Functional Analysis
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Lecture No. 18
Open Mapping and Closed Graph Theorems

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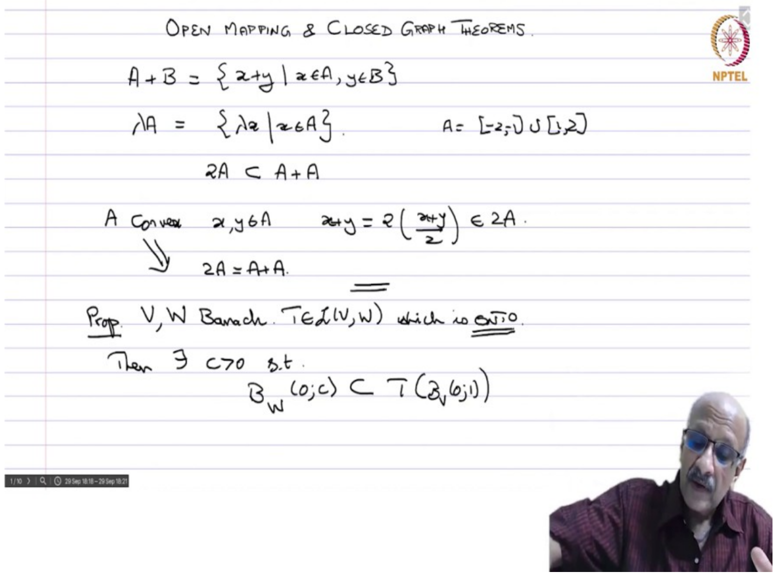

We will now look at the next two important applications of Baire theorem. These are the open mapping theorem and the closed graph theorem.

Before we start let us recall some notation. If you have a vector space and you have two sets A, B then $A+B=\{x+y: x\in A, y\in B\}$ is nothing but the algebraic sum. Similarly if λ is any scalar then $\lambda A=\{\lambda x: x\in A\}$.

If you take $2A$ is a set of all elements of the form $2x$ and $2x=x+x$ and so, it is contained in $A+A$. So, $2A\subseteq A+A$. But the converse need not be true. For instance, if you take $A=[-2,-1]\cup[1,2]$. Then 0 will be in $A+A$, but it is not in A. Because 0 is not in A, so it is not in $2A$ also. But if A is convex and if you have $x, y\in A$, then $x+y=2\left(\frac{x+y}{2}\right)\in 2A$. A being convex $\frac{x+y}{2}\in A$, so this belongs to $x+y\in 2A$. Thus, $A+A=2A$.

So, these are some notations which we will need in the proof of the open mapping theore. We will first prove a proposition.

Proposition. Let V, W be two Banach spaces and $T \in L(V, W)$which onto. Then there exists a $C > 0$ such that $B_W(0; C) \subseteq T(B_V(0;1))$.

What does this say? This says that if you take a neighborhood of the origin and T is an onto continuous linear map, then the image is also a neighborhood of the origin. From this we will be able to show the T maps open sets into open sets and that is called open mapping theorem.

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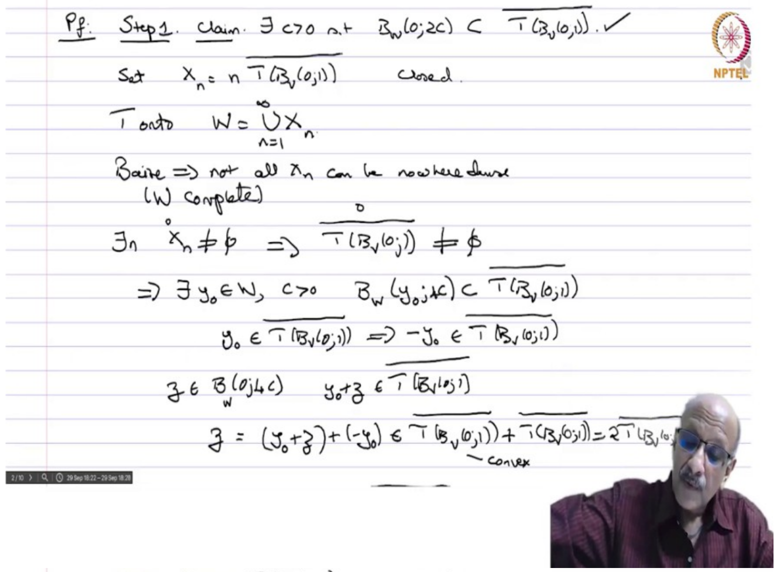

