Functional Analysis
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Lecture No. 19
Annihilators

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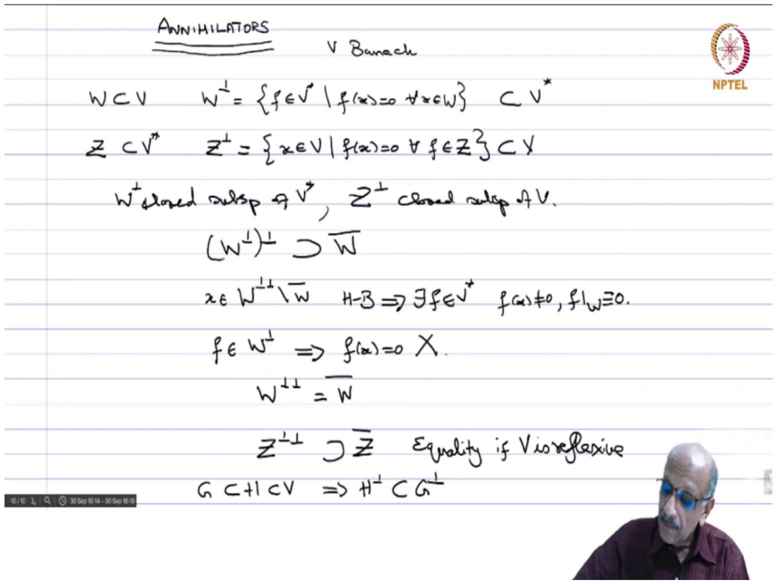

Today, we will discuss about annihilators. This will play an important role, when we study the relationship between the kernel of an operator and the range, the kernel and range of the adjoint of an operator and so on.

Let V be a Banach space and $W\subseteq V$. Then the annihilator of W is defined as $W^{\perp} = \left\{f\in V^{*}:\ f(x) = 0,\ \forall x\in W\right\} \subseteq V^{*}$. Similarly, if you have $Z\subseteq V^{*}$, then we can define the annihilator $Z^{\perp} = \{x\in V: f(x),\ \forall f\in Z\}\subseteq V$. $W^{\perp}$ is a closed subspace of V^{*} and $Z^{\perp}$ is a closed subspace of V and we have already seen the first one in one of the exercises earlier, the second one is absolutely the same.

$W^{\perp\perp}\supseteq W$ because if $x\in W$, then for every element in $f\in W^{\perp}$, $f(x) = 0$ and therefore, $x\in W^{\perp\perp}$. Since $W^{\perp\perp}$ is a closed subspace so $W^{\perp\perp} \supseteq \overline{W}$.

Now, what about the equality? Yes, we do have. Suppose we have $x\in W^{\perp\perp}\setminus\overline{W}$ then Hahn Banach theorem tells you there exists a $f\in V^{*}$ such that $f(x)\neq 0$ but $f|_{W} = 0$. Thus, $f\in W^{\perp}$. Since $x\in W^{\perp\perp}$ this means that $f(x) = 0$, which is a contradiction. Therefore, we have that $W^{\perp\perp} = \overline{W}$.

Now, in the same way, if you take $Z^{\perp\perp}\backslash\overline{Z}$. But now you try to imitate the same argument, you will fail because now you have to apply Hahn Banach something in V^{**}, and that does not come in anywhere in our calculations here. So, we only can say this about equality if V is reflexive because V is reflexive, V^{**} is nothing but V and therefore, you can repeat the previous argument and therefore, you have $Z^{\perp\perp} = \overline{Z}$.

Then finally, if $G \subseteq H \subseteq V$, then $H^{\perp} \subseteq G^{\perp}$. This is really obvious. So, now we will go through a proposition.

