Functional Analysis
Professor S. Kesavan
Department of Mathematics
Institute of Mathematical Sciences
Lecture 17
Application to Fourier series

We will now see a nice application of the uniform bondedness theorem.

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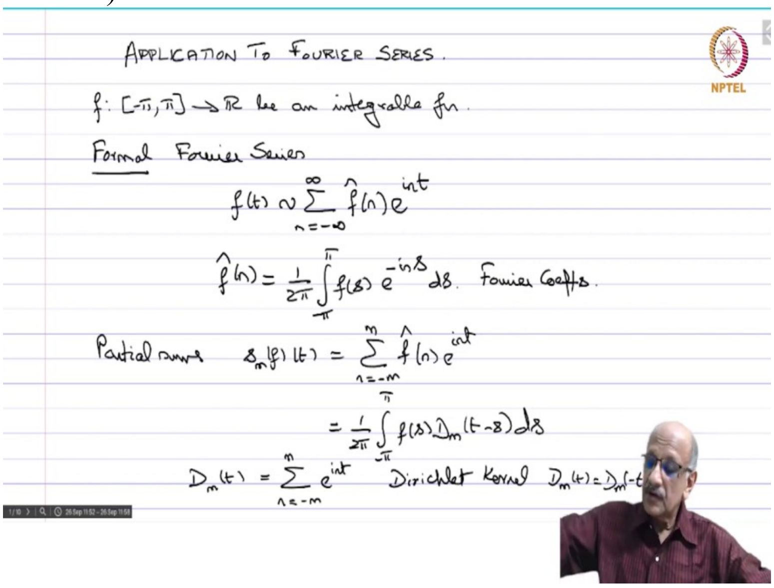

We are going to see an application to Fourier series.

Let $f: [-\pi, \pi] \mapsto R$ be an integrable function. We write its formal Fourier series as

$$f(t) \sum_{-\infty \le n \le \infty} \hat{f}(n) e^{\int ¿}, ¿$$

$$\hat{f}(n) = \frac{1}{2\pi} \int_{[-\pi, \pi]} f(s) e^{-ins} ds$$ are the Fourier coefficient.

The question which we are going to ask is, to what extent does this Fourier series represent the function itself. In particular, if f is continuous, 2π-periodic function, will the Fourier series converge to $f(t)$ at every point $t \in [-\pi, \pi]$? Unfortunately, the answer is no and it was a big controversy in the later half of the 18th century, for nearly 70 years, till in the beginning of the 19th century Dirichlet established in 1829, the first sufficient condition for the series to converge to the value of the function or whatever it converges to and then it was strengthened by Jordan and in fact, the study of Fourier series justification of its convergence and all this, led to a lot of mathematical development and like making precise what is the notion of a function? What is Cantor's theory of infinite series, the theories of integration of Riemann and Lebesgue and the theories of summability of series. All this lot led to a lot of research, a lot of mathematics got developed in just trying to understand what this means.

In this talk, we will now use the Banach-Steinhauss theorem to show that there exists a very large class of continuous, 2π-periodic functions, where the Fourier series will fail to converge and the set of points where it converges is also very big.

So, that means the situation is quite bad and therefore, one has to see in what sense we want to use the Fourier series or not. So, what do we do to do, to study the Fourier series. So, we have to study the convergence of the partial sums of the Fourier series which is

$$S_m(f)(t) = \sum_{-m \leq n \leq m} \hat{f}(n) e^{\int ¿} = \frac{1}{2\pi} \int_{[-\pi,\pi]} f(s) D_m(t-s) ds, ¿$$

where $D_m(t) = \sum_{-m \leq n \leq m} e^{\int ¿} ¿$ is called the Dirichlet Kernel. The Dirichlet Kernel is an even function.

