Functional Analysis
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Lecture 16
Baire's Theorem and Applications

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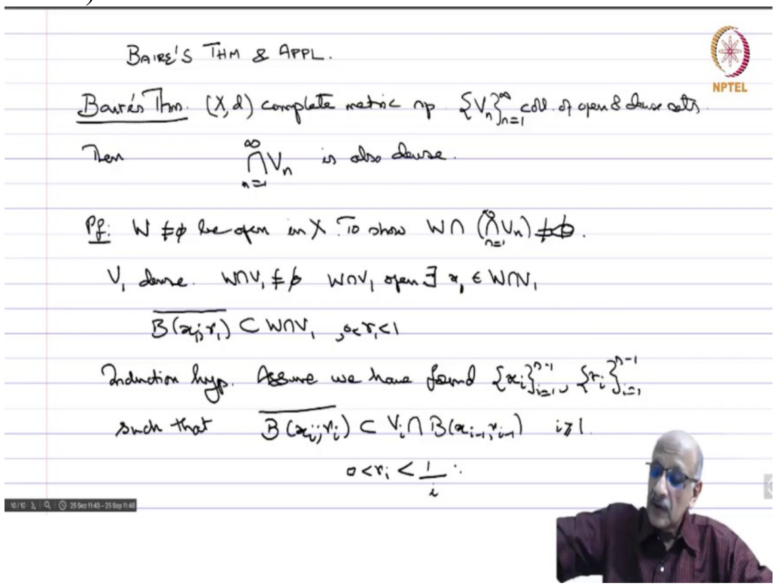

We will now start a new topic. This I will call **Baire's theorem and applications**.

There are four grand theorems of functional analysis. The first one is the Hahn–Banach theorem which we have already seen and three others follow from Baire's theorem.

Let us first recall what is Baire's theorem.

Theorem. Let (X, d) be a complete metric space and you have a collection of open and dense sets (V_n) then $\cap_{n=1,2,..,\infty} V_n$ is also dense.

So, in the complete metric space, if you have a countable family of open dense sets then the intersection is also dense (countable intersection of open sets is not an open set in general). We know that only finite intersections of open sets are open. But these are what are called G_δ-sets (a G_δ set is a countable intersection of open sets). So, if you have a countable collection of open dense sets then the intersection is a dense G_δ.

Proof. Let $W \neq \phi$ be open in X. We need to show that $W \cap (\cap_{n=1,2,..\infty} V_n) \neq \phi$. Then, every open set will meet the intersection and therefore, the intersection is automatically dense. First, since V_1 is dense, $W \cap V_1 \neq \phi$ and also open. So there exists $x_1 \in W \cap V_1$ and $0 < r_1 < 1$ such that $\overline{B(x_1; r_1)} \subseteq W \cap V_1$.

So, now we are going to use the induction hypothesis**.** Assume that we have found $(x_i)_{i=1,2,\ldots n-1}, (r_i)_{i=1,2,\ldots n-1}$such that $\overline{B(x_i;r_i)} \subseteq V_i \cap B(x_{i-1};r_{i-1}) \wedge 0<r_i<\frac{1}{i}; i \geq 1$. So, these are the two conditions in the induction hypothesis.

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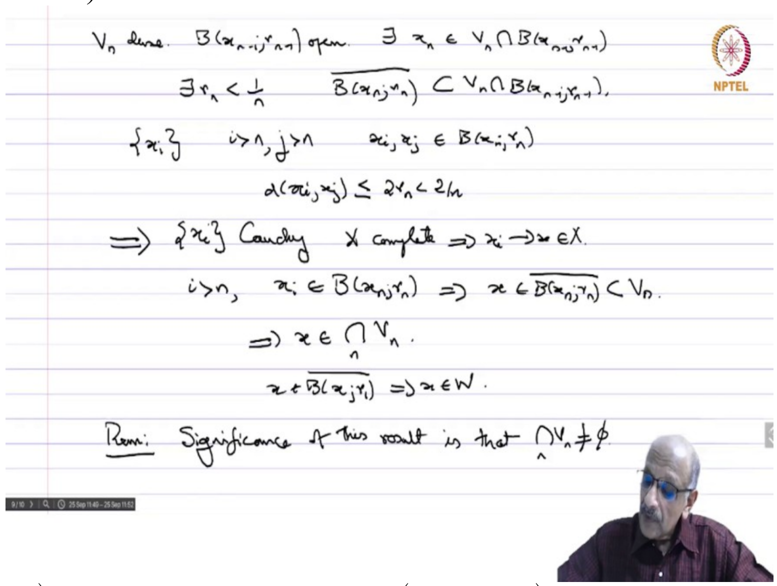