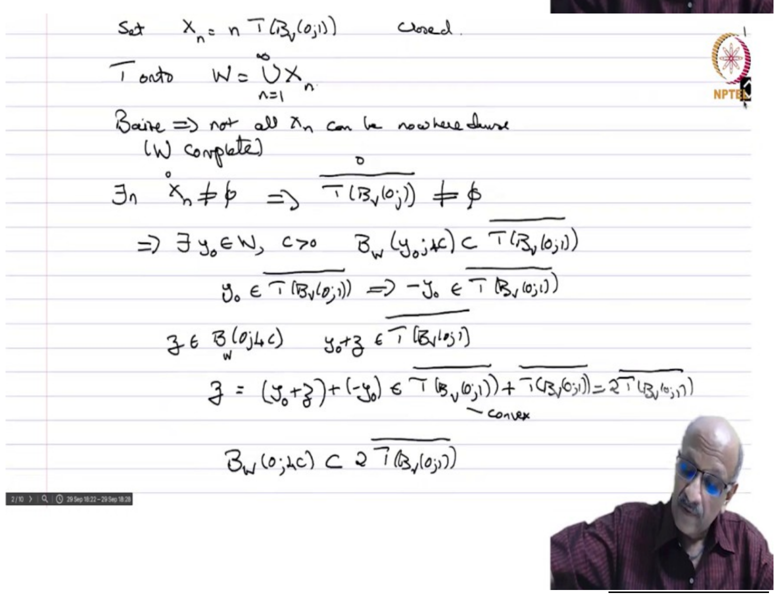

Proof. Step 1. Claim:There exists $C > 0$ such that $B_W(0; 2C) \subseteq \overline{T(B_V(0;1))}$. So, it is a slightly bigger set. Let us set $X_n = n\overline{T(B_V(0;1))}$. Then each X_n is closed. Also T is onto. So, every element in W is a image of something in V and you can scale it and therefore, you can show that in fact, $W = \cup_{n=1,2,\ldots\infty} X_n$. So, the Baire's theorem implies that not all the X_n can be nowhere dense (as W is complete). So, there exists an n such that $\int(X_n) \neq \phi$. This implies that $\int\left(\overline{T(B_V(0;1))}\right) \neq \phi$. So, there exists $y_0 \in W$ and a $C>0$ such that $B_W(y_0;4C) \subseteq \overline{T(B_V(0;1))}$. Now, in particular, take $y_0 \in \overline{T(B_V(0;1))}$. Then $-y_0 \in \overline{T(B_V(0;1))}$. For $z \in B_W(y_0;4C)$, $y_0 + z \in \overline{T(B_V(0;1))}$. Now, $y_0 \in \overline{T(B_V(0;1))}$ implies $-y_0 \in \overline{T(B_V(0;1))}$. Then, $z = (y_0 + z) - y_0 \in \overline{T(B_V(0;1))} + \overline{T(B_V(0;1))} = 2\overline{T(B_V(0;1))}$. So, $B_W(y_0;4C) \subseteq 2\overline{T(B_V(0;1))}$. Therefore, if you scale it by 2, then you get whatever the claim.