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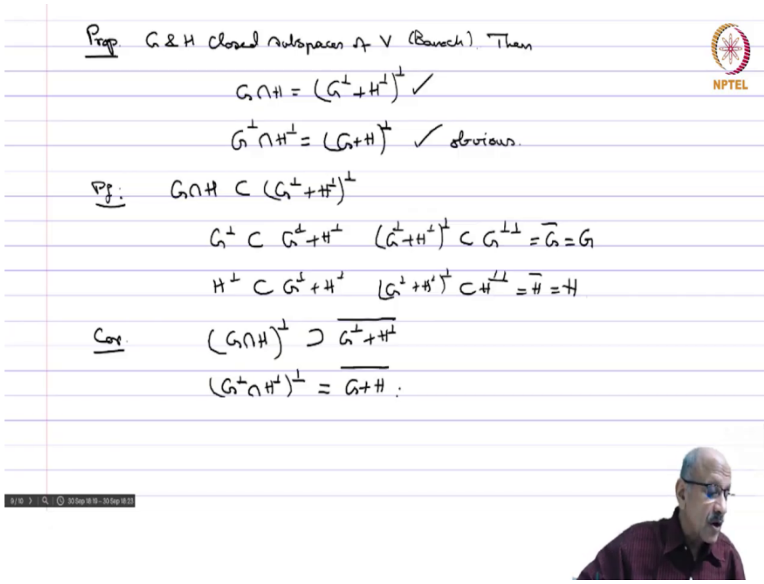

Proposition. Let G, H be closed subspaces of V which is Banach. Then $G \cap H = \left(G^{\perp} + H^{\perp}\right)^{\perp}$ and $G^{\perp} \cap H^{\perp} = (G + H)^{\perp}$.

Proof. The second one is almost obvious. If something kills both G and H, it will kill all of $G + H$. So, it will belong to $(G + H)^{\perp}$. Conversely, if something is in $(G + H)^{\perp}$, so, it will kill all of $G + H$. Hence, it kills all of G and it will kill all of H. So, it is in $G^{\perp} \cap H^{\perp}$. So, this one is obvious. So, now, let us prove the first one.

We already have that $(G \cap H) \subseteq \left(G^{\perp} + H^{\perp}\right)^{\perp}$. Why? Because, if you have something in $G \cap H$, then it is killed by everything in $G^{\perp}$ as well as by everything in $H^{\perp}$. So, it is killed by everything in $G^{\perp} + H^{\perp}$ i.e., it belongs to $\left(G^{\perp} + H^{\perp}\right)^{\perp}$. For the converse, we have that $G^{\perp} \subseteq G^{\perp} + H^{\perp}$ and $\left(G^{\perp} + H^{\perp}\right)^{\perp} \subseteq G^{\perp\perp} = \overline{G} = G$. Similarly, $H^{\perp} \subseteq G^{\perp} + H^{\perp}$. By the same reason $\left(G^{\perp} + H^{\perp}\right)^{\perp} \subseteq H^{\perp\perp} = \overline{H} = H$. Therefore, we have the reverse inclusion.

Now, you combine these two to get the following corollary.

Corollary. $(G \cap H)^{\perp} \supseteq \overline{G^{\perp} + H^{\perp}}$ and $\left(G^{\perp} \cap H^{\perp}\right)^{\perp} = \overline{G + H}$.

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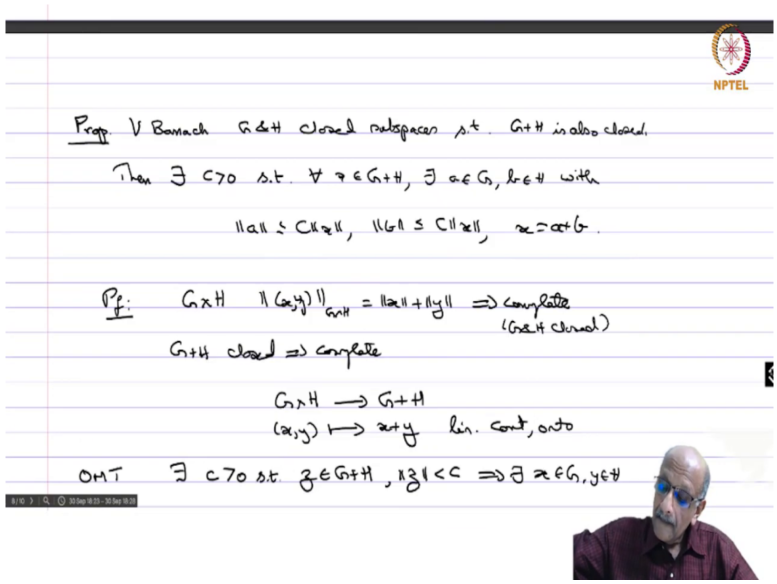

Proposition. Let V be Banach and G, H be closed subspaces such that $G + H$ is also closed. Then there exists a constant $C > 0$ such that for every $x \in G + H$ there exists $a \in G$, $b \in H$ with $||a|| \leq C||x||$ and $||b|| \leq C||x||$ and $x = a + b$.