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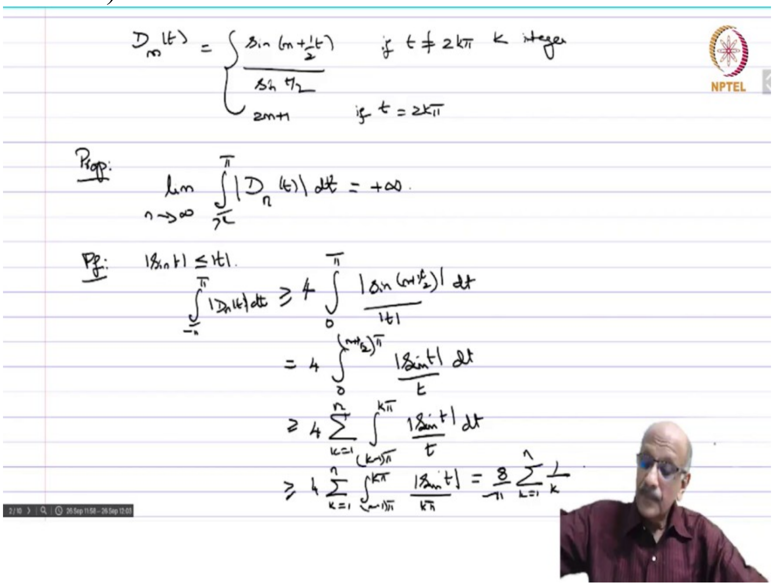

This is just calculation and I am not going to do that.

$D_m(t) = \frac{\sin(m + \frac{t}{2})}{\sin\frac{t}{2}}$ if $t \neq 2k\pi$, k is an integer and $D_m(t) = 2m+1$ if $t = 2k\pi$.

Proposition. $\lim_{n\to\infty} \int_{[-\pi,\pi]} |D_n(t)|\, dt = \infty.$

Proof. We have that $|\sin(t)| \leq |t|$. So

$$\int_{[-\pi,\pi]} |D_n(t)|\, dt \geq 4 \int_{[0,\pi]} \frac{\left|\sin\left(n+\frac{t}{2}\right)\right|}{|t|}\, dt = 4\int_{[0,(n+\frac{1}{2})\pi]} \frac{|\sin t|}{t}\, dt$$

$$\geq 4 \sum_{k=1,2,\ldots n} \int_{[(k-1)\pi,k\pi]} \frac{|\sin(t)|}{t}\, dt$$

$$\geq 4 \sum_{k=1,2,\ldots n} \int_{[(k-1)\pi,k\pi]} \frac{|\sin(t)|}{k\pi}\, dt = \frac{8}{\pi} \sum_{k=1,2,\ldots n} \frac{1}{k}.$$

Now $\sum_{k=1,2,\ldots\infty} \frac{1}{k}$ is a divergent series and therefore this partial sum should go to infinity. Thus,

$\lim_{n\to\infty} \int_{[-\pi,\pi]} |D_n(t)|\, dt = \infty.$ So, that proves this proposition.

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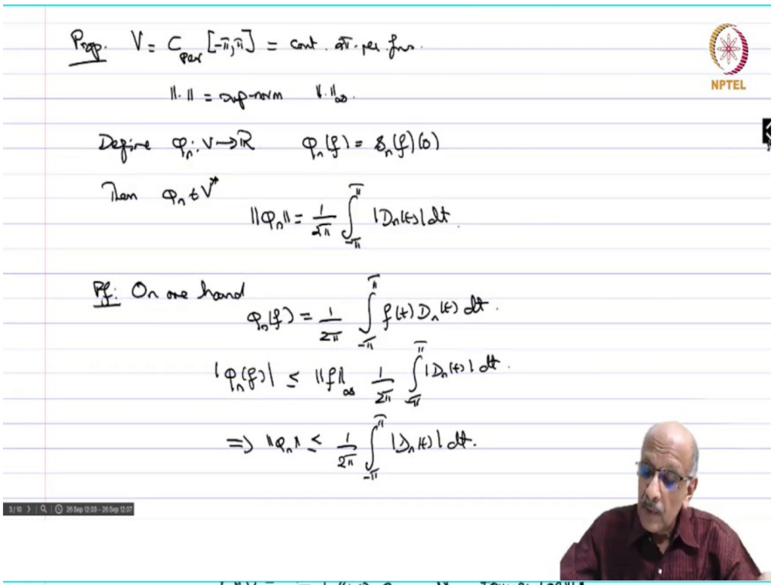

