Functional Analysis
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Lecture 15
Exercises-Part 2

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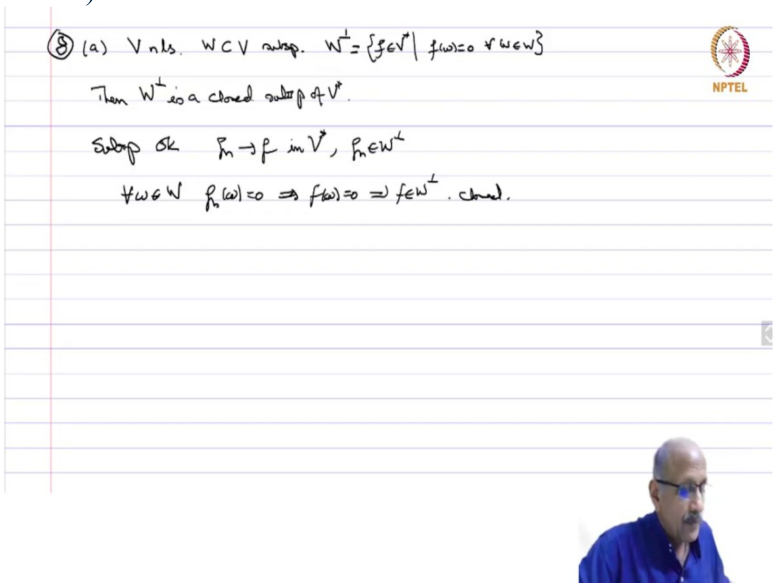

Exercise 8. We will do in several parts.

(a). Let V be a non-linear space and W contained in V is a subspace. I am going to define $W^{\perp}=\{f\in V^{¿}:f(w)=0\ \forall\ w\in W\}$. Then $W^{\perp}$ is a closed subspace.

Solution. This is almost immediate. By linearity, $W^{\perp}$ is a subspace. What about closure? Suppose $f_n\to f\in V^{¿}, f_n\in W^{\perp}$. Then, for every $w\in W^{\perp}$, you have $f_n(w)=0$ implies $f(w)=0$. This implies $f\in W^{\perp}$. So, $W^{\perp}$ is closed.

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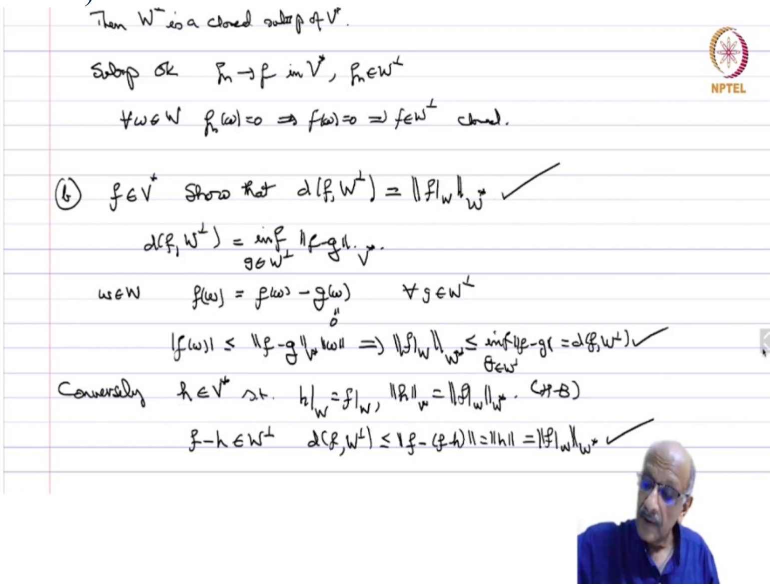

Next part of this exercise very interesting.

(b). For $f \in V^{*}$ show that $d(f, W^{\perp}) = \| f|_W \|_{W^{*}}$, where $d(f, W^{\perp}) = \inf_{g \in W^{\perp}} \|f - g\|_{V^{*}}$.