So, what does this mean? If the sum of two close subspaces is closed, then you can decompose any vector into a sum of a vector in G and a vector in H but in a continuous fashion. So, there are many decompositions possible. But you will be able to select one in a fair manner that is uniformly continuous.

Proof. You consider the spaces $G \times H$ G with the norm $||x, y||_{G \times H} = ||x|| + ||y||$. Both are close subspaces of a Banach space, so, each one is a Banach space in its own right. So, the product is a Banach space with this norm. Now, $G + H$ is also closed. So, it is also a Banach space. So, we have two complete Banach spaces and we are going to take a mapping from $G \times H$ into $G + H$. We are going to take $(x, y) \mapsto x + y$, which is linear, continuous and above all it is onto. Then, by open mapping theorem, there exists a $C > 0$ such that $z \in G + H$, $||z|| < C$ implies $\exists x \in G,\ y \in H$ such that $z = x + y$ and $||x + y|| < 1$. Now, if $z \in G + H$ is arbitrary, then $\frac{C}{2} \frac{z}{||z||} = x' + y'$ and $\left|\left|x' + y'\right|\right| < 1,\ x' \in G, y' \in H$.

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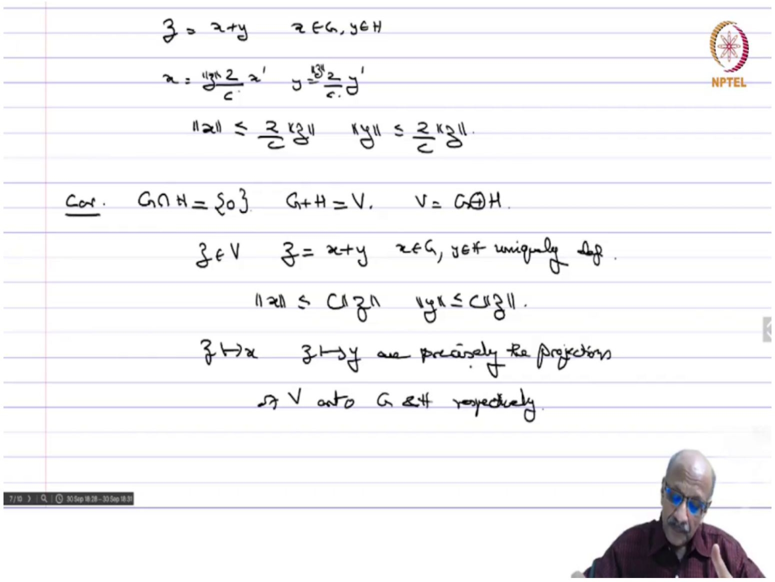

Thus, $z = x + y$, where $x = \frac{2||z||}{C}x^{'} \in G,\ y = \frac{2||z||}{C}y^{'} \in H$. Therefore, $||x|| \leq \frac{2}{C}||z||$ and $||y|| \leq \frac{2}{C}||z||$.

So, now, let us see a corollary to this thing.

Corollary. Let $G \cap H = \{0\}$ and $G + H = V$. Then you have $V = G \oplus H$.

V equals direct sum of G, H, but in this case the decomposition is unique there is no several decompositions possible. By the previous theorem, if you have that $z \in V$ then z has a unique decomposition $z = x + y$, $x \in G,\ y \in H$ and you will have $||x|| \leq C||z||,\ ||y|| \leq C||z||$ and $z \mapsto x,\ z \mapsto y$ are precisely the projections of V onto G and H respectively. If you have a decomposition, if $G + H$ is whole of V and $G \cap H = \{0\}$ then the projection is continuous. This is a consequence of the open mapping theorem.