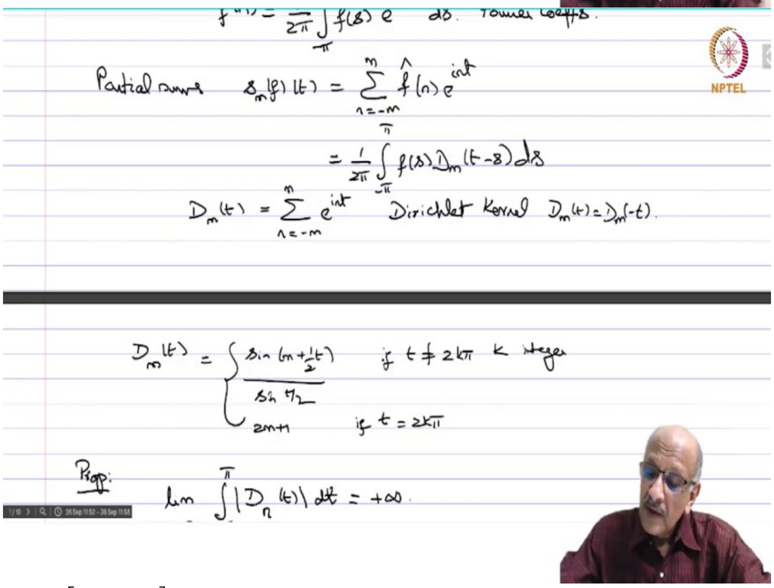

Proposition. $V = C_{per}[-\pi, \pi]$= Continuous 2π-periodic functions. This is a vector space and then you are going to put the usual norm. Norm here is the sup-norm as usual and that I am going to call it as norm infinity.

Now, define $\phi_n : V \mapsto R$. So, $\phi_n(f) = S_n(f)(0)$. Then, $\phi_n \in V^{¿}$and we can actually compute $\|\phi_n\| = \frac{1}{2\pi} \int_{[-\pi,\pi]} |D_n(t)| dt$. That is why we calculated that integral a little earlier, because this is in fact the norm.

Proof. On one hand, we have

$$\phi_n(f) = \frac{1}{2\pi} \int_{[-\pi,\pi]} f(t) D_n(t) dt.$$

$|\phi_n(f)| \le \|f\|_\infty \frac{1}{2\pi} \int_{[-\pi,\pi]} ¿ D_n(t) \vee dt$. $\|\phi_n\| \le \frac{1}{2\pi} \int_{[-\pi,\pi]} |D_n(t)| dt$

Now we want to show that it is actually attained. So, this supremum will have to be attained. So, we have to find a function or sequence of functions for which it will actually go to this optimal value.