Now we take (V_n) as dense and since $B(x_{n-1};r_{n-1})$ is open, therefore, there exists $x_n \in V_n \cap B(x_{n-1};r_{n-1})$. Again the set $V_n \cap B(x_{n-1};r_{n-1})$ is open and therefore, there exists $r_n<\frac{1}{n}$, such that $\overline{B(x_n;r_n)} \subseteq V_n \cap B(x_{n-1};r_{n-1})$. So, what have we constructed? We have constructed a sequence (x_i) such that, if you take any $i>n, j>n$, then you have $x_i, x_j \in B(x_n;r_n)$ and therefore, $d(x_i,x_j) \leq 2r_n<\frac{2}{n}$. Therefore, (x_i) Cauchy. X is complete implies $x_i \to x \in X$. So, we have found a candidate. Now, for any $i>n$, you have $x_i \in B(x_n,r_n)$ and therefore, you have $x_i \to x$. So, $x \in \overline{B(x_n,r_n)} \subseteq V_n$. So, $x \in \cap_{n=1,2,\ldots\infty} V_n$. Also, $x \in \overline{B(x_1,r_1)}$ and this implies it $x \in W$. Therefore, we have found a common element in W and the intersection. This completes the proof.

Remark. The significance of this result is not really the denseness of $\cap_{n=1,2,\ldots\infty} V_n$. The significance of this result is that intersection $\cap_{n=1,2,\ldots\infty} V_n$is non-empty. That is how it is often used.

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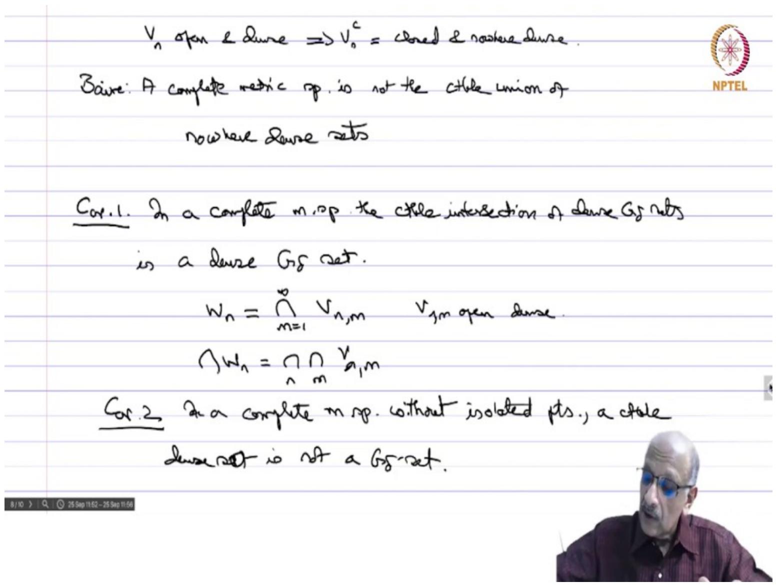

We can state this theorem in a different way. If V_n is open dense implies V_n^c is closed and nowhere dense and therefore, the theorem can also be stated as follows.

Barie's theorem. A complete metric space is not the countable union of nowhere dense sets.

The countable union of nowhere dense sets is usually called a first category and everything which is not first category is called second category and Baire's theorem says that complete metric space is therefore a set of second category and that is why it is called the sometimes the Baire's category theorem.

Corollary. In a complete metric space, the countable intersection of dense G_δ sets is a dense G_δ set. Now, what does this mean? So, G_δ set is the countable intersection of open sets. Let (W_n) *be dense* G_δ sets. So, $W_n = \cap_{m=1,2..\infty} V_{n,m}$; $V_{n,m}$ are open. Since W_n are dense, $V_{n,m}$ are also dense. So $\cap_{n=1,2\ldots\infty} W_n = \cap_{n=1,2\ldots\infty} \cap_{m=1,2..\infty} V_{n,m}$and that is again a countable intersection of open dense sets and therefore, it is also dense and since it is a countable intersection of open sets, it is also a G_δ. Therefore, in a complete metric space, the countable intersection of dense G_δ sets is a dense G_δ set.