Solution. Let us take $w \in W$. Then $f(w) = f(w) - g(w), \forall\, g \in W^{\perp}$. Therefore, $|f(w)| \le \|f - g\|_{V^{*}} \|w\|$ and therefore, you have $\| f|_W \|_{W^{*}} \le \inf_{g \in W^{\perp}} \|f - g\|_{V^{*}} = d(f, W^{\perp})$. So, we have one-sided inequality.

Conversely, let $h \in V^{*}$ is such that $h|_W = f|_W$ and $\|h\|_{V^{*}} = \| f|_W \|_{W^{*}}$. Then $f - h \in W^{\perp}$ and you have $d(f, W^{\perp}) \le \|f - (f - h)\| = \|h\| = \| f|_W \|_{W^{*}}$. So, you have the reverse inequality. So, $d(f, W^{\perp}) = \inf_{g \in W^{\perp}} \|f - g\|_{V^{*}}$.

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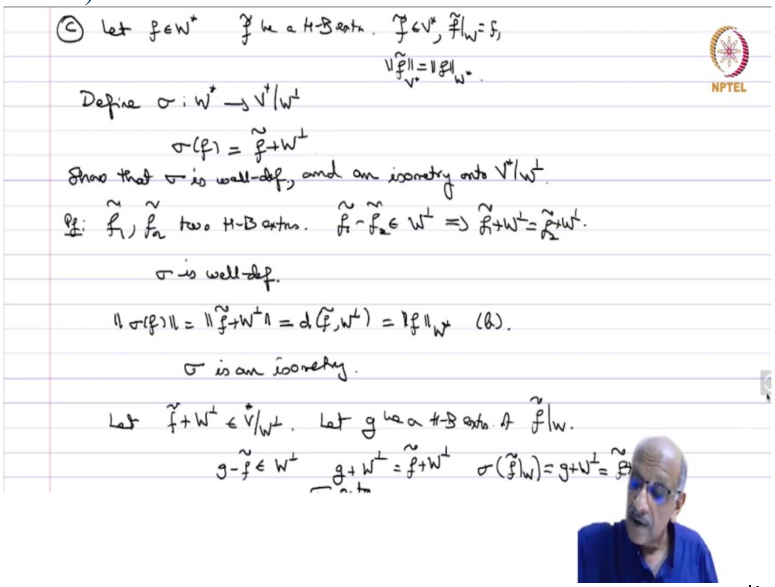

(c). Let $f \in W^{¿}$ and $\tilde{f}$ be Han Banach extension of f i.e., $\tilde{f} \in V^{¿}, \tilde{f}\,|_W = f, \|\tilde{f}\|_{V^{¿}} = \|f\|_{W^{¿}}$. Define $\sigma : W^{¿} \mapsto \frac{V^{¿}}{W^{\perp}}$ as $\sigma(f) = \tilde{f} + W^{\perp}$.Show that σ is well defined and isometry onto $\frac{V^{¿}}{W^{\perp}}$.

What does this theorem say? $\sigma : W^{¿} \mapsto \frac{V^{¿}}{W^{\perp}}$is isometric isomorphism. So, it is one to one, onto, and it is isometry. So, it is continuous. Consequently, the dual of the space W can be thought of as $\frac{V^{¿}}{W^{\perp}}$. So, we have identified $W^{¿}$.

Solution. Let $\tilde{f}_1, \tilde{f}_2$ be two Hahn Banach extensions of f. That mean, they agree on W. So, $\tilde{f}_1 - \tilde{f}_2 \in W^{\perp}$. This implies that $\tilde{f}_1 + W^{\perp} = \tilde{f}_2 + W^{\perp}$ and therefore, σ is well defined.

