)\in l_p,¿$ which is the sequence got by component wise addition. Fix k, then by the Minkowski's inequality

$$\left(\sum_{i=1,\dots k}|x_i+y_i|^p\right)^{\frac{1}{p}}\leq\left(\sum_{i=1,\dots k}|x_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1,\dots k}|y_i|^p\right)^{\frac{1}{p}}\leq\|x\|_p+\|y\|_p$$

So, this is true for every k and therefore I can let k tend to infinity, the right side is independent of k, and hence we get that $\|x+y\|_p\leq\|x\|_p+\|y\|_p$, which is finite of course.

This shows that $x+y\in l_p$ and we have also proved the triangle inequality simultaneously. Therefore, all these sequence spaces are normed linear spaces.

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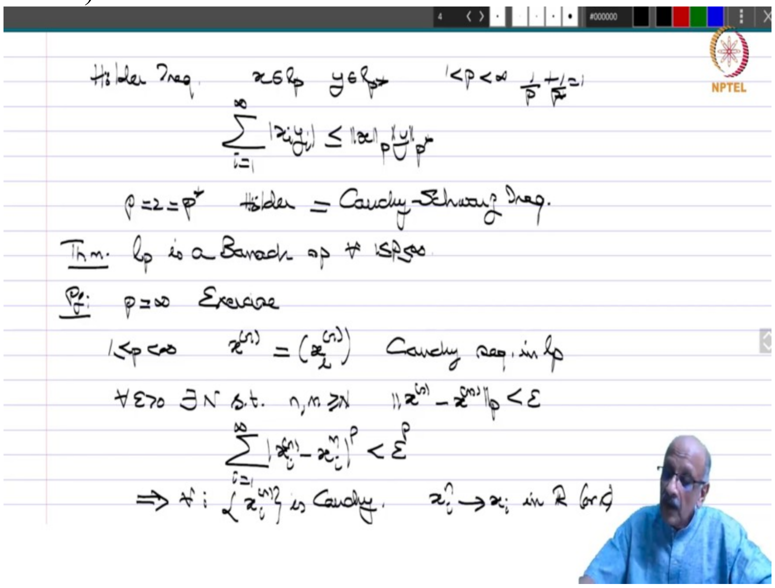

Using the same k trick, you can prove the following Holder's inequality:

Holder's inequality: Let $1<p<\infty$ and $x\in l_p, y\in l_{p^{¿}}$. Then, $\sum_{i=1,\ldots N}|x_i y_i|\leq\|x\|_p\|y\|_{p^{¿}}$.

As I said, do this for any k, you will get it up to k; and then you can write this on left hand side, right hand side. And therefore now you can pass to the limit as $k$tends to infinity, so you get this; so, you have the Holder's inequality.

I forgot to tell you that if $p=2=p^{¿}$, then the Holder's inequality is the same as the famous Cauchy Schwarz inequality, which you might have already come across.

So, finally we have the following theorem.

Theorem. l_p is a Banach space for every $1\leq p\leq\infty$.

Proof. For $p=\infty$, I will leave it as an exercise; it is again much easier than the other cases. Let $1\leq p<\infty$. We want to show that every Cauchy sequence in l_p is convergent. Let us take a Cauchy sequence (x_n) in l_p. For every fixed n, I will write its coordinates as x_n^i, $1\leq i<\infty$. Thus, $x_n=(x¿¿n^i),1\leq i<\infty¿$. The sequence (x_n) is a Cauchy sequence. What does it mean? For every $\epsilon>0$ there exists $N\in N$, such that if $n,m\geq N$ then we have $¿|x_n-x_m||_p<\epsilon$. In other words,

$$\sum_{i=1,2\ldots\infty}|x_n^i-x_m^i|^p<\epsilon^p$$

This implies for each i we have that $(x¿¿n^i)¿$ is itself Cauchy, and therefore you have that (x_n^i) will converge to some x^i in R or C, if you are looking at complex sequences; this will be C.

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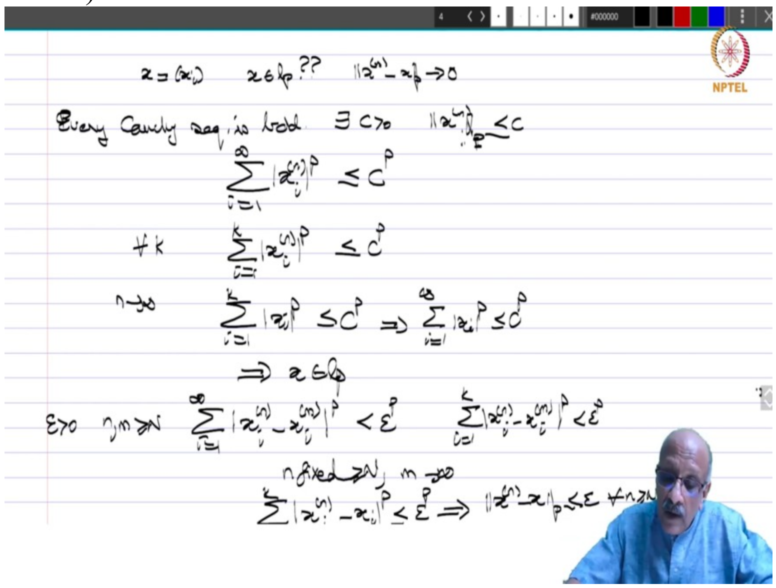

So, we have a candidate now for the limit x which is $x=(x^i)$. So, we want to know if the candidate is eligible that means you have to answer two questions. 1. $x \in l_p$ and 2. $\|x_n - x\|_p \to 0$. If you answer both these questions affirmatively, then that means every Cauchy sequence is convergent; and therefore l_p would be a Banach space. Now let us do this. First of all, just as real line, every Cauchy sequence is bounded. Therefore, there exists a positive $C>0$ such that $\|x_n\|_p \le C$. This means that

$$\sum_{i=1,\ldots\infty} |x_n^i|^p \le C^p$$

Therefore, for every k we have

$$\sum_{i=1,\ldots k} |x_n^i|^p \le C^p,$$

because this sum is smaller than the previous infinite sum. Now one can allow $n \to \infty$ to get

$$\sum_{i=1,\ldots k} |x^i|^p \le C^p.$$

Left hand side is independent of k. Therefore, one can let k tend to infinity to get

$$\sum_{i=1,\ldots\infty} |x^i|^p \le C^p,$$

which implies that x belongs to l_p. So, we have answered the first question.

Now we need to answer the second question. We already saw that for given $\epsilon > 0$, there exists $n, m \geq N$ such that

$$\sum_{i=1,\ldots\infty} \left| x_n^i - x_m^i \right|^p \leq \epsilon^p$$

So, we do the k trick once more. For fixed k, we have

$$\sum_{i=1,\ldots k} \left| x_n^i - x_m^i \right|^p \leq \epsilon^p$$

Now, you keep n fix greater than or equal to capital N, and m tend to infinity. So, you get that

$$\sum_{i=1,\ldots k} \left| x_n^i - x^i \right|^p \leq \epsilon^p$$

This is true for every k and this means that $\left\| x_n - x \right\|_p \leq \epsilon, \forall n \geq N$. This means $\left\| x_n - x \right\|_p \to 0$. Therefore, we have shown that the candidate is indeed suitable. This completes the proof that l_p is a Banach space. So, with this we will stop sequence spaces; our next example would be function spaces.