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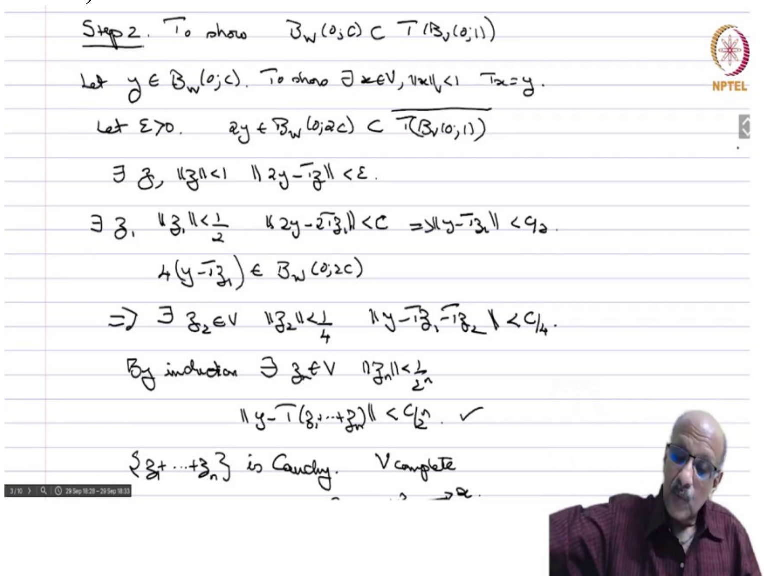

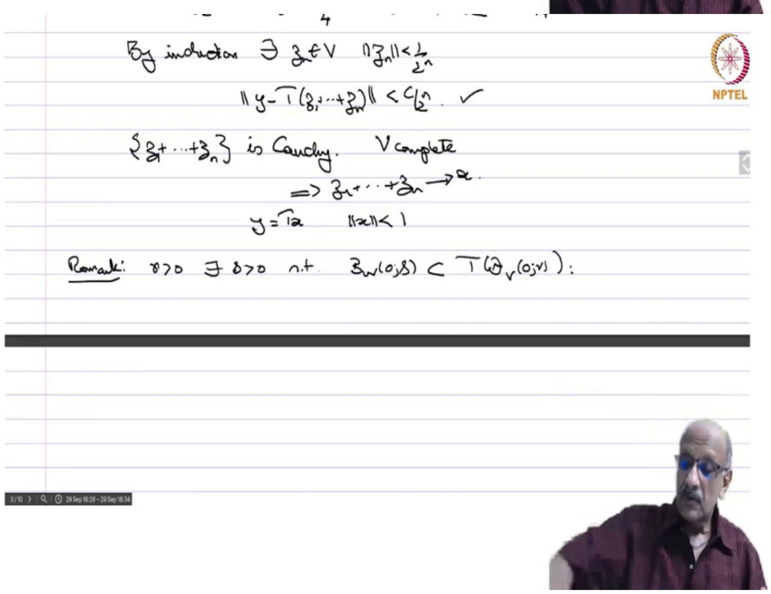

Step 2. Now we are going to use the fact that V is complete. We are going to show that $B_W(0;C) \subseteq T(B_V(0;1))$. So, the closure by halfing the radius we are able to remove the closure.

Let $y \in B_W(0;C)$. To show that $\exists\, x \in V, \|x\|_V < 1$ such that $Tx = y$. Let $\epsilon > 0$ be arbitrary. Then, since $y \in B_W(0;C)$, $2y \in B_W(0;2C) \subseteq \overline{T(B_V(0;1))}$.

So, there exists a z such that $\|z\| < 1$ and $\|2y - T(z)\| < \epsilon$. From this we say that there exists z_1 such that $\|z_1\| < \frac{1}{2}$ and $\|2y - 2T(z_1)\| < C$. This implies that $\|y - T(z_1)\| < \frac{C}{2}$. So 4¿ So, from this you have that there exists $z_2 \in V$ and $\|z_2\| < \frac{1}{4}$ and $\|y - T(z_1) - T(z_2)\| < \frac{C}{4}$. Therefore, eventually by induction, there exists $z_n \in V$ with $\|z_n\| < \frac{1}{2^n}$ and $\|y - T(z_1 + z_2 + \ldots + z_n)\| < \frac{C}{2^n}$. Look at the sequence $(z_1 + z_2 + \ldots + z_n)$.What is the difference of two consecutive terms? The difference of two consecutive terms is z_{n+1} and its norm is less than $\frac{1}{2^n}$ and $\sum_{n=1,2\ldots\infty} \frac{1}{2^n} < \infty$. Therefore, the sequence $(z_1 + z_2 + \ldots + z_n)$ is Cauchy. Hence, the sequence $(z_1 + z_2 + \ldots + z_n)$ converges to some x and then $y = Tx$. Further, $\|x\| \leq 1$. So, that proves the proposition completely.