Corollary 2. In a complete metric space without isolated points, countable dense set is not a G_δ set.

So, when I say in such a metric space without isolated points, when I say dense G_δ, that means, the set is automatically uncountable. So, that is how this Corollary is going to be used. Namely, if you have a complete metric spaces out isolated points and you have a dense G_δ, then it is automatically uncountable.

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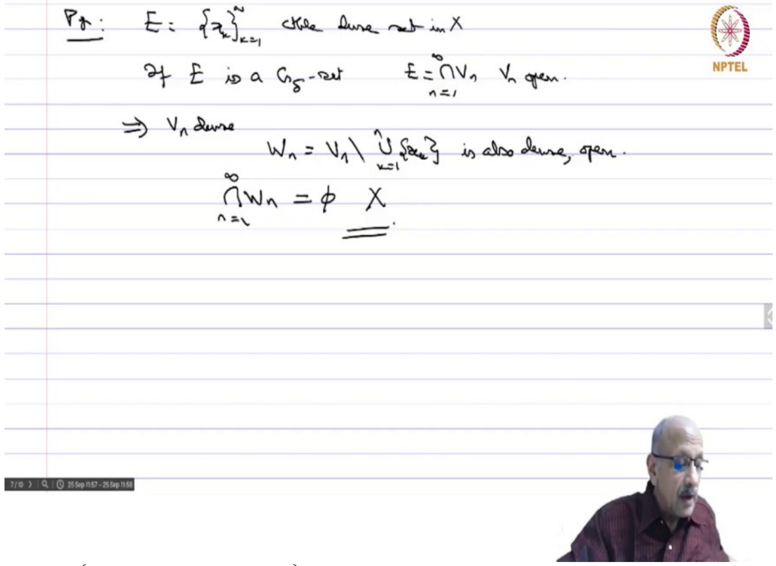

Proof. Let us take $E=\{x_k : k=1,2,\ldots\infty\}$ a countable dense set. If E is a G_δ, then E can be witness as $E=\cap_{n=1,2,..\infty} V_n; V_n open$ and then obviously, V_n is also dense (because it is bigger than the dense sets). Now, if you take $W_n=V_n \setminus \cup_{k=1,2,\ldots n}\{x_k\}$ is also dense. Then $\cap_{n=1,2,\ldots\infty} W_n=\phi$ which is a contradiction to Baire's theorem and therefore, this is not possible.

So, if you do not have isolated points, then dense G_δ will automatically be a uncountable.

The Baire's category theorem has several applications. In this section, we will see its application to Banach spaces.

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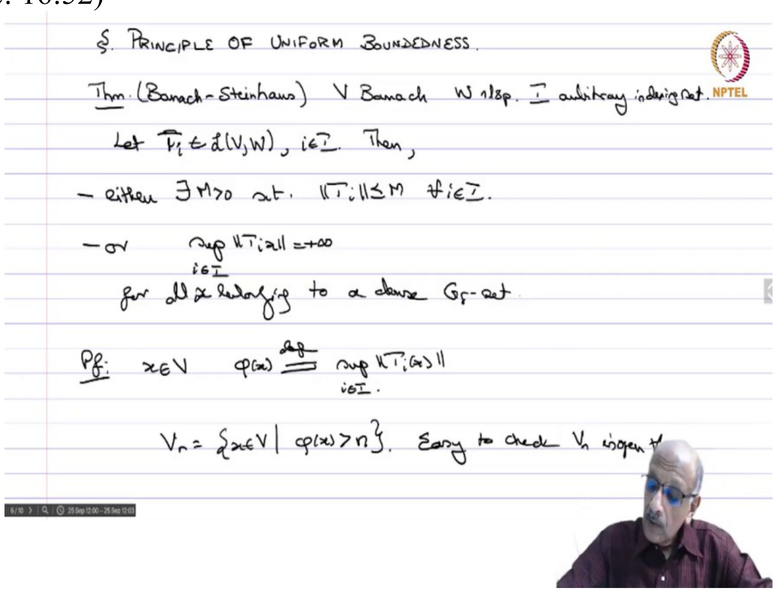

Anyway, so now, we will do some applications of Baire's category theorem in function analysis. So, the first application of Baire's category theorem is the principle of uniform boundedness.