Now, what is $¿ \vee \sigma(f) \vee ¿$? $\|\sigma(f)\| = \|\tilde{f} + W^{\perp}\| = d(\tilde{f}, W^{\perp}) = \|f\|_{W^{¿}}$. Therefore, σ is an isometry.

Finally, we have to check that it is surjective. So, let $\tilde{f} + W^{\perp} \in \frac{V^{¿}}{W^{\perp}}$. Let g be a Hahn Banach extension of $\tilde{f}\,|_W$. So, g also agrees with $\tilde{f}$ on W and therefore, $g - \tilde{f} \in W^{\perp}$. Therefore, $g + W^{\perp} = \tilde{f} + W^{\perp}$. But then, $\sigma\left(\tilde{f}\,|_W\right) = g + W^{\perp} = \tilde{f} + W^{\perp}$. This proves the theorem completely.

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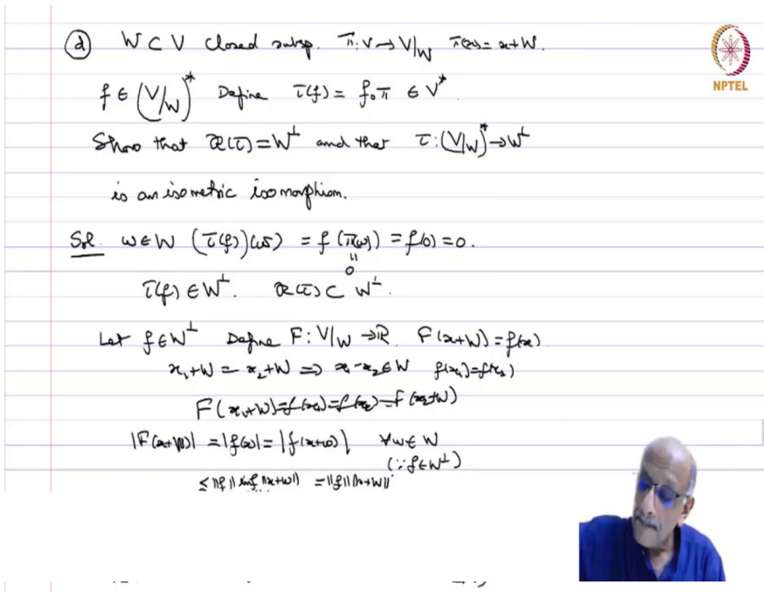

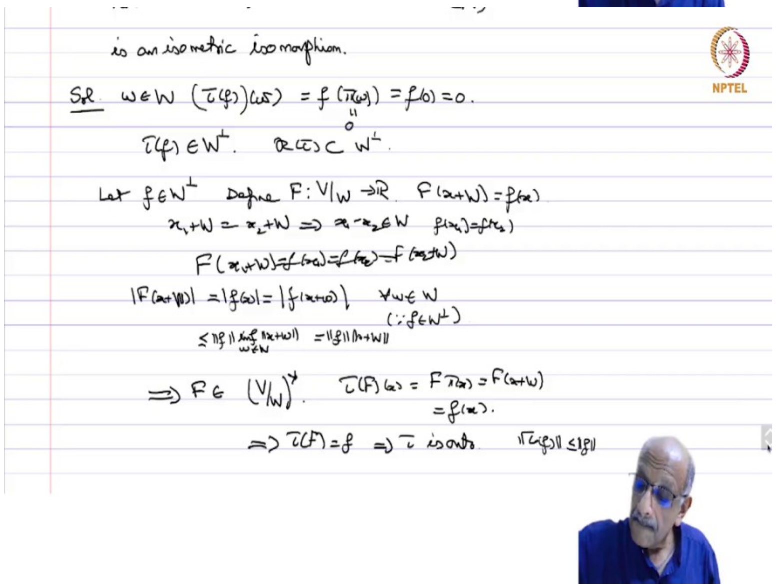

We have identified the dual of a product we have identified the dual of subspace. Now, we want to identify the dual of a quotient space. That is the next item.