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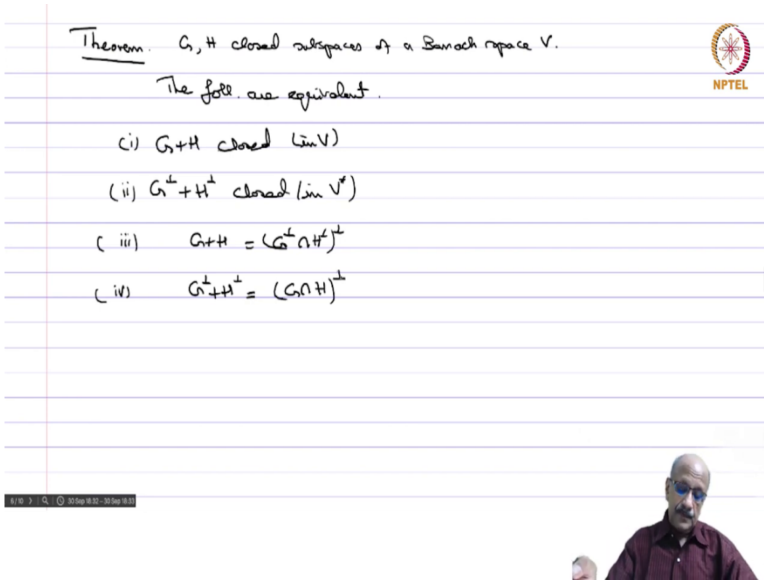

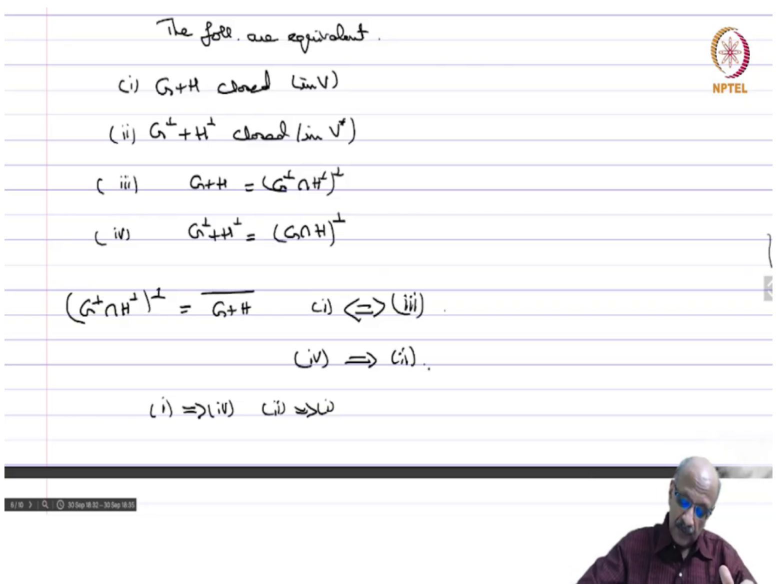

Theorem Let G and H be two closed subspaces of a Banach space V. Then the following are equivalent.

1: $G + H$ is closed in V.

2: $G^{\perp} + H^{\perp}$ is closed in V^{*}.

3: $G + H = \left(G^{\perp} + H^{\perp}\right)^{\perp}$

4: $G^{\perp} + H^{\perp} = (G \cap H)^{\perp}$.

Now, we already know that $\left(G^{\perp} + H^{\perp}\right)^{\perp} = \overline{G + H}$. So, if $G + H$ is closed, then we have 1 $\Longrightarrow$ 3. And if you have 3, then you have $\overline{G + H} = G + H$ and therefore, $G + H$ is closed. So, 1$\Longleftrightarrow$ 3.

Also 4 $\Longrightarrow$ 2 is obvious. So, to complete the proof we need to show that 1 $\Longrightarrow$ 4 and 2 $\Longrightarrow$ 1. So, let us prove 1 $\Longrightarrow$ 4.

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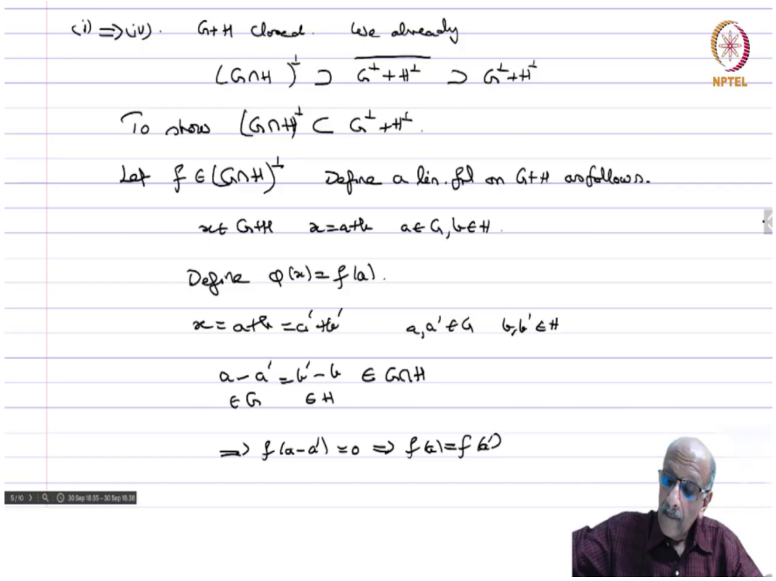