Theorem. (Banach-Steinhaus theorem). Let V be Banach and W be a normed-linear space. Let I be an arbitrary indexing set and let $T_i \in L(V, W); i \in I$. Then,

(i). either there exists $M > 0$ such that $\|T_i\| \leq M, \forall i \in I$ (so (T_i) are uniformly bounded)

or (ii). $\underset{i \in I}{\text{¿}} \text{¿} |T_i(x)| \vee \text{¿} = \infty \text{¿}$ for all x belonging to dense G_δ set. (So, you see that the $\underset{i \in I}{\text{¿}} \text{¿} |T_i(x)| \vee \text{¿}\text{¿}$ will blow up for a large number of points).

Proof. For $x \in V$, define $\phi(x) = ^{\text{¿}} i \in I \text{¿} |T_i(x)| \vee \text{¿}\text{¿}$ and you set $V_n = \{x \in V : \phi(x) > n\}$. Now, norm is a continuous function, T_i's are all continuous functions and therefore, one can easily check V_n is open for all n. Now one of the two things can only happen, namely, all the V_n's may be dense or there may be V_n which is not dense.

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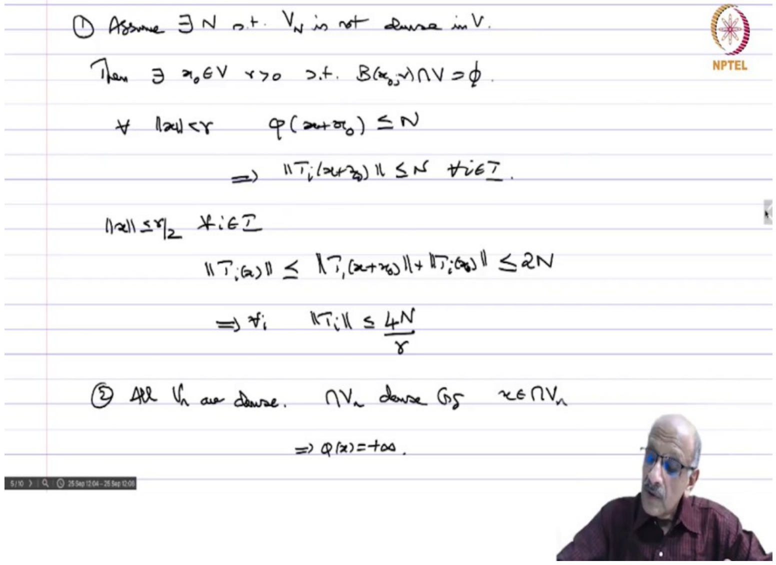

Assume that there exists N such that V_n is not dense in V. Then there exists $x_0 \in V \wedge r > 0$ such that $B(x_0; r) \cap V = \phi$. Therefore, for all $\|x\| < r$, $\phi(x + x_0) \leq N$. This implies that $\|T_i(x + x_0)\| \leq N, \forall i \in I$. Now, you choose $\|x\| \leq \frac{r}{2}$ and then for all $i \in I$, $\|T_i(x)\| \leq \|T_i(x + x_0)\| + \|T_i(x_0)\| \leq 2N$ and this implies that for all i, $\|T_i\| \leq \frac{4N}{r}$. So, this is the first alternative.

So, the second alternative is all V_n are dense, therefore, $\cap_n V_n$ is a dense G_δ set and if $x \in \cap_n V_n$ then $\phi(x) = +\infty$ and that is precisely what the theorem is saying. So, this is called the **Banach-Steinhaus theorem**.

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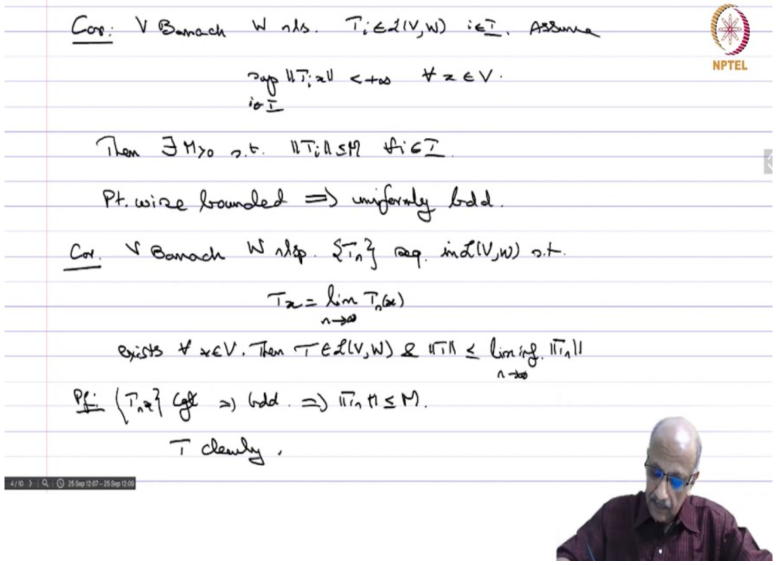