Now we will make a remark.

Remark. Because everything is going to be scaled, if $r > 0$, then there exists an $s > 0$ such that $B_W(0;s) \subseteq T(B_V(0;r))$. This is obvious. We did it for 1, then it is just a question of scaling it.

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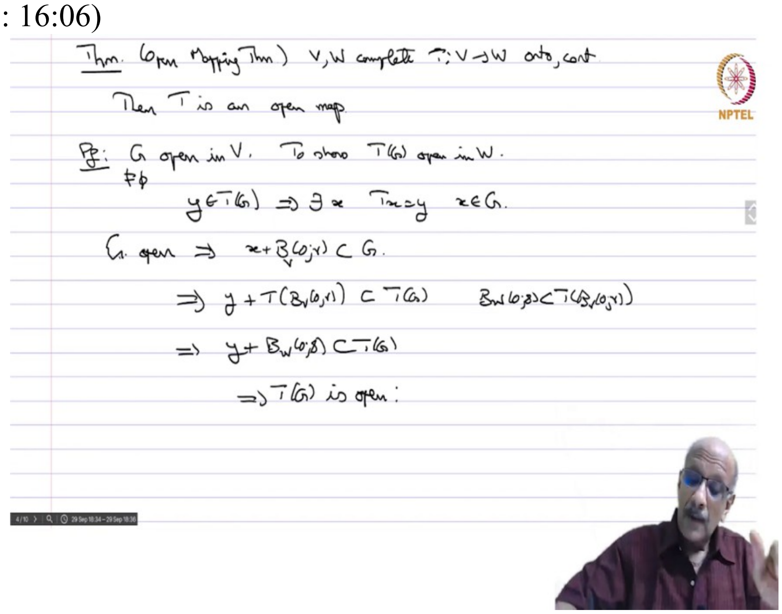

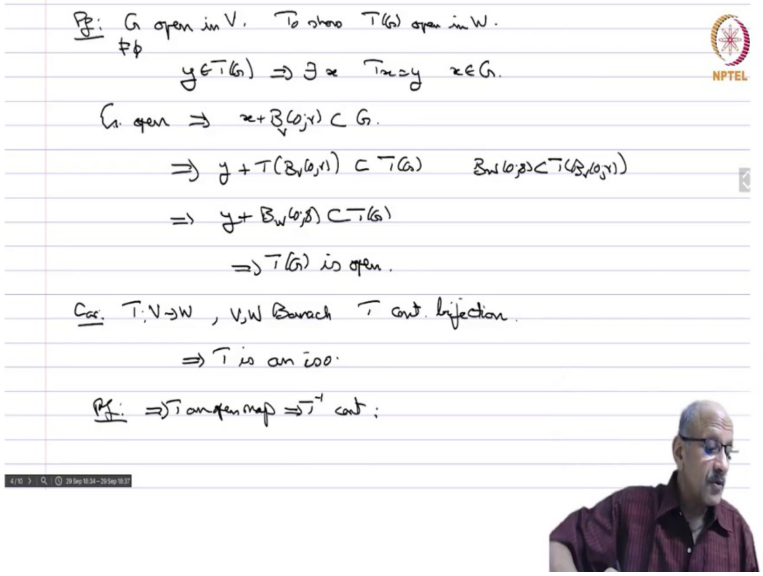

Now we can prove the open mapping theorem.