So, what does 1 say? $G + H$ is closed. We already proved that $(G + H)^{\perp} \supseteq \overline{G^{\perp} + H^{\perp}} \supseteq G^{\perp} + H^{\perp}$. To show $(G \cap H)^{\perp} \subseteq G^{\perp} + H^{\perp}$. Let $f \in (G \cap H)^{\perp}$. Define a linear functional on $G + H$ as follows:

If $x \in G + H$, then $x = a + b$; $a \in G, b \in H$. Define $\phi(x) = f(a)$.

We have to first check that this is well defined. If $x = a + b = a' + b'$ where $a, a' \in G$ and $b, b' \in H$ then you have that $a - a' = b - b'$. Hence $a - a', b - b' \in G \cap H$. But $f \in (G \cap H)^{\perp}$ and therefore, $f\left(a - a'\right) = f\left(b - b'\right) = 0$ which implies $f(a) = f\left(a'\right)$.

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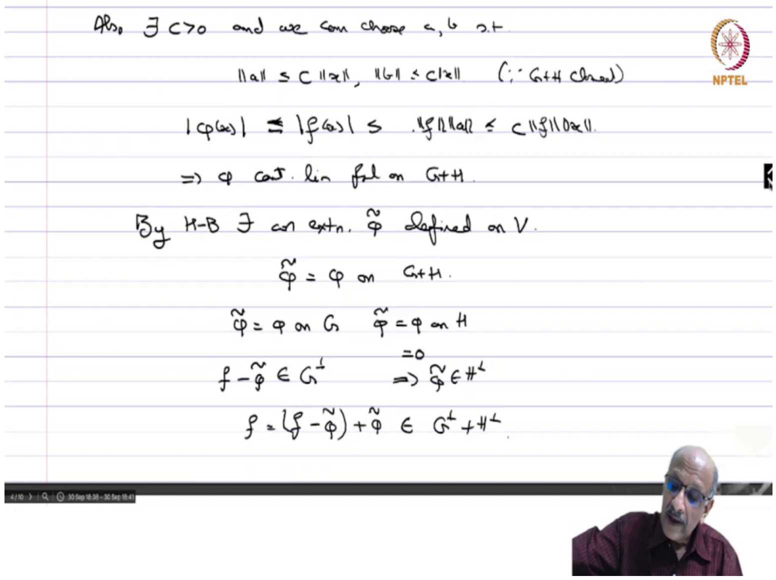

Also there exists a $C > 0$ and we can choose a, b such that $||a|| \leq C||x||$, $||b|| \leq C||x||$ (*by* previous propositions which we already proved based on the open mapping theorem). Therefore, $|\phi(x)| = |f(a)| \leq ||f||||a|| \leq C||f||||x||$, which implies that ϕ is a continuous linear functional on $G + H$. Therefore, by Hahn Banach theorem, there exists an extension $\widetilde{\phi}$ on V such that $\widetilde{\phi} = \phi$ on the $G + H$. Therefore, $\widetilde{\phi} = \phi$ on G and $\widetilde{\phi} = \phi = 0$ on H. Now, notice that $f - \widetilde{\phi} \in G^{\perp}$ and $\widetilde{\phi} \in H^{\perp}$. Therefore, we can write $f = \left(f - \widetilde{\phi}\right) + \widetilde{\phi} \in G^{\perp} + H^{\perp}$. So, that completes the proof.

The last inclusion I am not going to do namely $2 \Longrightarrow 1$. This is a bit long and very technical. And there were really no new ideas but someone can try and prove. So, it is best if you read it yourself in the book. So, this theorem is more or less proved except for this 2 implies 1, which I am going to omit the proof for lack of time.