Corollary Let V be a Banach space and W be a normed-linear space and $T_i \in L(V,W); i \in I$. Assume that $\sup_{i\in I}\|T_i(x)\| < \infty, \forall x \in V$. Then that exists $M > 0$ such that $\|T_i\| \le M, \forall i \in I$. So, we have simply excluded the second possibility and therefore, this is just the first possibility in the previous theorem and therefore, you have point-wise bounded implies uniformly bounded.

So, that is why this is called the **Uniform Boundedness Principle**.

Corollary. Let V be a Banach space and W be a normed-linear space and $(T_n) \in L(V,W)$ such that $T(x) = \lim_{n\to\infty} T_n(x)$ exists for all $x \in V$. Then $T \in L(V,W)$ and $\|T\| \le \liminf_{n\to\infty} \|T_n\|$.

Proof. $(T_n(x))$ is convergent implies that it is bounded. Therefore, $\|T_n\| \le M$ as we know from the previous corollary. T is clearly linear.

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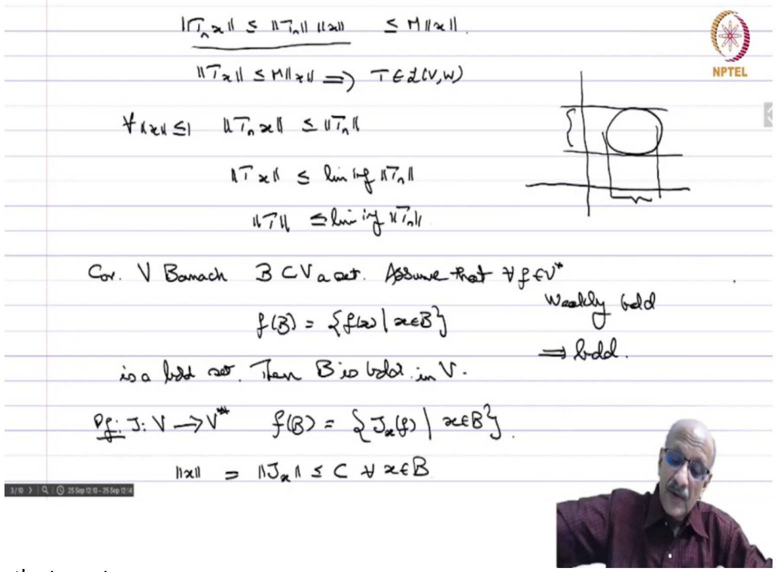

Therefore, $\|T_n(x)\| \leq \|T_n\| \|x\| \leq M \|x\|$. Therefore, if you pass to the limit, you get $\|T(x)\| \leq M \vee |x| \vee ¿$ and this implies that $T \in L(V, W)$. So, pointwise limit of bounded linear operators is also a bounded linear operator, if V is Banach.

Now, if you take for all $\|x\| \leq 1$, you have $\|T_n(x)\| \leq \vee |T_n| \vee ¿$ and therefore, if you took limit on each side, you get $\|T(x)\| \leq \lim inf ¿ |T_n| \vee ¿¿$ and therefore, you have $\|T\| \leq \lim inf ¿ |T_n| \vee ¿¿$.

Corollary. Let V be a Banach space and $B \subseteq V$. Assume that for every $f \in V^{¿}$, $f(B) = \{f(x) : x \in B\}$ is a bounded set in $R \vee C$, whichever is the scalar field. Then B is bounded in V.

Proof. You will look at $J : V \mapsto V^{¿*¿¿}$. $f(B) = \{J_x(f) : x \in B\}$. All the J_x are bounded. So, by the Hahn–Banach theorem $\|J_x\| \leq C, \forall x \in B$. But J_x is an isometry, so $\|J_x\| = ¿|x| \vee ¿$ and therefore B is bounded in V.

So, we will now look at a very nice application of the uniform bonded principle to analysis.