(d). Let W be a closed subspace in V. We define $\pi : V \mapsto \frac{V}{W}$ as $\pi(x) = x + W$. For $f \in \left(\frac{V}{W}\right)^{*}$ define $\tau(f) = f \circ \pi \in V^{*}$. Show that $Range(\tau) = W^{\perp}$ and $\tau : \left(\frac{V}{W}\right)^{*} \mapsto W^{\perp}$ is an isometric isomorphism.

So, τ is onto and an isometry and therefore, we have now the dual of the quotient space $\frac{V}{W}$ is nothing but $W^{\perp}$. That is a very beautiful characterization.

Solution. You take $w \in W$ and let us take $\tau(f)(w) = f(\pi(w)) = f(0) = 0$. Therefore, $\tau(f) \in W^{\perp}$. So, $Range(\tau) \subseteq W^{\perp}$. Now, let $f \in W^{\perp}$. Define $F : \frac{V}{W} \mapsto R$ as $F(x+W) = f(x)$. We have to check that it is well defined. Let $x_1 + W = x_2 + W$. Then $x_1 - x_2 \in W$. So, f ¿. So, $F(x_1 + W) = f(x_1) = f$ ¿ So, if you have two same cosets given by different representatives, it does not matter, the answer is finally the same. So, this is well defined and also $|F(x+W)| = |f(x+w)|, \forall w \in W$ (as $f \in W^{\perp}$¿.

Therefore, $|F(x+W)| \leq \|f\| \inf_{w \in W}$ ¿$|x+w| \vee$ ¿$= \|f\| \vee |x+W| \vee$ ¿¿. Therefore, $F \in \left(\frac{V}{W}\right)^{¿}$. Now $\tau(F)(x) = F(\pi(x)) = F(x+W) = f(x)$ and therefore, $\tau(F) = f$. This implies that τ is onto. You also have that $\|\tau(F)\| \leq \|f\|$..

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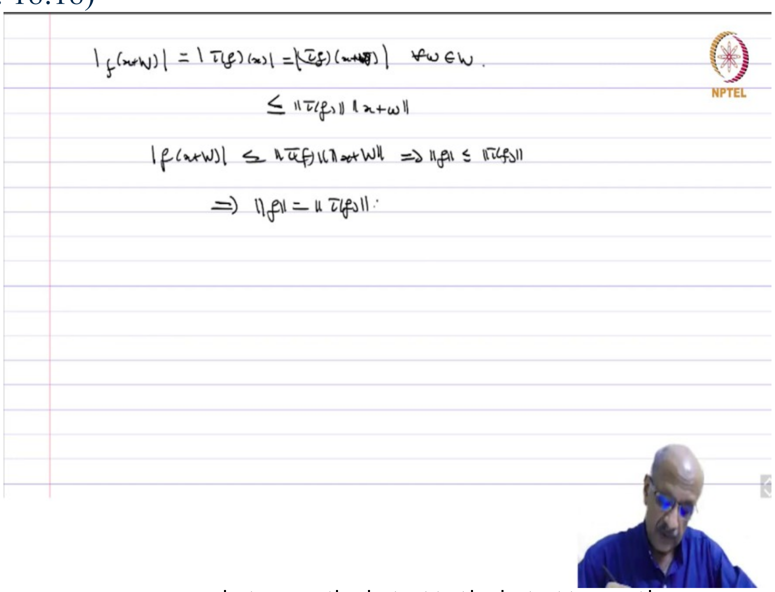

Finally, we have $|f(x+W)| = |\tau(F)(x)| = |\tau(F)(x+w)|, \forall w \in W$. Therefore, $|f(x+W)| \leq \|\tau(F)\| \vee |x+W| \vee$ ¿ and this implies that $\|f\| \leq \vee |\tau(F)| \vee$ ¿. Therefore, we have $\|f\| =$ ¿$|\tau(F)| \vee$ ¿. Therefore, the dual of the space $\left(\frac{V}{W}\right)^{¿}$ is nothing but $W^{\perp}$.So, that is dual of the quotient space.