Theorem. (Open mapping). Let V, W be complete and $T: V \mapsto W$ be onto, continuous. Then T is an open map, that means, it maps open sets into open sets.

Proof. Let G be non-empty open in V. To show $T(G)$ is open in W. Let us assume that $y \in T(G)$. This means that there exists an $x \in G$ such that $T(x) = y$. G is open implies $x + B_V(0;r) \subseteq G$. Therefore, $y + T(B_V(0;r)) \subseteq T(G)$. But there exists $B_W(0;s) \subseteq T(B_V(0;r))$ (that is what we remarked) and therefore, this implies that $y + B_W(0;s) \subseteq T(G)$ and this implies that $T(G)$ is open and this proves the open map theorem.

There are several applications of this theorem and some of them are really very interesting.

Corollary. Let V, Wbe Banach and $T: V \mapsto W$ be a continuous bijection. Then T is an isomorphism.

Proof. T is an open map implies that T^{-1} inverse is continuous. Therefore, it is an isomorphism. So, this is just that. Now, this leads to another very interesting corollary.

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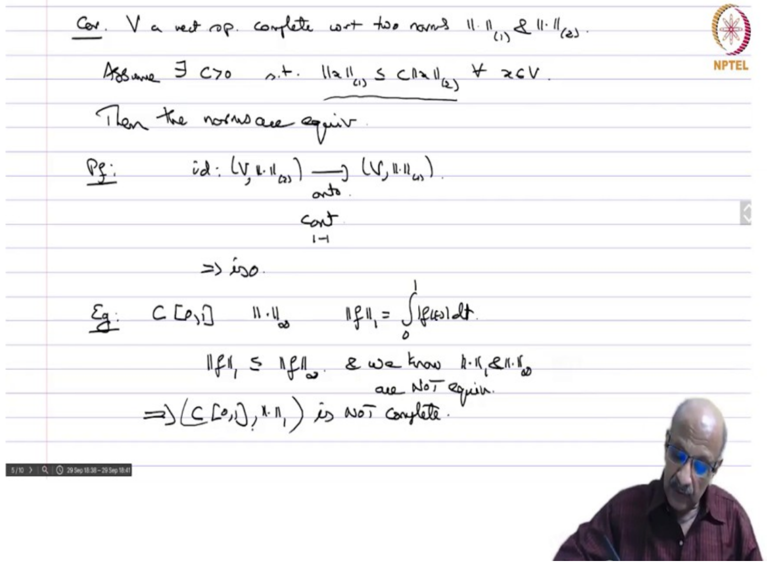

Corollary. Let V be a vector space complete with respect to two norms, $\|.\|_1$ and $\|.\|_2$. Assume that there exists $C>0$ C such that $\|x\|_1 \leq C\|x\|_2, \forall x \in V$. Then the norms are equivalent.

Proof. You take the identity map $I:(V, \|.\|_2) \mapsto (V, \|.\|_1)$. So, this is an onto map and further it is also continuous because of the given condition $\|x\|_1 \leq C\|x\|_2, \forall x \in V$. This is also 1 to 1 map. So, identity is an isomorphism and therefore, the norms are equivalent.

Let us see an example of an application of this very nice result.

Example. Let us take $C[0,1]$, then you have two norms: (a) $\|.\|_\infty$ and (b) $\|f\|_1 = \int_{[0,1]} |f(t)|\,dt$. You have that $\|f\|_1 \leq \|f\|_\infty$ and we also saw that these two norms are not equivalent. We know that these are not equivalent. We already know that $C[0,1]$, with $\|.\|_\infty$ is complete. Thus, if $C[0,1]$, with $\|.\|_1$ is also complete, then by the previous corollary the two norms must be equivalent, which is a contradiction. Therefore, $C[0,1]$ with $\|.\|_1$ is not complete.