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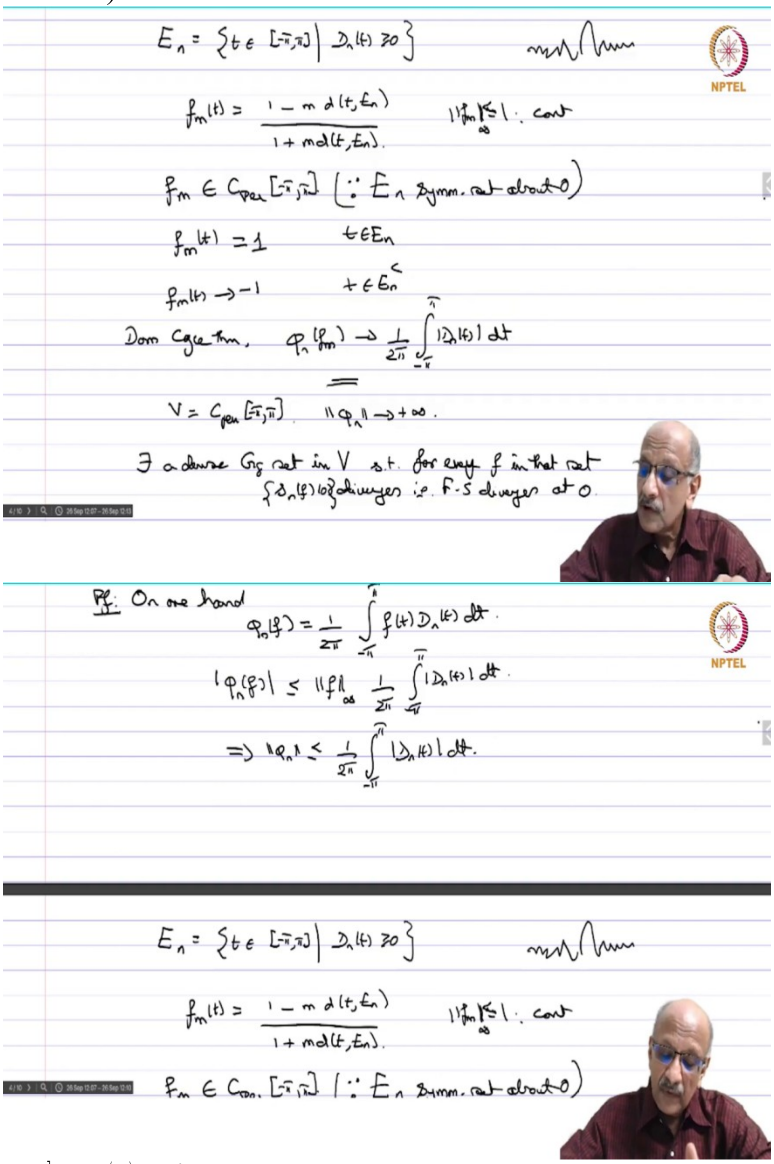

Let $E_n = \{t \in [-\pi, \pi] : D_n(t) \geq 0\}$. So, now, you define

$$f_m(t) = \frac{1 - m\,dist(t, E_n)}{1 + m\,dist(t, E_n)}$$

So, $\|f_m\|_\infty \leq 1$ and continuous. Now, I claimed that $f_m \in C_{per}([-\pi, \pi])$. The distance function is a continuous function and denominator does not vanish and therefore, f_m is a continuous function.

Why is this 2π-periodic? That means, you must show that at π and $-\pi$, they take the same value. The set E_n itself is symmetric about the origin as D_n is even function. So, E_n is a symmetric set about 0 and therefore, you have f_m is in fact a periodic function.

Now, what does $f_m(t)$ if t belongs to you E_n? If $t\in E_n$, $dist(t,E_n)=0$ and therefore, $f_m(t)=1$ and if $t\in E_n^c$, then I can divide through it by m and let m tend to infinity, so $f_m(t)\to -1$. So, point wise $f_m(t)$ goes to the function, which is 1 on E_n and -1 on E_n^c and the integral is dominated by D_n which is a integrable function. Therefore, by the dominated convergence theorem, $\phi_n(f_m)\to \frac{1}{2\pi}\int_{[-\pi,\pi]}|D_n(t)|dt$. Therefore, you have a sequence which goes to the supremum and therefore, this is in fact the norm. This completes the proof.

Now let us apply this to the space. Take the space $V=C_{per}([-\pi,\pi])$ V equals C periodic, Then what do you know? we have that $\|\phi_n\|\to\infty$. Therefore, there exists a dense G_δ set in V such that for every f, we have that $S_n(f)(0)\to\infty$ S n of f at 0 diverges. That is Fourier series diverges at 0. So, there is a huge set of continuous functions which are 2π-periodic for whom the Fourier series is divergent at the origin.