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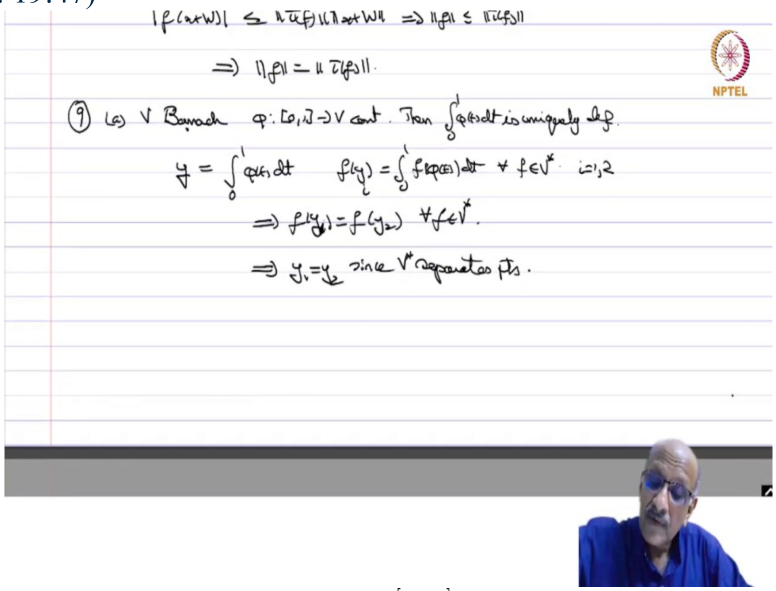

Exercise 9. (a). Let V be a Banach space and $\phi:[0,1]\mapsto V$ be continuous. Then, we know that the integral exists. Show that $\int_{[0,1]} \phi(t)\,dt$ is uniquely defined.

Solution. Let you had two vectors y_1, y_2 such that $f(y_i)=\int_{[0,1]} f(\phi(t))\,dt, \forall f \in V^{¿}; i=1,2.$ This implies that $f(y_1)=f(y_2), \forall f \in V^{¿}$. But we know that $V^{¿}$ separates points. Hence, $y_1=y_2$.

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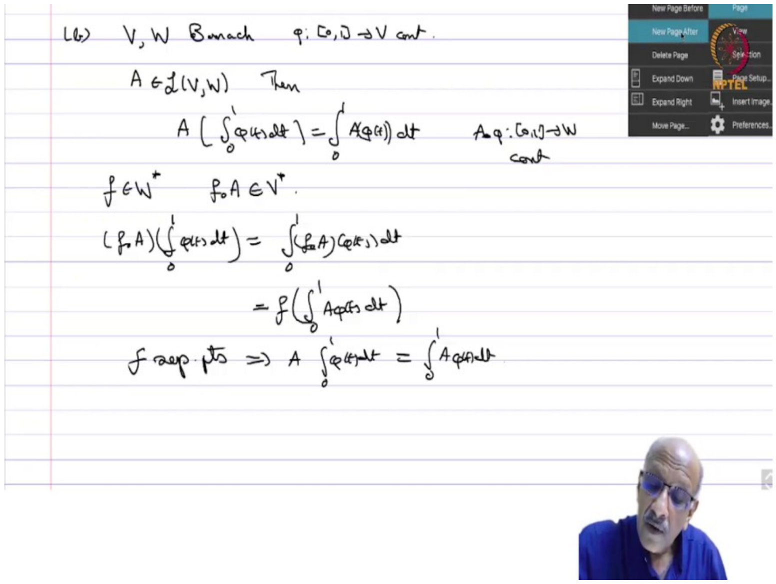