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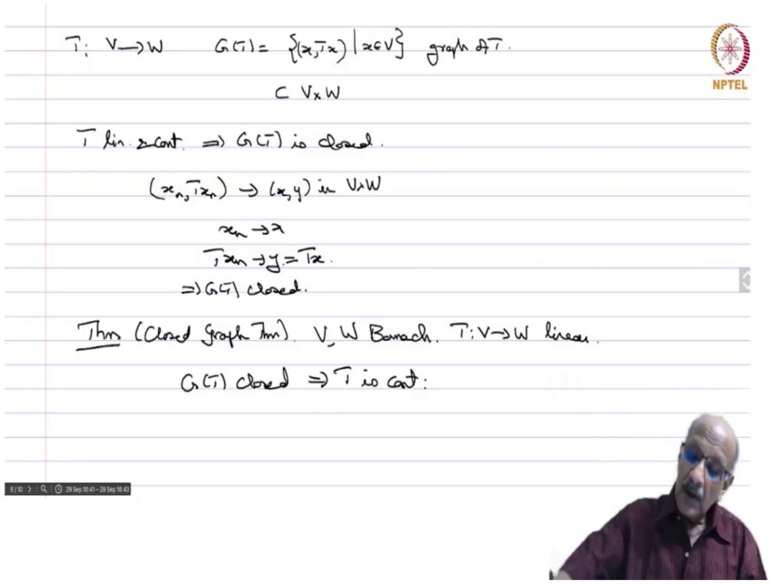

We are ready to move to the next theorem, namely, the close graph theorem.

Let us first define the graph. Let $T: V \mapsto W$ be a map. We define $G(T) = \{(x, Tx) : x \in V\}$ is called the graph of T. So, this is some subspace of $V \times W$.

So, if T is linear and continuous, then $G(T)$ is closed. Why? Let $(x_n, T(x_n)) \to (x, y)$ in $V \times W$. Then $x_n \to x$, $T(x_n) \to y = T(x)$ (as T is continuous). Therefore, $(x, y) = (x, T(x)) \in G(T)$. So, this implies that $G(T)$ is closed.

The beautiful thing is the converse is also true. So, this is the theorem.

Theorem. (Closed graph theorem). Let V, W be two Banach spaces and $T: V \mapsto W$ linear. Then, $G(T)$ is closed implies T is continuous.

We have seen many ways in which you can prove the continuity of an operator. This is now a new one. It says we have a linear map between Banach spaces, then, if the graph is closed, then the map has to be continuous. So, this is one more nice result. So, let us prove this theorem.

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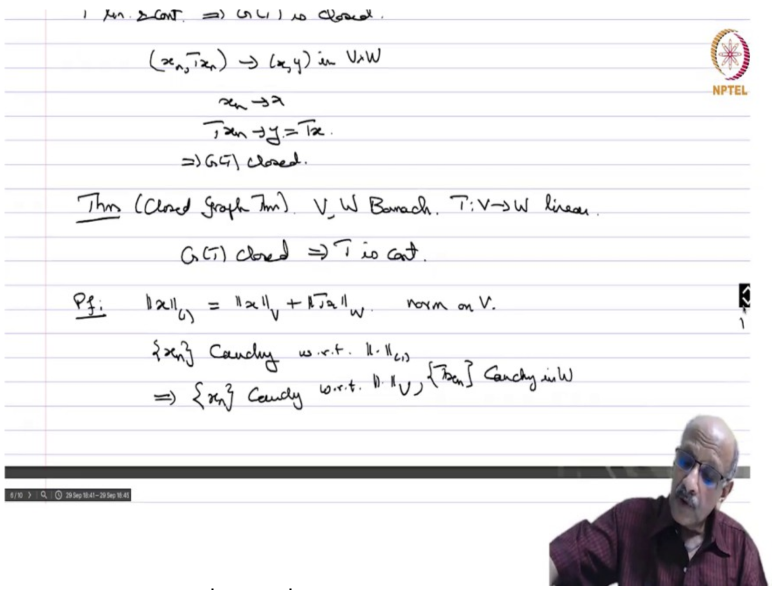

Proof. Let us define $\|x\|_1 = \|x\|_V + \|T(x)\|_W$. This is trivially a norm that you can check without any problem. Let us take a Cauchy sequence (x_n) with respect to $\|.\|_1$. This implies (x_n) Cauchy with respect to V and $T(x_n)$ is Cauchy in W.