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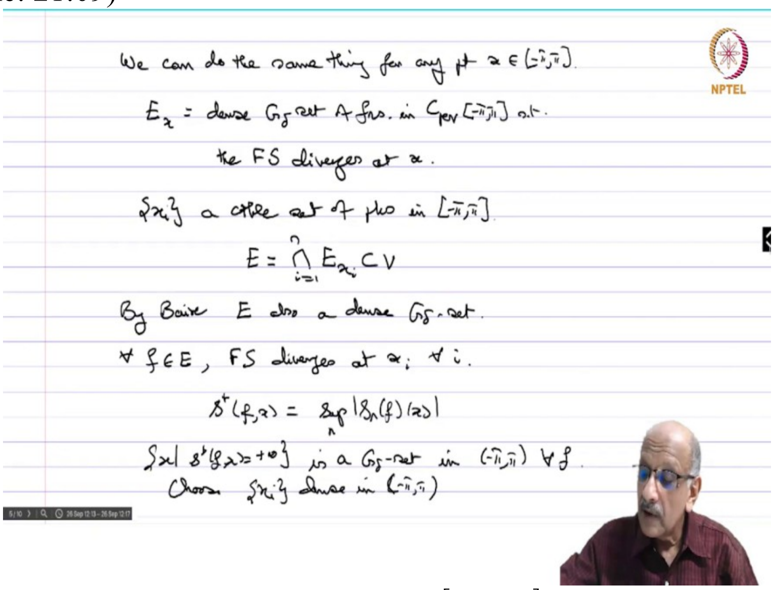

Now, we can do the same thing for any point $x\in[-\pi,\pi]$, 0 is not special. So, we now call $E_x = dense\, G_\delta\, set\, of\, functions \in C_{per}([-\pi,\pi])$ such that the Fourier series diverges at x.

So, now you take $\{x_i\}$ a countable set of points in $[-\pi,\pi]$ and you write $E=\cap_{i=1,2,\ldots,n}E_{x_i}\subseteq V$ and by Baire's theorem, E is also dense G_δ set. Now, for every $f\in E$, the Fourier series will diverge at x_i for all i. You define $S^{¿}(f,x)={}^{¿}n¿S_n(f)(x)\vee¿¿$. We have that $\{x:S^{¿}(f,x)=\infty\}$ is a G_δ set in $[-\pi,\pi]$ for each f. Now, choose $\{x_i\}$ dense in $[-\pi,\pi]$. So, in fact you can just take the rationals for instance. We have the following proposition.

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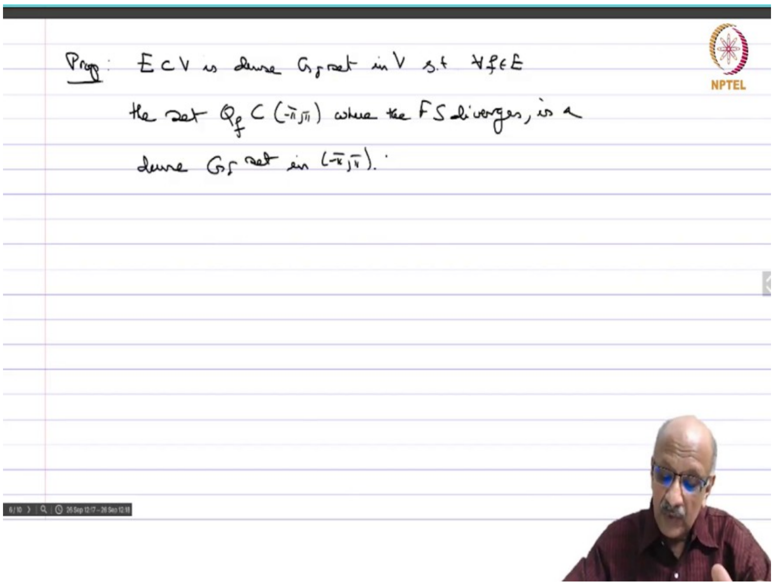

Proposition. Let $E \subseteq V$ is dense G_δ set in V such that for all $f \in E$, the set $Q_f \subseteq [-\pi, \pi]$ where the Fourier series diverges is a dense G_δ set in $[-\pi, \pi]$.

So, we saw a corollary of Baire's theorem, which says if you do not have isolated points, (in $C_{per}[-\pi, \pi]$ you do not have isolated points in $[-\pi, \pi]$), then dense G_δ set has to be uncountable.

Therefore, there exists unaccountably many 2π-periodic functions, for each of them the Fourier series will diverge at an uncountable number of points. So, the Fourier series convergence point-wise is not at all something which you can take for granted. So, that is everything comes from Baire's theorem and application of the Banach Steinhaus theorem and so this is a new application. So, next we will look at some other theorems which work in Banach spaces.