(b). What did we see? Namely, every continuous linear functional should go through the integral. Now, we will show that in fact, every continuous linear operator also goes through the integrals. Let V, W be two Banach spaces and $\phi : [0,1] \mapsto V$ be continuous. Let $A \in L(V, W)$. Then

$$A\left(\int_{[0,1]} \phi(t)\, dt\right) = \int_{[0,1]} A(\phi(t))\, dt.$$

Solution. Let us take any $f \in W^{¿}$. Then $f \circ A \in V^{¿}$. So,

$(f \circ A)\left(\int_{[0,1]} \phi(t)\, dt\right) = \int_{[0,1]} (f \circ A)(\phi(t))\, dt = f\left(\int_{[0,1]} A(\phi(t))\, dt\right)$. Now f separates points implies $A ¿$.

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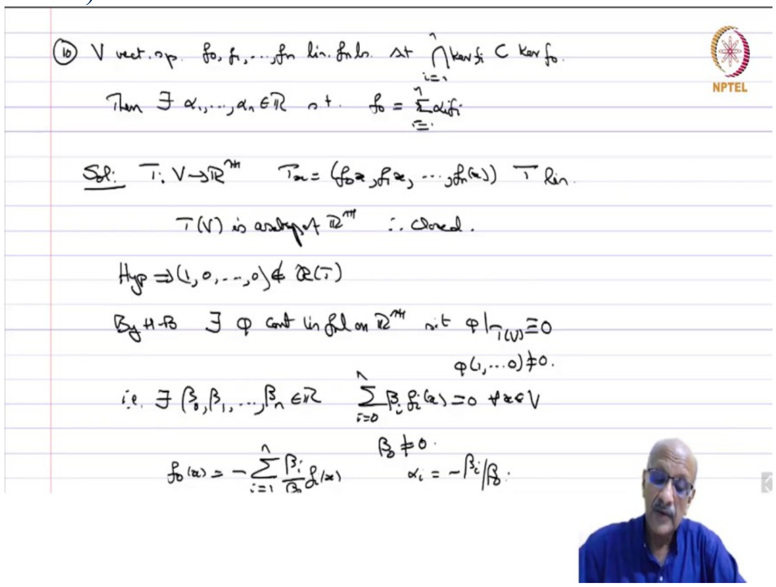

Exercise 10. It is a very nice exercise, which has a lot of applications.

Let V be a vector space and $f_0, f_1, \ldots, f_n$ be linear functional such that $\cap_{i=1,2\ldots n} Ker(f_i) \subseteq f_0$. Then there exists $\alpha_1, \ldots \alpha_n \in R$ such that $f_0 = \sum_{i=1,2,\ldots,n} \alpha_i f_i$.

Solution. Define $T: V \mapsto R^{n+1}$ as $T(x) = (f_0(x), f_1(x), \ldots, f_n(x))$. So, T is linear. So, $T(V)$ is a subspace of R^{n+1} and everything is in finite dimensional, therefore closed. Now, hypothesis implies $(1,0,0,\ldots 0) \notin Range(T)$. So, by Hahn Banach, there exists a continuous linear functional ϕ on R^{n+1} such that $\phi|_{T(V)} = 0$ and $\phi((1,0,0,\ldots 0)) \neq 0$. But what is the continuous linear functional on R^{n+1}. The dual of R^{n+1} is R^{n+1} itself. So, there exists $\beta_0, \ldots \beta_n \in R$ such that $\sum_{i=0,1,\ldots,n} \beta_i f_i(x) = 0$ for all $x \in V$ but this should not vanish on $(1,0,0\ldots 0)$ that means $\beta_0 \neq 0$. So, now it is immediate. You have that $f_0(x) = -\sum_{i=1,2,\ldots,n} \frac{\beta_i}{\beta_0} f_i(x)$. So, you just have to take $\alpha_i = \frac{-\beta_i}{\beta_0}$.

So, these are some of the exercises there are many more exercises in the book which I mentioned and in many other books too. So, it is good for you to try some of them. Thank you.