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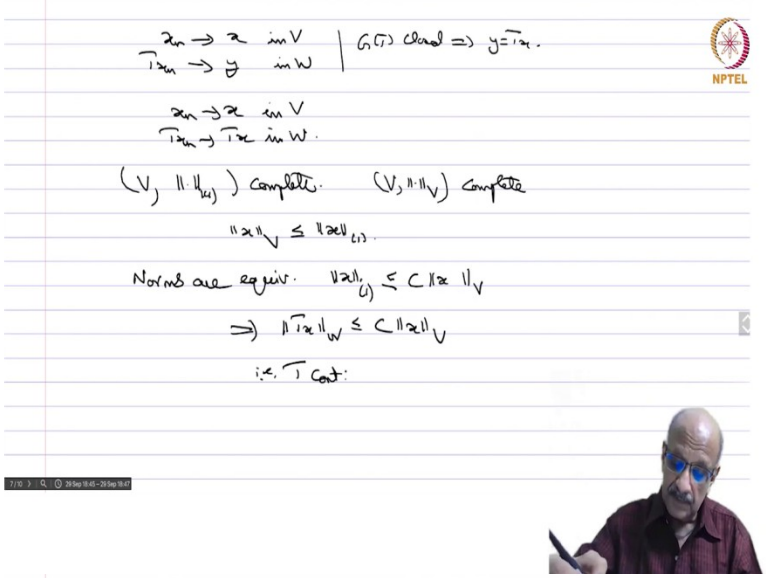

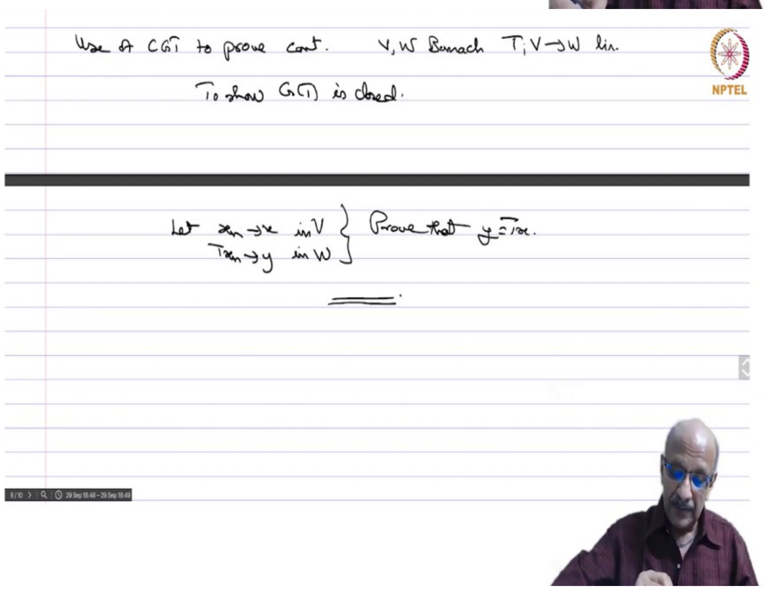

We have both the spaces are complete and therefore you have $x_n \to x$ for some $x \in V$ and $T(x_n) \to y$ for some $y \in W$. Now, you know that $G(T)$ is closed. This implies $y = T(X)$. So, you have that V with $\|.\|_1$ is complete. V with $\|.\|_V$ is also complete. Now, you have two norms on V and V is complete with respect to both the norms and you also know that $\|.\|_V \leq \|.\|_1$. Therefore, these two norms are equivalent. Thus, $\|.\|_1 \leq C\|.\|_V$. So, in particular, $\|T(x)\|_W \leq C\|x\|_1$ i.e., T is continuous and therefore, we are through. So, we this completes the proof.